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ON A MINIMUM CYCLE BASIS OF A GRAPH

1. Introduction. Let $G = \langle V(G), E(G)\rangle$ denote a *simple graph*, i.e., a graph without loops and multiple edges. We define a *cycle* C in G as a sequence of edges

$$C = \{(x_{i_1}, x_{i_2}), (x_{i_2}, x_{i_3}), \ldots, (x_{i_{h-1}}, x_{i_h}), (x_{i_h}, x_{i_1})\}, \qquad \text{where } h \geqslant 3,$$

such that if $l \neq k$, then $x_{i_l} \neq x_{i_k}$. The *length* $|C|$ of the cycle C is the number of its edges. Cycles of a graph G generate the *cycle space* with symmetric difference of C_1 and C_2 (i.e., $C_1 \oplus C_2 = (C_1 \cup C_2) - (C_1 \cap C_2)$) as an addition of cycles C_1 and C_2. The *dimension* of the cycle space is equal to

$$\lambda = \nu(G) = m(G) - n(G) + p(G),$$

where $n(G) = |V(G)|$, $m(G) = |E(G)|$, and $p(G)$ is the number of connected components of G. A *cycle basis* of G is defined as a basis for the cycle space of G which consists entirely of cycles.

Let $\mathscr{C} = \{C_1, C_2, \ldots, C_\lambda\}$ denote a cycle basis; then we define the *length* of $\mathscr{C}$ by

$$|\mathscr{C}| = \sum_{i=1}^{\lambda} |C_i|.$$

Without loss of generality we may assume that G is a *biconnected graph*, i.e., every edge belongs to a cycle of G and the cycle space of G is not a direct sum of subspaces.

We are interested in finding a minimum cycle basis. Such bases have been considered by Stepanec [2], Zykov [5] and then by Hubicka and Sysło [1]. In this paper we formulate the problem in terms of matroids and then we consider a particular case of the general problem, namely, of finding a minimum cycle basis which is a *fundamental cycle set*, i.e., which can be generated by a spanning tree.

In papers [2] and [5] methods for finding extremal cycle bases were presented, but they do not give a solution for all graphs. Paper [1] contains the characterization of those graphs for which the methods of [2] and [5] fail to find a solution and includes algorithms for finding subminimum cycle bases and subminimum fundamental cycle sets.

In this paper we present counterexamples for the conjectures that the algorithms of [1] find a minimum cycle basis and a minimum fundamental cycle set of a graph. In the last section we propose a method for finding a subminimum fundamental cycle set of a graph, which is an iteration of the method of [1] for pseudo-random starting solutions.

2. Matroid formulation. The problem of finding an extremal cycle basis of a graph can be formulated in terms of matroids.

A *matroid* $M = (X, \mathscr{I})$ consists of a finite set X of elements and of a family $\mathscr{I}$ of subsets of X which satisfy the following conditions:

(a) $\emptyset \in \mathscr{I}$ and all proper subsets of a set I in $\mathscr{I}$ are in $\mathscr{I}$.

(b) If I and J are sets in $\mathscr{I}$ containing p and $p+1$ elements, respectively, then there exists an element $x \in J-I$ such that $I \cup \{x\} \in \mathscr{I}$.

A subset I in $\mathscr{I}$ is said to be an *independent set*. Let $M = (X, \mathscr{I})$ be a matroid whose elements X have obtained non-negative weights w, i.e., $w\colon X \to R_+ \cup \{0\}$. We are interested in finding a maximal independent set of a matroid for which the sum of weights of its elements is minimum. This problem can be solved by the following greedy algorithm:

GREEDY ALGORITHM. Choose the elements of the matroid in the order of their weights, the lightest element first, rejecting an element only if its selection would destroy independence of elements chosen so far.

The problem of finding a minimum cycle basis of a graph G can be formulated now in terms of matroids as follows.

Let $M = (X, \mathscr{I})$ be the matroid induced by G, i.e.

(a) X is the set of all simple cycles of G;

(b) the set $I \subseteq X$ belongs to the family $\mathscr{I}$ if and only if I is a cycle or I is a set of independent cycles.

A maximal independent set of $\mathscr{I}$ is a cycle basis of G. Let $|C|$ be the weight of a cycle C. The problem of finding a minimum cycle basis of a graph G is now equivalent to that of finding the maximal element of $\mathscr{I}$ which has the minimum weight. We can find such an element of $\mathscr{I}$ by the greedy algorithm.

The main disadvantage of the greedy algorithm applied to the problem of finding a minimum cycle basis is that we must know all cycles of a graph G.

