On a modified conjecture of De Giorgi

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Abstract We study the Γ-convergence of functionals arising in the Van der Waals-Cahn-Hilliard theory of phase transitions. The corresponding limit is given as the sum of the area and the Willmore functional. The problem under investigation was proposed as modification of a conjecture of De Giorgi and partial results were obtained by several authors. We prove here the modified conjecture in space dimensions $n = 2, 3$.

Keywords Γ-limit · Willmore surfaces · Cahn-Hilliard theory · Geometric measure theory.

Mathematics Subject Classification (2000) 49 J 45 · 49 Q 15 · 35 J 60

1 Introduction

In 1991 De Giorgi stated the following conjecture (see [dG91] *Conjecture 4*).

Conjecture 1.1 (De Giorgi) Consider for $\Omega \subset \mathbb{R}^n$, $\lambda > 0$ functionals $G_p : L^1(\Omega) \to \mathbb{R}$, $p > 0$, defined by

$$G_p(u) := \int_\Omega \Big[\Big(\frac{\Delta u}{p} - p \sin u\Big)^2 + \lambda\Big] \Big[\frac{|\nabla u|^2}{p} + p(1 - \cos u)\Big]\, dx$$

if $u \in W^{2,1}(\Omega)$, $G_p(u) := \infty$ if $u \in L^1(\Omega) \setminus W^{2,1}(\Omega)$. Then there exists a constant $k \in \mathbb{R}$ depending only on n, such that for any $u = 2\pi\mathcal{X}_E$ with $E \subset \Omega$, $\partial E \cap \Omega \in C^2$,

$$\Gamma(L^1(\Omega)) - \lim_{p\to\infty} G_p(u) = 8\sqrt{2}\lambda\mathcal{H}^{n-1}(\partial E \cap \Omega) + k \int_{\partial E\cap\Omega} |\mathbf{H}_{\partial E}|^2\, d\mathcal{H}^{n-1},$$

where $\mathbf{H}_{\partial E}$ denotes the mean curvature vector of ∂E.

This work was supported by the European Community's Human Potential Programme under contract HPRN-CT-2002-00274, FRONTS-SINGULARITIES.

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In the context of the Van der Waals-Cahn-Hilliard theory of phase transitions a modification of De Giorgis conjecture for closely related functionals was proposed by several authors and has drawn much attention, due to both the widespread use of that theory and the mathematical interest in the conjecture. See for example [BePa93], [LoMa00], [BeMu04] and the references therein.

To describe the problem setting, we consider a set $\Omega \subset \mathbb{R}^n$, let $W(t) := (1-t^2)^2$ be a standard double well potential and define for $\varepsilon > 0$ functionals $\mathcal{F}_\varepsilon : L^1(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}_\varepsilon(u) := \int\limits_\Omega \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\Big)\,\mathrm{d}\mathcal{L}^n + \int\limits_\Omega \frac{1}{\varepsilon}\Big(-\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\Big)^2\,\mathrm{d}\mathcal{L}^n \tag{1.1}$$

if $u \in L^1(\Omega) \cap W^{2,2}(\Omega)$ and $\mathcal{F}_\varepsilon(u) := \infty$ if $u \in L^1(\Omega) \setminus W^{2,2}(\Omega)$.
Further we put $\sigma := \int_{-1}^1 \sqrt{2W}$, and for $\mathcal{X} = 2\mathcal{X}_E - 1$ with $E \subseteq \Omega$ and $\partial E \cap \Omega \in C^2$ we define

$$\mathcal{F}(\mathcal{X}) := \sigma\mathcal{H}^{n-1}(\partial E \cap \Omega) + \sigma \int\limits_{\partial E\cap\Omega} |\mathbf{H}_{\partial E}|^2\,\mathrm{d}\mathcal{H}^{n-1}. \tag{1.2}$$

The aim of this paper is to prove, in small space dimensions, the proposed modification of De Giorgis Conjecture 1.1, as stated in the following theorem.

Theorem 1.2 (Modified De Giorgi Conjecture) *Let* $n = 2, 3$. *For any* $\mathcal{X} = 2\mathcal{X}_E - 1$ *with* $E \subset \Omega$, $\partial E \cap \Omega \in C^2$,

$$\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0}\ \mathcal{F}_\varepsilon(\mathcal{X}) = \mathcal{F}(\mathcal{X}) \tag{1.3}$$

holds.

Compared to the original conjecture of De Giorgi the structure of the approximate functionals $\mathcal{F}_\varepsilon$ is different in the choice of the double well potential and, more importantly, in the the second term of $\mathcal{F}_\varepsilon$, where instead of the 'energy density' $\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)$ the factor $\frac{1}{\varepsilon}$ appears.

The Γ-convergence of the first part of the functionals $\mathcal{F}_\varepsilon$ to the first term of $\mathcal{F}$, which is basically the area functional, was already proved by Modica and Mortola [MoMor77], see also [Mo87]. The second part of $\mathcal{F}$ is up to a constant identical to the Willmore functional.

The modified De Giorgi conjecture as stated above was investigated by several authors. Bellettini and Paolini [BePa93] (see also [BeMu04]) proved the *limsup-estimate* necessary for the Gamma-convergence (1.3). Loreti and March considered in [LoMa00] the gradient flows corresponding to the functionals $\mathcal{F}_\varepsilon$, $\mathcal{F}$ and proved the convergence as $\varepsilon \to 0$ by formal asymptotic expansions.

The *liminf-estimate* belonging to (1.3) turns out to be the difficult part in the proof of the Modified De Giorgi Conjecture and only recently partial results were obtained. In [BeMu04] Bellettini and Mugnai proved the Gamma-convergence for rotationally symmetric data in $\mathbb{R}^2$ and Moser proved in [Mos04] the *liminf-estimate* in three space dimensions if the data are monotone in one direction. The lower-semicontinuity of $\mathcal{F}$, which is a necessary condition for $\mathcal{F}$ being a Γ-limit, follows from a recent result of the second author in [Sch04], where the lower semi-continuity of the Willmore functional under weak convergence of currents is proved.

To prove the Modified De Giorgi Conjecture in space dimensions $n = 2, 3$ for general data we combine the approach of Hutchinson and Tonegawa in [HT00], [T02] with arguments used by Chen in [C96]. As limit of appropriately defined *energy measures* we obtain a rectifiable varifold whose multiplicity is an integer multiple of σ. This limit varifold has a weak mean curvature and satisfies the liminf estimate for its integrated squared mean curvature. The major challenge to derive these results is the control of the so called *discrepancy measures*, which is much more delicate here than in [HT00], [T02] and which requires a careful analysis and some additional arguments. The liminf estimate for the Willmore functional of ∂E is then deduced using a Theorem from [Sch04], relating the mean curvature of the limit varifold to the local geometry given by ∂E.

