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ON A METRIC INDUCED BY ANALYTIC CAPACITY

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Dedicated to Professor Yukinari Tôki on his 60th birthday

1. In our previous paper [6] we gave a conjecture that the metrics $c_\beta(z)|dz|$ and $\sqrt{\pi \tilde{K}(z, z)}|dz|$ have negative curvatures $\leqq -4$; here $c_\beta(z)$ and $\tilde{K}(z, z)$ are the capacity and the Bergman kernel of exact analytic differentials on an open (non-trivial) Riemann surface. In the present paper we shall show that the curvature of the metric $c_B(z)|dz| \leqq -4$ for plane regions $\Omega \notin O_{AB}$, where $c_B(z)$ denotes the analytic capacity of Ω at z. In order to verify $c_B(z) \in C^2$ we prove that $c_B(z)$ is real analytic. This enables us to answer a question of Havinson [4], namely "Does the sequence of extremal functions ψ_n in the dual problem of Schwarz's lemma in Ω_n converge as $\{\Omega_n\}$ exhausts Ω?".

2. Let Ω be a plane region $\notin O_{AB}$. The analytic capacity $c_B(\zeta)$ is given by $\sup |f'(\zeta)|$ in the family of analytic functions satisfying $f(\zeta)=0$ and $|f(z)| \leqq 1$. Let $\{\Omega_n\}$ be a canonical exhaustion of Ω such that the boundary of Ω_n consists of a finite number of analytic curves. Let $c_n(\zeta)$ be the analytic capacity of Ω_n. Then $\{c_n(\zeta)\}$ is decreasing and tends to $c_B(\zeta)$. There exist extremal functions f_n such that $f'_n(\zeta)=c_n(\zeta)$ and $f'_0(\zeta)=c_B(\zeta)$. It is known that those extremal functions are unique [4]. The function $\log c_B(\zeta)$ is subharmonic [2].

In every Ω_n there exists the Szegö kernel $k_n(z, \zeta)$ and its adjoint kernel $l_n(z, \zeta)$ [3]. $k_n(z, \zeta)$ is hermitian and analytic with respect to z and $\bar{\zeta}$. Further the following facts are known [3]:

$$f_n(z) = \frac{k_n(z, \zeta)}{l_n(z, \zeta)} \quad \text{with} \quad c_n(\zeta) = 2\pi k_n(\zeta, \zeta) \tag{1}$$

and

$$|k_n(z, \zeta)|^2 \leqq |k_n(z, z)||k_n(\zeta, \zeta)|.$$

Thus $k_n(z, \zeta)$ is uniformly bounded on every compact subset and hence forms a normal family of analytic functions of two variables $z, \bar{\zeta}$. We will show that $\{k_n(z, \zeta)\}$ converges to a function $k(z, \zeta)$ uniformly on every compact subset of Ω.

Suppose that there exist two limit functions $k(z, \zeta)$ and $k^*(z, \zeta)$. We may assume $0 \in \Omega$. The difference $k(z, \zeta) - k^*(z, \zeta)$ has an expansion in a polydisc $\{|z|<r\} \times \{|\zeta|<r\}$

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$$k(z,\zeta)-k^*(z,\zeta)=\sum_{\nu,\mu=0}^{\infty} a_{\nu\mu}z^{\nu}\bar{\zeta}^{\mu} \qquad \text{with} \qquad a_{\nu\mu}=\bar{a}_{\mu\nu}.$$

Since $k(z, z)=k^*(z, z)=c_B(z)/2\pi$, by setting $z=|z|e^{i\theta}$ we have $a_{00}=0$ and by induction

$$\sum_{\nu+\mu=n} a_{\nu\mu}e^{i(\nu-\mu)\theta}=0.$$

Hence $a_{\nu\mu}=0$ and $k(z,\zeta)$ coincides with $k^*(z,\zeta)$. $\{f_n(z)\}$ clearly converges to $f_0(z)$ uniformly on every compact subest of Ω. Thus the sequence $\{l_n(z,\zeta)\}$ converges to a meromorphic function $l(z,\zeta)$ uniformly on every compact subset of $\Omega-\zeta$. This yields

THEOREM 1. *The sequence $\{k_n(z,\zeta)\}$ converges uniformly on every compact subset of Ω. The analytic capacity $c_B(z)$ is real analytic. Further the sequence $\{l_n(z,\zeta)\}$ converges uniformly on every compact subset of $\Omega-\zeta$.*

3. The last statement gives an affirmative answer to a question of Havinson [4]. For a moment suppose $\infty\in\Omega$. Take an exhaustion $\{\Omega_n\}$ of Ω with $\infty\in\Omega_1$. Let $\tilde{E}_1(\Omega_n)$ be the family of functions ϕ analytic in Ω_n satisfying $\phi(\infty)=0$ and

$$\int_{\partial\Omega_n}|\phi|ds\leqq 1,$$

where ds is the length element on $\partial\Omega_n$.