3. Let $C_1, C_2, \ldots, C_k$ be a set of independent cycles of C. Let us put

$$G_k^c = \langle V(G_k^c), E(G_k^c)\rangle, \quad G_k = \langle V(G_k^c), E(G_k)\rangle,$$

where

$$V(G_k^c) = \bigcup_{i=1}^{k} V(C_i), \quad E(G_k^c) = \bigcup_{i=1}^{k} E(C_i)$$

and

$$E(G_k) = \big(V(G_k^c) \times V(G_k^c)\big) \cap E(G).$$

In other words, G_k^c is the subgraph of G generated by the edges of cycles $C_1, C_2, \ldots, C_k$, and G_k is the subgraph of G induced by the set of vertices of these cycles.

ALGORITHM 1 (Hubicka and Sysło [1]). Initialization. Let $u_1 \in E(G)$. Find the shortest cycle C_1 containing u_1.

General Step. Let $C_1, C_2, \ldots, C_{k-1}$ be a system of independent cycles chosen so far and such that $\nu(G_{k-1}^c) = k-1$. For $u_k \in E(G)-E(G_{k-1}^c)$ find in G the shortest cycle C_k which contains u_k and satisfies $\nu(G_k^c) = k$. Repeat the General Step as long as possible.

If k is a number such that for every $u_k \in E(G)-E(G_{k-1}^c)$ every shortest cycle containing u_k satisfies $\nu(G_k^c) > k$, then perform the Completion of the Algorithm 1 for one edge from $E(G)-E(G_{k-1}^c)$ and go back to the General Step.

Completion. We have $\nu(G_{k-1}^c) = k-1$ and $\nu(G_k^c) = k+p > k$. Find a sequence of vertices $x_1, y_1, x_2, y_2, \ldots, x_{p+1}, y_{p+1}$ such that they belong to C_k and G_{k-1}^c and the cycle C_k from x_i to y_i $(i = 1, 2, \ldots, p+1)$ is not contained in G_{k-1}^c. Find e_i and f_i, the shortest paths from x_i to y_i in G_{k-1}^c and C_k, respectively. The current system of independent cycles is of the form

$$C_1, C_2, \ldots, C_k, C_{k+1} = (e_{i_1}, f_{i_1}), \ldots, C_{k+p} = (e_{i_p}, f_{i_p}),$$

where $(e_{i_1}, f_{i_1}), \ldots, (e_{i_p}, f_{i_p})$ are p shortest cycles chosen among $p+1$ cycles $(e_1, f_1), \ldots, (e_{p+1}, f_{p+1})$.

Algorithm 1 finds a subminimum cycle basis of a graph G.

Example 1.

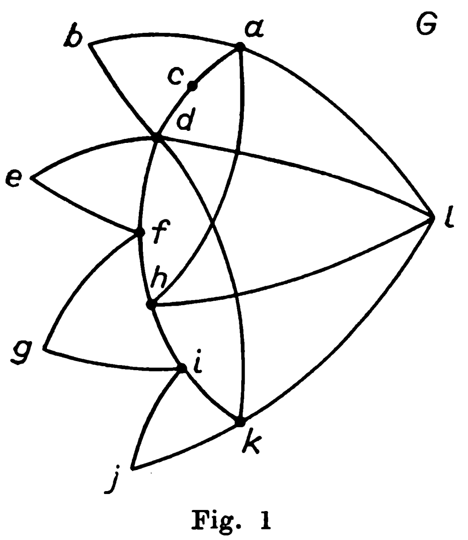

Fig. 1

The graph G shown in Fig. 1 has $\lambda = \nu(G) = 20-12+1 = 9$ independent cycles. Applying Algorithm 1 to G we can perform the General Step four times. Namely,

$$u_1 = (a, b), \quad C_1 = \{a, b, d, c, a\}, \quad \nu(G_1^c) = 1;$$
$$u_2 = (d, e), \quad C_2 = \{d, e, f\}, \quad \nu(G_2^c) = 2;$$
$$u_3 = (f, g), \quad C_3 = \{f, g, i, h, f\}, \quad \nu(G_3^c) = 3;$$
$$u_4 = (i, j), \quad C_4 = \{i, j, k, i\}, \quad \nu(G_4^c) = 4.$$