In the following paragraph we fix some notation and state the liminf estimate, that is the remaining part for the proof of the Modified De Giorgi Conjecture. Auxiliary estimates and in particular a refined version of a Theorem from [C96] are given in section 3. The rectifiability of the limit of the energy measures and the liminf estimate for this varifold limit is proved in section 4. The last paragraph deals with the integrality up to a factor σ of this limit which finally enables us, using the above mentioned result from [Sch04], to deduce the liminf estimate.

2 The liminf estimate

Since the limsup estimate corresponding to the Modified De Giorgi Conjecture was already established in [BePa93], Theorem 1.2 follows if we prove the following Theorem.

Theorem 2.1 (Modified De Giorgi conjecture; liminf estimate) *Let* $n = 2, 3,$ $E \subseteq \Omega$ *with* $\partial E \cap \Omega \in C^2$, $\mathcal{X} = 2\mathcal{X}_E - 1$ *and consider* $(u_\varepsilon)_{\varepsilon>0} \subset L^1_{loc}(\Omega)$ *with* $u_\varepsilon \to \mathcal{X}$ *in* $L^1_{loc}(\Omega)$*. Then*

$$\mathcal{F}(\mathcal{X}) \leq \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

□

By a standard approximation argument it is sufficient to consider $u_\varepsilon \in C^2(\Omega), \varepsilon > 0$. We let $v_\varepsilon \in C^0(\Omega)$ such that

$$-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon) = v_\varepsilon \quad \text{in } \Omega. \tag{2.1}$$

As in [HT00], [T02], [C96] we define *energy measures* μ_ε and *discrepancy measures* ξ_ε, in addition we define measures α_ε corresponding to the second term in the functionals $\mathcal{F}_\varepsilon$,

$$\mu_\varepsilon := \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\mathcal{L}^n, \tag{2.2}$$

$$\xi_\varepsilon := \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\mathcal{L}^n, \tag{2.3}$$

$$\alpha_\varepsilon := \frac{1}{\varepsilon}v_\varepsilon^2\mathcal{L}^n. \tag{2.4}$$

Observe that ξ_ε measures the deviation of the somehow 'ideal situation' of equipartition of energy $\varepsilon/2|\nabla u_\varepsilon|^2 = \varepsilon^{-1}W(u_\varepsilon)$. To prove Theorem 2.1 we can assume that $\liminf_{\varepsilon\to 0}\mathcal{F}_\varepsilon(u_\varepsilon) < \infty$ and, eventually restricting ourselves to a subsequence, that

$$\mu_\varepsilon(\Omega) + \alpha_\varepsilon(\Omega) \leq C \tag{2.5}$$

and

$$\mu_\varepsilon \to \mu, \quad \xi_\varepsilon \to \xi, \quad \alpha_\varepsilon \to \alpha \quad \text{weakly}^* \text{ in } C_0^0(\Omega)^*. \tag{2.6}$$

A major difficulty in proving the rectifiablility and integrality of μ is the control of the discrepancy measures ξ_ε.

Anticipating the results from Chapter 4 and Chapter 5 we prove Theorem 2.1.

Proof of Theorem 2.1:
First we get by [MoMor77] (see also [Mo87])

$$\nu = \mathcal{H}^{n-1} \lfloor \partial^* E \leq \sigma^{-1} \mu, \tag{2.7}$$

and ν is a unit-density, in particular an integral $(n-1)-$varifold even without assuming regularity for ∂E .

By Theorem 4.1, we get that μ is rectifiable and

$$\int_\Omega |\mathbf{H}_\mu|^2 \, \mathrm{d}\mu \leq \alpha(\Omega) \leq \liminf_{\varepsilon \to 0} \alpha_\varepsilon(\Omega) =$$

$$= \liminf_{\varepsilon \to 0} \int_\Omega \frac{1}{\varepsilon} \Big(-\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) \Big)^2 \, \mathrm{d}\mathcal{L}^n < \infty. \tag{2.8}$$

By Theorem 5.1, $\sigma^{-1}\mu$ is integral and, as ∂E is assumed to be smooth, we can apply [Sch04] Corollary 4.3 to obtain

$$\mathbf{H}_{\partial E} = \mathbf{H}_\nu = \mathbf{H}_{\sigma^{-1}\mu} \quad \nu - \text{almost everywhere}. \tag{2.9}$$

Combining (2.7) - (2.9) yields

$$\sigma \Bigg(\mathcal{H}^{n-1}(\partial E \cap \Omega) + \int_{\partial E \cap \Omega} |\mathbf{H}_{\partial E}|^2 \, \mathrm{d}\mathcal{H}^{n-1} \Bigg) \leq \mu(\Omega) + \int_\Omega |\mathbf{H}_\mu|^2 \, \mathrm{d}\mu \leq$$

$$\leq \liminf_{\varepsilon \to 0} \mu_\varepsilon(\Omega) + \liminf_{\varepsilon \to 0} \int_\Omega \frac{1}{\varepsilon} \Big(-\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) \Big)^2 \, \mathrm{d}\mathcal{L}^n \leq$$

$$\leq \liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

This proves the theorem.

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3 Auxiliary Estimates

In this section we prove some estimates we need later. The next lemma, which is a refined version of a result in [C96], is an important step to control the discrepancy measures and the key to improve a monotonicity formula for the measures μ_ε that we derive in section 4.

Lemma 3.1 ([C96] Theorem 3.6)
Let $n = 2,3, 0 < \delta \le \delta_0, 0 < \varepsilon \le \varrho$,

$$\varrho_0 := \max(2, 1 + \delta^{-M}\varepsilon)\varrho,$$

$u_\varepsilon \in C^2(B_{\varrho_0}(0)), v_\varepsilon \in C^0(B_{\varrho_0}(0)), M$ *universal large and*

$$-\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) = v_\varepsilon \quad \text{in } B_{\varrho_0}(0).$$

Then

$$\varrho^{1-n} \int\limits_{B_\varrho(0)} \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\Big)_+ \le C\delta \varrho^{1-n} \int\limits_{B_{2\varrho}(0)} \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big) +$$
$$+ C\delta^{-M}\varepsilon\varrho^{1-n} \int\limits_{B_{\varrho_0}(0)} v_\varepsilon^2 + C\delta^{-M}\varrho^{1-n} \int\limits_{B_{\varrho_0}(0)\cap[|u_\varepsilon|\ge 1]} \frac{1}{\varepsilon} W'(u_\varepsilon)^2 + C(\varepsilon/\varrho)\delta.$$

□

Compared to Theorem 3.6 in [C96] this lemma makes dependences on the small parameter δ explicit. We give the proof of Lemma 3.1 at the end of this section.