As the duality relation in Schwarz's lemma, Havinson [4] showed

$$c_n(\infty)=\min_{\phi\in\tilde{E}_1(\Omega_n)}\int_{\partial\Omega_n}|1+\phi|ds$$

and that the extremal function ϕ_n exists uniquely in $\tilde{E}_1(\Omega_n)$ and satisfies

$$-if_n(z)(1+\phi_n(z))dz=|1+\phi_n(z)|ds$$

where f_n is the extremal function in no. **2** with $\zeta=\infty$. He proved that every subsequence of $\{\phi_n\}$ contains a convergent subsequence and conjectured that $\{\phi_n\}$ itself is convergent as $n\to\infty$.

On the other hand the square of $l_n(z,\infty)$ is expressed as

$$l_n(z,\infty)^2=\frac{1}{4\pi^2}\left(1+\frac{b_1}{z}+\cdots\right)$$

near the point at infinity and has no zeros in Ω_n. The fundamental relation $l_n(z,\infty)ds=i\overline{k_n(z,\infty)dz}$ along $\partial\Omega_n$ [3] and (1) show $-il_n(z,\infty)^2f_n(z)dz>0$ along $\partial\Omega_n$ which implies that $(1+\phi_n(z))/l_n(z,\infty)^2$ is real and that $\phi_n(z)=4\pi^2l_n(z,\infty)^2-1$. The validity of his conjecture is deduced from Theorem 1.

4. We now turn to the estimation of the curvature of the metric $c_B(z)|dz|$. Let $\lambda|dz|$ and $\mu|dz|$ with $\lambda,\mu\geqq 0$ be two metrics on a plane region Ω. Following

Ahlfors [1], we call $\mu|dz|$ a *supporting metric of* $\lambda|dz|$ at the point ζ if 1) $\lambda=\mu$ at ζ and 2) $\lambda-\mu\geqq 0$ in a neighborhood of ζ. Then we prepare

LEMMA. *If* $\mu|dz|$ *is a supporting metric of* $\lambda|dz|$ *and if both* λ *and* μ *are of class* C^2, *then the curvature of* $\mu|dz|$ *dominates that of* $\lambda|dz|$.

Proof. The curvature $\kappa(\lambda; z)$ of $\lambda|dz|$ at z is given by

$$\kappa(\lambda; z)=\frac{-\Delta\log\lambda}{\lambda^2}.$$

Since $\log(\lambda/\mu)$ assume a local minimum at $z=\zeta$, we have $\Delta\log(\lambda/\mu)\geqq 0$, which implies the assertion.

We state

THEOREM 2. *The curvature of the metric* $c_B(z)|dz|\leqq -4$.

Proof. Let $f_0(z)$ be the extremal function with $c_B(\zeta)=f_0'(\zeta)$. Set $F(z,\eta)=(f_0(z)-f_0(\eta))/(1-\overline{f_0(\eta)}f_0(z))$. We have $|F(z,\eta)|\leqq 1$ and

$$\frac{|f_0'(\eta)|}{1-|f_0(\eta)|^2}\leqq c_B(\eta).$$

The metric $F'(z,z)|dz|$ has the curvature -4 at every point z except for the zeros z_ν of f_0'. Thus $c_B(z)|dz|$ has a supporting metric $|F'(z,z)||dz|$ at every point $\zeta\in\Omega$. We have $\kappa(c_B; z)\leqq -4$.

There remains a problem to decide the equality statement in Theorem 2. It is plausible that if $\kappa(c_B; z)=-4$ at one point z, Ω is conformally equivalent to the unit disc less a (possible) closed set expressed as a countable union of compact $N_{\mathfrak{B}}$ sets.

5. A recent development of the theory of conformal metrics was given by Heins [5]. He defined an S-K metric by a metric $\lambda|dz|$ such that

1) λ is nonnegative and upper semi-continuous, and

2) $$\varliminf_{r\to 0}\frac{1}{r^2}\left(\frac{1}{2\pi}\int_0^{2\pi}\log\lambda(\zeta+re^{i\theta})d\theta-\log\lambda(\zeta)\right)\geqq\lambda(\zeta)^2.$$

As he pointed out, the operator in the left hand side in 2) is one fourth of the generalized lower Laplacian and if $\log\lambda\in C^2$, the condition 2) reduces to

$$\Delta\log\lambda(\zeta)\geqq 4\lambda(\zeta)^2.$$

A sufficient condition for a metric to be S-K is the existence of a supporting metric with curvature $\leqq -4$ at every point [5]. From this Theorem 2 again follows.

This remark works for the metric $c_{SB}(z)|dz|$ associated with bounded univalent functions. Let $\mathfrak{F}$ be the family of univalent functions satisfying $|g|\leqq 1$ on Ω. $c_{SB}(\zeta)$ is defined by

$$c_{SB}(\zeta)=\sup_{g\in\mathcal{F}}|g'(\zeta)|.$$

If $\mathcal{F}\neq\phi$, there exists an extremal function satisfying $c_{SB}(\zeta)=g_0'(\zeta)$ with $g_0(\zeta)=0$ in $\mathcal{F}$. Thus we can construct a supporting metric at every point $\zeta\in\Omega$ as before. However $c_{SB}(z)$ is not necessarily differentiable (an example is $c_{SB}(z)$ for an annulus $1<|z|<r$ at the points on the circle $|z|=\sqrt{r}$). Hence we must stop at the result that $c_{SB}(z)|dz|$ is an S-K metric on Ω.

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