Now, for every edge $u_5 \in E(G)-E(G_4^c)$, every shortest cycle containing u_5 satisfies $\nu(G_5^c) > 5$; therefore, we must perform the Completion. Let $u_5 = (a, h)$; then $C_5 = \{a, h, l, a\}$ and $\nu(G_5^c) = 6$. Then we have

$$x_1 = a, \quad y_1 = h, \quad x_2 = h, \quad y_2 = a,$$
$$e_1 = \{a, c, d, f, h\}, \quad f_1 = \{a, h\},$$
$$e_2 = \{h, f, d, c, a\}, \quad f_2 = \{h, a\},$$
$$|(e_1, f_1)| = |(e_2, f_2)| = 5.$$

Thus, we augment the current subbasis by

$$C_6 = (e_1, f_1) = \{a, c, d, f, h, a\}$$

and then we have $\nu(G_6^c) = 6$. For every edge $u_7 \in E(G)-E(G_6^c)$, every shortest cycle containing u_7 satisfies $\nu(G_7^c) > 7$; therefore, we perform the Completion again. Let $u_7 = (d, k)$; then $C_7 = \{d, k, l, d\}$ and $\nu(G_7^c) = 9$. Then we have

$$x_1 = d, \quad x_2 = k, \quad x_3 = l,$$
$$y_1 = k, \quad y_2 = l, \quad y_3 = d,$$
$$e_1 = \{d, f, h, i, k\}, \quad f_1 = \{d, k\},$$
$$e_2 = \{k, i, h, l\}, \quad f_2 = \{k, l\},$$
$$e_3 = \{l, a, c, d\}, \quad f_3 = \{l, d\},$$
$$|(e_1, f_1)| = 5, \quad |(e_2, f_2)| = 4, \quad |(e_3, f_3)| = 4.$$

The two shortest cycles are

$$C_8 = (e_2, f_2) = \{k, i, h, l, k\} \quad \text{and} \quad C_9 = (e_3, f_3) = \{l, a, c, d, l\}.$$

Finally, Algorithm 1 finds the cycle basis $\mathscr{C} = \{C_1, C_2, \ldots, C_9\}$ of length $|\mathscr{C}| = 4+3+4+3+3+5+3+4+4 = 33$ but it is not a minimum cycle basis. The set of independent cycles $\mathscr{C}' = \mathscr{C} - \{C_6\} \cup \cup \{l, d, f, h, l\}$ is shorter, namely $|\mathscr{C}'| = 32$.

Generally, let us consider the following case. Let

$$\mathscr{C} = \{C_1, C_2, \ldots, C_k, C_{k+1}, \ldots, C_\lambda\}$$

be a cycle basis found by Algorithm 1. Let k be such that $\nu(G_k) > k$. Possibly, in the graph G there exists a cycle C such that $E(C) \subset E(G_k)$ and there exists $j \in \{1, 2, \ldots, \lambda-k\}$ for which $|C| < |C_{k+j}|$. If the cycle C is not chosen by Algorithm 1, then the Algorithm 1 does not find a minimum cycle basis of G.

4. Let us consider a particular case of the general problem of finding a minimum cycle basis which can be derived from a spanning tree of a graph. Such cycle bases have been characterized in [1] and [3]. We have for instance the following lemma:

LEMMA 1 (Hubicka and Sysło [1]). *A cycle basis $\mathscr{C} = \{C_1, C_2, \ldots, C_\lambda\}$ of a graph G can be derived from a spanning tree of G if and only if $\mathscr{C}$ contains no cycle which consists of edges belonging to other cycles of $\mathscr{C}$.*

In general, a minimum cycle basis may not be generated by a spanning tree.

Now we present a method for finding a minimum fundamental cycle set.

Let T be a spanning tree of a graph G and let $w(u)$ and $t(u)$ for $u \in M(T) = E(G) - E(T)$ denote the length of a minimum cycle in G which contains u and the length of the cycle in $E(T) \cup \{u\}$, respectively. It is easy to see that the following lemma holds:

LEMMA 2 (Hubicka and Sysło [1]). *Let T be a spanning tree of a graph G. The fundamental cycle set $\mathscr{C}(T)$ generated by T is a minimum cycle basis of G if and only if $w(u) = t(u)$ for every edge $u \in M(T)$.*

Two spanning trees T and T_0 of a graph G are *adjacent* if there exist edges $u \in E(T)$ and $v \in E(T_0)$ such that $E(T) = E(T_0) - \{v\} \cup \{u\}$.