Lemma 3.2 ([C96] Lemma 4.3)
Let $n = 2,3, 0 < \delta \le \delta_0, R(\delta) = \delta^{-5}, \omega(\delta) = c_0\delta^{24}, U \in C^2(B_R), V \in C^0(B_R), B_R = B_R^n(0) \subseteq \mathbb{R}^n$ *satisfying*

$$\begin{aligned} -\Delta U + W'(U) &= V \quad \text{in } B_R, \\ |U| &\le 2 \quad \text{in } B_R, \\ \| V \|_{L^2(B_R)} &\le \omega. \end{aligned}$$

Then

$$\int\limits_{B_1} \Big(\frac{1}{2}|\nabla U|^2 - W(U)\Big)_+ \le C\delta \tag{3.1}$$

and for $\tau = \delta^{1/(2n+3)}$

$$\int\limits_{B_{1/2}} \Big(\frac{1}{2}|\nabla U|^2 - W(U)\Big)_+ \le C\tau \int\limits_{B_{1/2}} \Big(\frac{1}{2}|\nabla U|^2 + W(U)\Big) + \int\limits_{B_{1/2}\cap[|U|\ge 1-\tau]} \frac{1}{2}|\nabla U|^2. \tag{3.2}$$

Proof:
We may assume that U, V are smooth and consider

$$\begin{aligned} -\Delta \Psi &= -V \quad \text{in } B_R, \\ \Psi &= 0 \quad \text{on } \partial B_R. \end{aligned}$$

Putting $\Psi_R(x) = \Psi(Rx)$, $V_R(x) = R^2 V(Rx)$ we see that

$$\begin{aligned} -\Delta \Psi_R &= -V_R \quad \text{in } B_1, \\ \Psi_R &= 0 \quad \text{on } \partial B_1, \end{aligned}$$

hence by standard elliptic L^2-theory we obtain

$$\| \Psi_R \|_{W^{2,2}(B_1)} \le \| V_R \|_{L^2(B_1)} \le CR^{-n/2+2}\omega.$$

Rescaling yields

$$\begin{aligned}
&\| \Psi \|_{L^2(B_R)} + R \| \nabla\Psi \|_{L^2(B_R)} + R^2 \| D^2\Psi \|_{L^2(B_R)} \\
&= R^{n/2} \| \Psi_R \|_{W^{2,2}(B_1)} \\
&\le CR^2\omega
\end{aligned} \tag{3.3}$$

and by the embeddings $W^{2,2}(B_1) \hookrightarrow C^{0,1/2}(B_1) \hookrightarrow L^\infty(B_1)$ and $W^{1,2}(B_1) \hookrightarrow L^6(B_1)$ as $n \le 3$, we get

$$\begin{aligned}
\| \Psi \|_{L^\infty(B_R)} &= \| \Psi_R \|_{L^\infty(B_1)} \\
&\le C \| \Psi_R \|_{W^{2,2}(B_1)} \le CR^{-n/2+2}\omega \le 1,
\end{aligned} \tag{3.4}$$

due to our choice of ω. Moreover the inequality

$$\begin{aligned}
\| \nabla\Psi \|_{L^6(B_R)} &= R^{n/6-1} \| \nabla\Psi_R \|_{L^6(B_1)} \le CR^{n/6-1} \| \Psi_R \|_{W^{2,2}(B_1)} \\
&\le CR^{-n/3+1}\omega
\end{aligned} \tag{3.5}$$

holds. Next we see that $\| \Delta U \|_{L^2(B_R)} \le CR^{n/2}$ and putting $U_R(x) = U(Rx)$ we obtain by Friedrich's Theorem

$$\begin{aligned}
\| U_R \|_{W^{2,2}(B_{1/2})} &\le C\big(\|\Delta U_R\|_{L^2(B_1)} + \|U_R\|_{L^2(B_1)}\big) \\
&\le C\big(R^{-n/2+2}\|\Delta U\|_{L^2(B_R)} + 1\big) \\
&\le CR^2.
\end{aligned}$$

By the embedding $W^{1,2}(B_{1/2}) \hookrightarrow L^6(B_{1/2})$, as $n \le 3$, we get further

$$\begin{aligned}
\|\nabla U\|_{L^6(B_{R/2})} &= R^{n/6-1}\|\nabla U_R\|_{L^6(B_{1/2})} \\
&\le CR^{n/6-1}\|U_R\|_{W^{2,2}(B_{1/2})} \\
&\le CR^{n/6+1}.
\end{aligned} \tag{3.6}$$

Now we put $U_0 := U + \Psi \in W^{2,2}(B_R)$ and see by (3.4) and the assumptions on U that

$$|U_0| \le 3, \tag{3.7}$$

$$-\Delta U_0 = -W'(U) \tag{3.8}$$

holds. As

$$\begin{aligned}
\frac{1}{2}|\nabla U|^2 - W(U) &= \frac{1}{2}|\nabla U_0 - \nabla\Psi|^2 - W(U_0 - \Psi) \\
&\le \big(\frac{1}{2} + \varsigma\big)|\nabla U_0|^2 - W(U_0) + C|\Psi| + \big(\frac{1}{2} + \frac{1}{\varsigma}\big)|\nabla\Psi|^2,
\end{aligned} \tag{3.9}$$

we see by (3.4), (3.5), (3.9) for $0 < \beta \le 1$

$$\begin{aligned}
&\int_{B_1} \big(\frac{1}{2}|\nabla U|^2 - W(U)\big)_+ \\
&\le \int_{B_1} \big(\frac{1}{2}|\nabla U_0|^2 - W(U_0)\big)_+ + \int_{B_1} \Big(\beta|\nabla U_0|^2 + C|\Psi| + (\frac{1}{2} + \frac{1}{\beta})|\nabla\Psi|^2\Big) \\
&\le \int_{B_1} \big(\frac{1}{2}|\nabla U_0|^2 - W(U_0)\big)_+ + C\big(\beta + R^{-n/2+2}\omega + \frac{1}{\beta}R^{-2n/3+2}\omega^2\big).
\end{aligned} \tag{3.10}$$