LEMMA 3. *If a fundamental cycle set $\mathscr{C}(T_0)$ is a minimum cycle basis of a graph G, then T_0 is a locally minimum spanning tree of G, i.e.,*

$$\sum_{g \in E(T_0)} w(e) = \min_{\{T: d(T, T_0) = 1\}} \sum_{e' \in E(T)} w(e').$$

Proof. Let us suppose that T_0 is not a locally minimum spanning tree of G. Then there exists a tree T which is adjacent to T_0 and its length is smaller than that of T_0, i.e., there exist edges $u \in M(T_0)$ and $v \in E(T_0)$ such that

$$E(T) = E(T_0) - \{v\} \cup \{u\} \quad \text{and} \quad w(T) < w(T_0),$$

where

$$w(T) = \sum_{e \in E(T)} w(e).$$

Therefore

$$w(v) > w(u) = t(u) = |C_{T_0}(u)|.$$

Since $v \in E(C_{T_0}(u))$, we obtain $w(v) \leqslant |C_{T_0}(u)|$ and we arrive at a contradiction.

The following example shows that the converse does not hold.

Example 2.

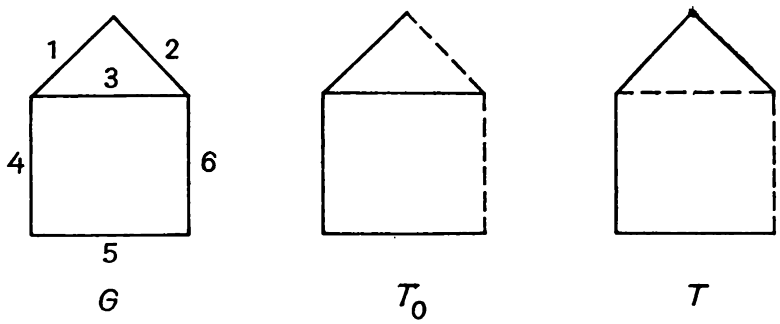

Fig. 2

The weights of edges of G are

$$w(1) = w(2) = w(3) = 3, \quad w(4) = w(5) = w(6) = 4.$$

Then for the spanning tree T_0 we have

$$w(T_0) = 3+3+4+4 = 14, \quad |\mathscr{C}(T_0)| = 3+4 = 7,$$

and for T

$$w(T) = 3+3+4+4 = 14, \quad |\mathscr{C}(T)| = 3+5 = 8.$$

T is a minimum spanning tree of G and a locally minimum spanning tree of G, but $\mathscr{C}(T)$ is not a minimum cycle basis of G.

On the base of Lemma 3 we ought to look for minimum cycle bases which simultaneously are fundamental cycle sets among bases generated by locally minimum trees.

Algorithm 2 (Hubicka and Sysło [1]). Initialization. Let T be a spanning tree of a graph G. For instance, T may be a maximum spanning tree of the weighted graph $G_w \doteq \langle V(G), E(G); w\rangle$, where $w\colon W(G) \to R_+$ and $w(u)$ is the length of a minimum cycle in G containing edge u.

General Step. If $w(u) = t(u)$ for every edge $u \in M(T)$, then $\mathscr{C}(T)$ is a minimum cycle basis of G; therefore, the algorithm is terminated. In the opposite case, find edges $u^* \in M(T)$ and $v^* \in E(T)$ such that

$$|\mathscr{C}(T_{u^*v^*})| = \min_{u \in M(T)} \min_{v \in E(C_T(u))} |\mathscr{C}(T_{uv})|,$$

where T_{uv} is the spanning tree of G which consists of edges $E(T)-\{v\}\cup\{u\}$. Let T^* denote $T_{u^*v^*}$. The spanning tree T^* generates a minimum basis among the cycle bases which are derived from the spanning trees adjacent to T. If $|\mathscr{C}(T^*)| < |\mathscr{C}(T)|$, then replace T by T^* and return to the General Step. If $|\mathscr{C}(T^*)| \geqslant |\mathscr{C}(T)|$, then the algorithm is terminated. $\mathscr{C}(T)$ is the subminimum fundamental cycle set of G.

The next example shows that Algorithm 2 finds a subminimum fundamental cycle set of a graph G.

Example 3. Consider the graph G and its spanning trees shown in Fig. 3.

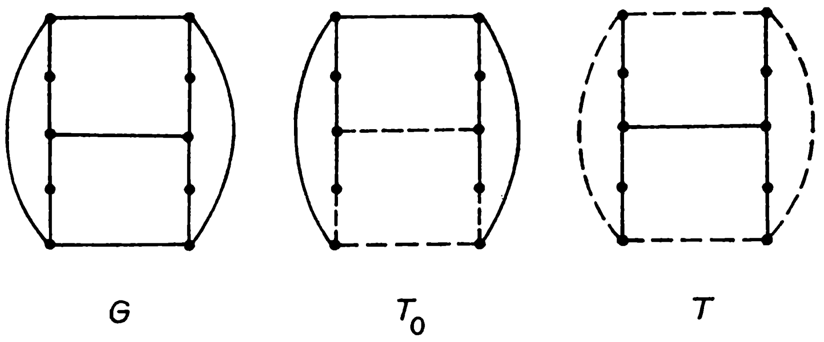

Fig. 3

It is easy to see that every tree T_1 which is adjacent to T satisfies

$$|\mathscr{C}(T)| < |\mathscr{C}(T_1)|.$$

We have $|\mathscr{C}(T)| = 22$ and $\mathscr{C}(T)$ is not a minimum fundamental cycle set, since $\mathscr{C}(T_0)$ is of length $|\mathscr{C}(T_0)| = 20$.

Algorithm 2 can be modified in such a way that it will search for the global minimum among the local minima obtained for different starting solutions. This approach has successfully been used in many heuristic algorithms, for instance, Lin used a similar idea in his method for solving the travelling salesman problem.

ALGORITHM 3. Step 1. Generate a pseudo-random spanning tree T_s of G.

Step 2. Improve T_s by applying Algorithm 2.

Step 3. If the found solution T' is better than the last solution T which has been obtained so far, i.e., if $|\mathscr{C}(T')| < |\mathscr{C}(T)|$, then replace T by T'.

Step 4. Repeat the algorithm from Step 1 until the computation time runs out or the answer is satisfactory.

This approach requires further investigations.

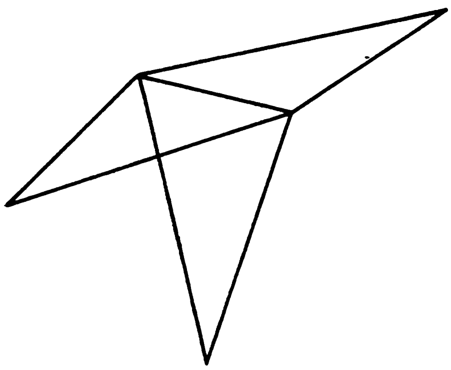

Fig. 4

5. In this section we shall consider only planar graphs. Following an idea of Hopcroft and Tarjan, and Hadlock who studied difficult algorithmic problems of graph theory for planar graphs, namely isomorphism and maximum cut, respectively, we may ask whether the problem of finding a minimum cycle basis can also be simplified for planar graphs. Our first observation is illustrated in Fig. 4; there exists a planar graph for which a minimum cycle basis is not a planar cycle basis, i.e., a set of interior faces of an embedding in the plane.

Thus, firstly, we simplify the general problem and we ask for a minimum planar basis of a planar graph. Another problem which appears is as follows: characterize planar graphs for which a minimum cycle basis is planar.

The importance of planar bases for planar graphs follows from the role which they play in characterization and determination of such graphs.

Let G be a planar biconnected graph with m edges and let $\mathscr{C} = \{C_i\}$ be a planar cycle basis of G. Then we have

$$|\mathscr{C}| = \sum_{C_i \in \mathscr{C}} |C_i| + |C_{\text{ex}}| - |C_{\text{ex}}| = 2m - |C_{\text{ex}}|,$$

where C_{ex} is the exterior cycle of the embedding of G with $\mathscr{C}$ as the set of interior faces.

A planar graph G is *k-outerplanar* if k is the maximum number of vertices which can lie on the boundary of the exterior face of a planar embedding of G in the plane. G is *outerplanar* if $k = |V(G)|$.

We have the following relation between the length of a minimum planar cycle basis and the k-outerplanarity.

THEOREM (Sysło [4]). *Every planar cycle basis $\mathscr{C}$ of a planar biconnected graph G with n vertices and m edges satisfies*

$$|\mathscr{C}| \geqslant 2m - n,$$

and a minimum planar cycle basis is of length $2m-k$ if and only if G is k-outerplanar.

Thus, the problem of finding a minimum planar cycle basis of a planar graph is closely related to that of testing k-outerplanarity. See [4] for further details.

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O MINIMALNEJ BAZIE CYKLI GRAFU

STRESZCZENIE

W pracy wykazano, iż przypuszczenia, że algorytmy z [1] znajdują minimalną bazę cykli i minimalny zbiór cykli fundamentalnych, są błędne. Zaproponowano ponadto metodę znajdowania subminimalnego zbioru cykli fundamentalnych, która jest iteracją metody opisanej w [1] dla pseudolosowych rozwiązań początkowych.