Choosing $\beta = R^{-n/3+1}\omega \le \delta \le 1$ and observing that $R^{-n/2+2}\omega \le R\omega \le \delta$ as $n \ge 2$, we see

$$\int_{B_1} \big(\frac{1}{2}|\nabla U|^2 - W(U)\big)_+ \le \int_{B_1} \big(\frac{1}{2}|\nabla U_0|^2 - W(U_0)\big)_+ + C\delta$$

and for proving (3.1) it suffices to show

$$\int_{B_1} \left(\frac{1}{2}|\nabla U_0|^2 - W(U_0)\right)_+ \le C\delta. \tag{3.11}$$

To this end, we put

$$\begin{aligned} V_0 &:= -\Delta U_0 + W'(U_0) \\ &= -\Delta\Psi - \Delta U + W'(U) \\ &\quad + W''(U)\Psi + \frac{1}{2}W'''(U)\Psi^2 + \frac{1}{6}W^{(iv)}(U)\Psi^3 \\ &= W''(U)\Psi + \frac{1}{2}W'''(U)\Psi^2 + \frac{1}{6}W^{(iv)}(U)\Psi^3. \end{aligned}$$

By (3.4)-(3.7) we obtain

$$\|V_0\|_{L^\infty(B_R)} \le CR^{-n/2+2}\omega \le 1 \tag{3.12}$$

and

$$\begin{aligned} \|\nabla V_0\|_{L^6(B_{R/2})} &\le C\Big(\|\nabla U\|_{L^6(B_{R/2})}\|\Psi\|_{L^\infty(B_R)} + \|\nabla\Psi\|_{L^6(B_R)}\Big) \\ &\le C\Big(R^{n/6+1-n/2+2}\omega + R^{-n/3+1}\omega\Big) \\ &\le CR^{-n/3+3}\omega. \end{aligned} \tag{3.13}$$

By (3.7), (3.8), $|U| \le 2$ and standard elliptic L^p-theory we get for any $B_1(x) \subseteq B_R$

$$\|U_0\|_{W^{2,p}(B_{1/2}(x))} \le C_p \qquad \text{for all } 1 < p < \infty$$

and hence

$$\|\nabla U_0\|_{L^\infty(B_{R-1}(0))} \le C. \tag{3.14}$$

Next we put

$$H := \frac{1}{2}|\nabla U_0|^2 - W(U_0) - G(U_0) - \varphi, \tag{3.15}$$

for some smooth $G : \mathbb{R} \to \mathbb{R}$ and $\varphi \in W^{2,2}(B_R)$, chosen below. We calculate

$$\begin{aligned} \Delta H &= |D^2U_0|^2 + \nabla U_0 \cdot \nabla\Delta U_0 - \Delta\varphi - \\ &\quad - \big(W' + G'\big)\Delta U_0 - \big(W'' + G''\big)|\nabla U_0|^2. \end{aligned}$$

Using $\Delta U_0 = W'(U_0) - V_0$ and $|\nabla U_0|^2 = 2(H + W + G + \varphi)$ we proceed

$$\begin{aligned} \Delta H &= |D^2U_0|^2 - \nabla U_0 \cdot \nabla V_0 + \big(W' + G'\big)V_0 - \Delta\varphi - \\ &\quad - W'\big(W' + G'\big) - 2G''(H + W + G + \varphi). \end{aligned}$$

From the definition of H we see

$$\nabla H = D^2 U_0 \nabla U_0 - \big(W' + G'\big)\nabla U_0 - \nabla\varphi$$

and

$$\begin{aligned} |\nabla U_0|^2 |D^2 U_0|^2 &\geq |D^2 U_0 \nabla U_0|^2 \\ &= \big|\nabla H + \big(W' + G'\big)\nabla U_0 + \nabla\varphi\big|^2 \\ &\geq 2\big(W' + G'\big)\nabla U_0 \cdot \nabla(H + \varphi) + \big(W' + G'\big)^2 |\nabla U_0|^2, \end{aligned}$$

hence in $B_R \cap \big[\nabla U_0 \neq 0\big]$

$$\begin{aligned} &\Delta H - \frac{2(W' + G')\nabla U_0}{|\nabla U_0|^2}\nabla H + 2G'' H \\ &\geq \big(W' + G'\big)^2 - W'\big(W' + G'\big) - 2G''(W + G) + \\ &\quad + \frac{2(W' + G')\nabla U_0}{|\nabla U_0|^2}\nabla\varphi - 2G''\varphi - \\ &\quad - \Delta\varphi - \nabla U_0 \cdot \nabla V_0 + \big(W' + G'\big)V_0 \\ &= \big(G'\big)^2 + \Big(G'W' - 2G''(W + G)\Big) + \\ &\quad + \frac{2(W' + G')\nabla U_0}{|\nabla U_0|^2}\nabla\varphi - 2G''\varphi - \\ &\quad - \Delta\varphi - \nabla U_0 \cdot \nabla V_0 + \big(W' + G'\big)V_0. \end{aligned}$$

Now we choose in (3.15) $G = G_\delta$, where

$$G_\delta(r) := \delta\Big(1 + \int_{-3}^{r} \exp\Big(-\int_{-3}^{t} \frac{|W'(s)| + \delta}{2(W(s) + \delta)}\, ds\Big)\, dt\Big).$$

Recalling (3.7) we obtain

$$\begin{aligned} \delta \leq\ & G_\delta(U_0) && \leq C\delta, \\ 0 <\ & G'_\delta(U_0) && \leq \delta && (3.16) \\ 0 <\ & -G''_\delta(U_0) = G'_\delta(U_0)\frac{|W'(U_0)| + \delta}{W(U_0) + \delta} && \leq C, && (3.17) \end{aligned}$$

and calculate

$$\begin{aligned} G'_\delta W' - 2G''_\delta(W + G_\delta) &= G'_\delta\Big(W' + \frac{|W'| + \delta}{W + \delta}(W + G_\delta)\Big) \\ &\geq \delta G'_\delta, \end{aligned}$$

as $G_\delta \geq \delta$. Therefore

$$\begin{aligned} &\Delta H - \frac{2(W' + G'_\delta)\nabla U_0}{|\nabla U_0|^2}\nabla H + 2G''_\delta H \\ &\geq \big(G'_\delta\big)^2 + \delta G'_\delta + \frac{2(W' + G'_\delta)\nabla U_0}{|\nabla U_0|^2}\nabla\varphi - 2G''_\delta\varphi \\ &\quad - \Delta\varphi - \nabla U_0 \cdot \nabla V_0 + \big(W' + G'_\delta\big)V_0 \qquad (3.18) \end{aligned}$$

in $B_R \cap \big[\nabla U_0 \neq 0\big]$. Next we choose φ to be the solution of the Dirichlet problem

$$\begin{aligned} -\Delta\varphi &= \big|\nabla U_0 \cdot \nabla V_0 - \big(W' + G_\delta'\big)V_0\big| && \text{in } B_{R/2}, \\ \varphi &= 0 && \text{on } \partial B_{R/2}. \end{aligned}$$

We observe that $\varphi \geq 0$ and by (3.7), (3.12)-(3.14), (3.16) we see

$$\begin{aligned} \|\Delta\varphi\|_{L^6(B_{R/2})} &= \|\nabla U_0 \cdot \nabla V_0 - \big(W' + G_\delta'\big)V_0\|_{L^6(B_{R/2})} \\ &\leq C\Big(R^{-n/3+3}\omega + R^{n/6-n/2+2}\omega\Big) \\ &\leq CR^{-n/3+3}\omega, \end{aligned}$$

hence putting $\varphi_R(x) = \varphi(Rx/2)$ by standard elliptic L^p-theory we obtain

$$\begin{aligned} \|\varphi_R\|_{W^{2,6}(B_1)} &\leq C\|\Delta\varphi_R\|_{L^6(B_1)} \\ &\leq CR^{2-n/6}\|\Delta\varphi\|_{L^6(B_{R/2})} \\ &\leq CR^{-n/2+5}\omega. \end{aligned}$$

By the embedding $W^{2,6}(B_1) \hookrightarrow W^{1,\infty}(B_1)$ as $n \leq 3$, we get

$$\begin{aligned} \|\varphi\|_{L^\infty(B_{R/2})} + \|\nabla\varphi\|_{L^\infty(B_{R/2})} &= \|\varphi_R\|_{W^{1,\infty}(B_1)} \\ &\leq C\|\varphi_R\|_{W^{2,6}(B_1)} \\ &\leq CR^{-n/2+5}\omega. \end{aligned} \tag{3.19}$$

Next, if $H > 0$ then $\nabla U_0 \neq 0$ and $|W'(U_0)|^2 \leq CW(U_0) \leq C|\nabla U_0|^2$ thus by (3.16) we obtain

$$\frac{|\big(W' + G_\delta'\big)(U_0)\nabla U_0|}{|\nabla U_0|^2} \leq C\Big(1 + \frac{\delta}{|\nabla U_0|}\Big).$$

From (3.18) together with (3.16)-(3.19) we get in $B_{R/2} \cap [H > 0]$

$$\begin{aligned} \Delta H &\geq \big(G_\delta'\big)^2 + \delta G_\delta' + \frac{2(W' + G_\delta')\nabla U_0}{|\nabla U_0|^2}\big(\nabla H + \nabla\varphi\big) \\ &\geq \big(G_\delta'\big)^2 + \delta G_\delta' - C\big(|\nabla H| + |\nabla\varphi|\big)(1 + \frac{\delta}{|\nabla U_0|}) \\ &\geq \big(G_\delta'\big)^2 + \delta G_\delta' - C(1 + \frac{\delta}{|\nabla U_0|})\big(|\nabla H| + R^{-n/2+5}\omega\big). \end{aligned} \tag{3.20}$$

Next we calculate by (3.7)

$$G_\delta'(U_0) \geq \delta \exp\Big(- \int_{-3}^{3} \frac{|W'(s)| + \delta}{2(W(s) + \delta)}\, ds\Big) \tag{3.21}$$

and

$$\begin{aligned} \int_{-3}^{3} \frac{|W'(s)| + \delta}{2(W(s) + \delta)}\, ds &\leq \int_{-3}^{0} \big|\frac{d}{ds} \log\big(W(s) + \delta\big)\big|\, ds + 3 \\ &= 3 + \log\big(W(3) + \delta\big) + \log\big(1 + \delta\big) - 2\log\delta \\ &\leq C - \log\delta^2 \end{aligned}$$

for $\delta < 1$. Therefore

$$G'_\delta(U_0) \geq c_0\delta^3$$

and by (3.20)

$$\Delta H \geq c_0(\delta^6 + \delta^4) - C\big(1 + \frac{\delta}{|\nabla U_0|}\big)\big(|\nabla H| + R^{-n/2+5}\omega\big) \tag{3.22}$$

in $B_{R/2} \cap [H > 0] \cap [\nabla U_0 \neq 0]$. Now we assume that

$$\eta := \sup_{B_1} H \geq \delta > 0. \tag{3.23}$$

We choose $\lambda \in C_0^2(B_{R/2})$ with $0 \leq \lambda \leq 1$, $\lambda = 1$ on $B_{R/4}$ and $|D^j\lambda| \leq CR^{-j}$ for $j = 1, 2$. Then there exists $x_0 \in B_{R/2}$ such that

$$(\lambda H)(x_0) \; = \; \max\big\{(\lambda H)(x) \; : \; x \in \overline{B_{R/2}}\big\} \; \geq \; \eta \; > \; 0.$$

As $H \leq C$ in B_{R-1} by (3.14), we get

$$\lambda(x_0) \geq c_0\eta \quad \text{ for some } c_0 > 0. \tag{3.24}$$

Further we obtain

$$H(x_0) \geq (\lambda H)(x_0) \; \geq \; \eta \; \geq \; \delta \; > \; 0 \tag{3.25}$$

and

$$|\nabla U_0(x_0)|^2 \geq 2H(x_0) \; \geq \; 2\eta \; \geq \; 2\delta \; > \; 0. \tag{3.26}$$

Next $\nabla(\lambda H)(x_0) = 0$, hence

$$\begin{aligned} |\nabla H(x_0)| &\leq \lambda(x_0)^{-1}|\nabla\lambda(x_0)|H(x_0) \\ &\leq C(R\eta)^{-1}. \end{aligned} \tag{3.27}$$

Finally

$$\begin{aligned} 0 &\geq \Delta(\lambda H)(x_0) \\ &= \lambda(x_0)\Delta H(x_0) + 2\nabla\lambda(x_0) \cdot \nabla H(x_0) + \Delta\lambda(x_0)H(x_0), \end{aligned}$$

hence by (3.14), (3.23), (3.24), (3.27)

$$\begin{aligned} \Delta H(x_0) &\leq \lambda(x_0)^{-1}\big(CR^{-1}|\nabla H(x_0)| + CR^{-2}\big) \\ &\leq C\eta^{-1}\big(R^{-2}\eta^{-1} + R^{-2}\big) \\ &= CR^{-2}\eta^{-1}(1 + \eta^{-1}) \\ &\leq CR^{-2}\delta^{-1}\eta^{-1}. \end{aligned} \tag{3.28}$$

On the other hand we obtain by (3.16), (3.22), (3.25)- (3.27) that

$$\Delta H(x_0) \geq \delta^4 - C\big((R\eta)^{-1} + R^{-n/2+5}\omega\big)$$

holds. Combining this inequality with (3.28) we obtain, as $R^{-n/2+5}\omega \ll \delta^4$

$$\begin{aligned}\frac{1}{2}\delta^4 &\leq \delta^4 - CR^{-n/2+5}\omega \\ &\leq CR^{-1}\eta^{-1}(1 + R^{-1}\delta^{-1}),\end{aligned}$$

hence

$$\begin{aligned}\eta &\leq CR^{-1}\delta^{-4}(1 + R^{-1}\delta^{-1}) \\ &\leq C\delta.\end{aligned}$$

In any case, assuming (3.23) or not, we arrive at

$$H \leq C\delta \quad \text{in } B_1$$

and (3.15), (3.16), (3.19) yield

$$\begin{aligned}\frac{1}{2}|\nabla U_0|^2 - W(U_0) &= H + G_\delta(U_0) + \varphi \\ &\leq C\delta + CR^{-n/2+5}\omega \\ &\leq \delta,\end{aligned}$$

which implies (3.11) and (3.1).
If $|U| \geq 1 - \tau$ on $B_{1/2}$ then (3.2) is immediate. Otherwise there is $x_0 \in B_{1/2}$ with $|U(x_0)| < 1 - \tau$. By standard elliptic estimates we get

$$\begin{aligned}\|U\|_{C^{0,1/2}(B_1)} &\leq \|U\|_{W^{2,2}(B_1)} \\ &\leq C\Big(\|\Delta U\|_{L^2(B_2)} + \|U\|_{L^2(B_2)}\Big) \\ &\leq C.\end{aligned}$$

Therefore $|U| \leq 1 - \tau/2$ and $W(U) \geq \tau^2/4$ in $B_{c_0\tau^2}(x_0) \subseteq B_1$ and

$$\int_{B_{1/2}(0)} W(U) \;\geq\; \big(c_0\tau^2\big)^n \frac{\tau^2}{4} \;=\; c_0\tau^{2(n+1)}.$$

Recalling $\tau^{2n+3} = \delta$ we get from (3.1)

$$\begin{aligned}\int_{B_{1/2}} \big(\frac{1}{2}|\nabla U|^2 - W(U)\big)_+ &\leq C\delta \\ &= C\tau\tau^{2n+2} \\ &\leq C\tau \int_{B_{1/2}} \big(\frac{1}{2}|\nabla U|^2 + W(U)\big)\end{aligned}$$

which yields (3.1).

///

Proposition 3.3 *For* $n = 2, 3, \varepsilon > 0, u_\varepsilon \in C^2(B_{\varepsilon/4}(0)), v_\varepsilon \in C^0(B_{\varepsilon/4}(0))$,

$$-\varepsilon \Delta u_\varepsilon + \varepsilon^{-1} W'(u_\varepsilon) = v_\varepsilon \quad \text{in } B_{\varepsilon/4}(0),$$

we have

$$|u_\varepsilon(0)| \leq 1 + C\varepsilon^{-n} \parallel (|u_\varepsilon| - 1)_+ \parallel_{L^1(B_{\varepsilon/4}(0))} + C\varepsilon^{1-n/2} \parallel v_\varepsilon \parallel_{L^2(B_{\varepsilon/4}(0))} .$$

Proof:
After rescaling $u(x) = u_\varepsilon(\varepsilon x), v(x) = \varepsilon v_\varepsilon(\varepsilon x)$ it suffices to prove the claim for $\varepsilon = 1$. We see

$$-\Delta(u-1)_+ \leq v_+ \qquad \text{in } B_{1/4}(0)$$

and consider the Dirichlet problem

$$\begin{aligned} -\Delta\psi &= v_+ \qquad \text{in } B_{1/4}(0),\\ \psi &= 0 \qquad \text{on } \partial B_{1/4}(0). \end{aligned}$$

Then the difference $(u-1)_+ - \psi$ is subharmonic, hence

$$\big((u-1)_+ - \psi\big)(0) \leq C\|(u-1)_+\|_{L^1(B_{1/4}(0))} + C\|\psi\|_{L^1(B_{1/4}(0))}.$$

By the Sobolev embedding $W^{2,2}(B_{1/4}(0)) \hookrightarrow L^\infty(B_{1/4}(0))$, as $n \leq 3$, and elliptic estimates, we get

$$\begin{aligned} \|\psi\|_{L^\infty(B_{1/4}(0))} &\leq C\|\psi\|_{W^{2,2}(B_{1/4}(0))}\\ &\leq C\|v_+\|_{L^2(B_{1/4}(0))} \end{aligned}$$

and

$$u(0) \leq 1 + C\|(u-1)_+\|_{L^1(B_{1/4}(0))} + C\|v\|_{L^2(B_{1/4}(0))}.$$

By symmetry the proposition follows.

///

The next two propositions give us control over ‘error terms’ as for example appearing in the estimate in Lemma 3.1.

Proposition 3.4 ([C96] Lemma 4.4)
For $n = 2, 3, 0 \leq \delta \leq \delta_0,\ \Omega \subseteq \mathbb{R}^n,\ \varepsilon > 0,\ u_\varepsilon \in C^2(\Omega),\ v_\varepsilon \in C^0(\Omega)$,

$$-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) = v_\varepsilon \quad \text{in } \Omega,$$

and $\Omega' \subset\subset \Omega, 0 < r \leq d(\Omega', \partial\Omega)$, *we have*

$$\int\limits_{[|u_\varepsilon| \geq 1-\delta]\cap\Omega'} \Big(\varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) + \frac{1}{\varepsilon}W'(u_\varepsilon)^2\Big) \leq$$

$$\leq C\delta \int\limits_{[|u_\varepsilon| \leq 1-\delta]\cap\Omega} \varepsilon|\nabla u_\varepsilon|^2 + C\varepsilon \int\limits_{\Omega} v_\varepsilon^2 + C(\delta r^{-1} + \delta^2 r^{-2})\varepsilon\mathcal{L}^n(\Omega) + Cr^{-2}\varepsilon \int\limits_{[|u_\varepsilon| \geq 1]\cap\Omega} W'(u_\varepsilon)^2.$$

Proof:
As in [C96] Lemma 4.4, we define $g(t) := W'(t)$ for $|t| \geq 1-\delta$ choose $t_0 = 1/\sqrt{3}$, that is $W''(\pm t_0) = 4(3t_0^2 - 1) = 0$, and set $\delta_0 = (1-t_0)/2$. We put $g(t) = 0$ for $|t| \leq t_0$ and g to be linear in $[-1+\delta, -t_0]$ and $[t_0, 1-\delta]$. Clearly $|g| \leq |W'|$.

For $\eta \in C_0^1(\Omega), 0 \leq \eta \leq 1, \eta = 1$ on $\Omega', |\nabla \eta| \leq Cr^{-1}$, we get

$$\int v_\varepsilon g(u_\varepsilon)\eta^2 = \int \Big(-\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) \Big) g(u_\varepsilon)\eta^2 =$$

$$= \int \varepsilon g'(u_\varepsilon)|\nabla u_\varepsilon|^2 \eta^2 + 2\int \varepsilon \nabla u_\varepsilon g(u_\varepsilon)\eta \nabla \eta + \int \frac{1}{\varepsilon} W'(u_\varepsilon) g(u_\varepsilon)\eta^2. \tag{3.29}$$

We calculate

$$\Big| \int v_\varepsilon g(u_\varepsilon)\eta^2 \Big| \leq \frac{\varepsilon}{2} \int_\Omega v_\varepsilon^2 + \int \frac{1}{2\varepsilon} g(u_\varepsilon)^2 \eta^2 \leq \frac{\varepsilon}{2} \int_\Omega v_\varepsilon^2 + \frac{1}{2\varepsilon} \int W'(u_\varepsilon) g(u_\varepsilon)\eta^2. \tag{3.30}$$

As $g(1-\delta) = |W'(1-\delta) - W'(1)| \leq C\delta$, we see $|g(t)|, |g'(t)| \leq C\delta$ for $|t| \leq 1-\delta$. Therefore

$$\Big| 2\int \varepsilon \nabla u_\varepsilon g(u_\varepsilon)\eta \nabla \eta \Big| \leq 2\delta \int_{[|u_\varepsilon| \leq 1-\delta]} \varepsilon |\nabla u_\varepsilon| \eta |\nabla \eta| + \Big| 2 \int_{[|u_\varepsilon| \geq 1-\delta]} \varepsilon \nabla u_\varepsilon W'(u_\varepsilon)\eta \nabla \eta \Big| \leq$$

$$\leq C\delta \int_{[|u_\varepsilon| \leq 1-\delta]} \varepsilon |\nabla u_\varepsilon|^2 + \varepsilon \delta r^{-1} \mathcal{L}^n(\Omega) + \tau \int_{[|u_\varepsilon| \geq 1-\delta]} \varepsilon |\nabla u_\varepsilon|^2 \eta^2 + C\varepsilon \tau^{-1} r^{-2} \int_{[|u_\varepsilon| \geq 1-\delta]} W'(u_\varepsilon)^2 \tag{3.31}$$

for $\tau > 0$. As $g'(t) = W''(t) \geq c_0$ for $|t| \geq 1-\delta$, we obtain from (3.29) - (3.31)

$$c_0 \int_{[|u_\varepsilon| \geq 1-\delta]} \varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon} \int W'(u_\varepsilon) g(u_\varepsilon)\eta^2 \leq$$

$$\leq C\delta \int_{[|u_\varepsilon| \leq 1-\delta]} \varepsilon |\nabla u_\varepsilon|^2 + \tau \int_{[|u_\varepsilon| \geq 1-\delta]} \varepsilon |\nabla u_\varepsilon|^2 \eta^2 + \frac{\varepsilon}{2} \int_\Omega v_\varepsilon^2 +$$

$$+ \varepsilon \Big(\delta r^{-1} + C\delta^2 r^{-2} \tau^{-1} \Big) \mathcal{L}^n(\Omega) + C\varepsilon \tau^{-1} r^{-2} \int_{[|u_\varepsilon| \geq 1]} W'(u_\varepsilon)^2.$$

Choosing $\tau = c_0/2$, we get

$$\int_{[|u_\varepsilon| \geq 1-\delta] \cap \Omega'} \Big(\varepsilon |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W'(u_\varepsilon)^2 \Big) \leq C\delta \int_{[|u_\varepsilon| \leq 1-\delta]} \varepsilon |\nabla u_\varepsilon|^2 +$$

$$+ C\varepsilon \int_\Omega v_\varepsilon^2 + C\varepsilon \Big(\delta r^{-1} + \delta^2 r^{-2} \Big) \mathcal{L}^n(\Omega) + C\varepsilon r^{-2} \int_{[|u_\varepsilon| \geq 1]} W'(u_\varepsilon)^2.$$

As $W(t) \leq CW'(t)^2$ for $|t| \geq 1-\delta$, the assertion follows.

///

Proposition 3.5 *For* $n = 2, 3, \Omega \subseteq \mathbb{R}^n, \varepsilon > 0, u_\varepsilon \in C^2(\Omega), v_\varepsilon \in C^0(\Omega)$,

$$-\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) = v_\varepsilon \quad in\ \Omega,$$

and $\Omega' \subset\subset \Omega, 0 < r \leq d(\Omega', \partial\Omega)$ *, we have*

$$\int\limits_{[|u_\varepsilon| \geq 1] \cap \Omega'} W'(u_\varepsilon)^2 \leq C_k (1 + r^{-2k} \varepsilon^{2k}) \varepsilon^2 \int\limits_{\Omega} v_\varepsilon^2 + C_k r^{-2k} \varepsilon^{2k} \int\limits_{[|u_\varepsilon| \geq 1] \cap \Omega} W'(u_\varepsilon)^2 \quad \forall k \in \mathbb{N}_0.$$

Proof:

We choose

$$\Omega' = \Omega'_k \subset\subset \Omega'_{k-1} \subset\subset \ldots \subset\subset \Omega'_0 = \Omega$$

with $d(\Omega'_l, \partial\Omega'_{l-1}) \geq r/k$ for $l = 1, \ldots, k$, put

$$I_l := \int\limits_{[|u_\varepsilon| \geq 1] \cap \Omega'_l} W'(u_\varepsilon)^2 \quad \text{for } l = 0, \ldots, k,$$

and see by Proposition 3.4 for $\delta = 0$

$$I_l \leq C \varepsilon^2 \int\limits_{\Omega} v_\varepsilon^2 + C k^2 r^{-2} \varepsilon^2 I_{l-1} \quad \text{for } l = 1, \ldots, k,$$

and the result follows by induction.

///

We are now in the position to prove Lemma 3.1.

Proof of Lemma 3.1:

First we consider $0 < \varepsilon \leq \varrho = 1$. For $0 < \delta \leq \delta_0$, we choose $R = R(\delta^{2n+3}) = \delta^{-10n-15} \gg 1$, $\omega = \omega(\delta^{2n+3}) = c_0 \delta^{25(2n+3)} \ll 1$ as in Lemma 3.2. Let $\{x_i\}_{i \in I} \subseteq B_1(0)$, $I \subset \mathbb{N}$ be a maximal collection of points satisfying

$$\min_{i \neq j \in I} |x_i - x_j| \geq \frac{\varepsilon}{2}.$$

Since $\varepsilon \leq 1$ we have

$$B_1(0) \subseteq \bigcup_{i \in I} \overline{B_{\varepsilon/2}(x_i)} \subseteq B_{3/2}(0), \tag{3.32}$$

$$\sum_{i \in I} \mathcal{X}_{B_\varepsilon(x_i)} \leq C_n \mathcal{X}_{B_2(0)}, \tag{3.33}$$

$$\sum_{i \in I} \mathcal{X}_{B_{2R\varepsilon}(x_i)} \leq C_n R^n \mathcal{X}_{B_{1+2R\varepsilon}(0)}. \tag{3.34}$$

For $i \in I$ and $x \in B_{2R}(0)$ we put

$$U_i(x) := u_\varepsilon(x_i + \varepsilon x),$$
$$V_i(x) := \varepsilon v_\varepsilon(x_i + \varepsilon x).$$

Observing that

$$x_i + \varepsilon x \in B_{1+2R\varepsilon}(0) \subseteq B_{1+\delta^{-M}\varepsilon}(0) \subseteq B_{\varrho_0}(0)$$

for $M \geq 10n+16$ and $\delta_0 \leq 1/2$, we see that

$$-\Delta U_i + W'(U_i) = V_i \qquad \text{in } B_{2R}(0). \tag{3.35}$$

We decompose I in

$$\begin{aligned} I_1 &:= \big\{ i \in I \,:\, \|v_\varepsilon\|_{L^2(B_{2R\varepsilon}(x_i))} < \varepsilon^{n/2-1}\omega \\ &\qquad\quad \text{and } \ \|(|u_\varepsilon|-1)_+\|_{L^1(B_{2R\varepsilon}(x_i))} < c_0\varepsilon^n \big\}, \\ I_2 &:= I \setminus I_1. \end{aligned}$$

For $i \in I_1$ we see

$$\begin{aligned} \|V_i\|_{L^2(B_{2R}(0))} \;&=\; \varepsilon^{-n/2}\|\varepsilon v_\varepsilon\|_{L^2(B_{2R\varepsilon}(x_i))} \;<\; \omega \;\leq\; c_0, \\ \|(|U_i|-1)_+\|_{L^1(B_{2R}(x_i))} \;&=\; \varepsilon^{-n}\|(|u_\varepsilon|-1)_+\|_{L^1(B_{2R\varepsilon}(x_i))} \;<\; c_0, \end{aligned}$$

hence by Proposition 3.3

$$\|U_i\|_{L^\infty(B_R(0))} \;\leq\; 1 + Cc_0 \;\leq\; 2,$$

if c_0 is small enough. Then Lemma 3.2 yields

$$\begin{aligned} &\int\limits_{B_{1/2}(0)} \big(\frac{1}{2}|\nabla U_i|^2 - W(U_i)\big)_+ \\ \leq\, &C\delta \int\limits_{B_{1/2}(0)} \big(\frac{1}{2}|\nabla U_i|^2 - W(U_i)\big)_+ + \int\limits_{B_{1/2}(0)\cap[|U_i|\geq 1-\delta]} \frac{1}{2}|\nabla U_i|^2. \end{aligned}$$

Transferring back to u_ε and v_ε this reads

$$\begin{aligned} &\int\limits_{B_{\varepsilon/2}(x_i)} \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\big)_+ \\ \leq\, &C\delta \int\limits_{B_{\varepsilon/2}(x_i)} \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\big) + \int\limits_{B_{\varepsilon/2}(x_i)\cap[|u_\varepsilon|\geq 1-\delta]} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2. \end{aligned}$$

Summing over $i \in I_1$, recalling (3.32)-(3.34) and using Proposition 3.4 we get

$$\begin{aligned} &\sum_{i\in I_1}\int\limits_{B_{\varepsilon/2}(x_i)} \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\big)_+ \\ \leq\, &C\delta \int\limits_{B_{3/2}(0)} \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\big) + C \int\limits_{B_{3/2}(0)\cap[|u_\varepsilon|\geq 1-\delta]} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 \\ \leq\, &C\delta \int\limits_{B_2(0)} \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\big) + C\varepsilon \int\limits_{B_2(0)} v_\varepsilon^2 + \\ &+ C\varepsilon\Big(\delta + \int\limits_{B_2(0)\cap[|u_\varepsilon|\geq 1]} W'(u_\varepsilon)^2\Big). \end{aligned} \tag{3.36}$$

For $i \in I_2$, we get from (3.35) and local elliptic estimates

$$\int\limits_{B_{1/2}(0)} |\nabla U_i|^2 \le C \int\limits_{B_1(0)} \big(W'(U_i)^2 + U_i^2 + V_i^2\big)$$
$$\le C + C\Big(\int\limits_{B_1(0)\cap[|U_i|\ge 1]} W'(U_i)^2 + \int\limits_{B_1(0)} V_i^2\Big).$$

As $i \in I_2$ and $W'(t)^2 \ge \big(|t|-1\big)_+^6$, we have

$$c_0 \le \Big(\int\limits_{B_{2R}(0)} \big(|U_i|-1\big)_+\Big)^6 + \omega^{-2} \int\limits_{B_{2R}(0)} V_i^2$$
$$\le CR^{5n} \int\limits_{B_{2R}(0)\cap[|U_i|\ge 1]} W'(U_i)^2 + \omega^{-2} \int\limits_{B_{2R}(0)} V_i^2$$

hence

$$\int\limits_{B_{1/2}(0)} |\nabla U_i|^2 \le C\Big(R^{5n} \int\limits_{B_{2R}(0)\cap[|U_i|\ge 1]} W'(U_i)^2 + \omega^{-2} \int\limits_{B_{2R}(0)} V_i^2\Big).$$

Transferring back to u_ε and v_ε and summing over $i \in I_2$, we obtain by (3.32)-(3.34)

$$\sum_{i\in I_2} \int\limits_{B_{\varepsilon/2}(x_i)} \varepsilon|\nabla u_\varepsilon|^2$$
$$\le CR^n\Big(R^{5n}\int_{B_{1+2R\varepsilon}(0)} \frac{1}{\varepsilon}W'(u_\varepsilon)^2 + \varepsilon\omega^{-2} \int\limits_{B_{1+2R\varepsilon}(0)} v_\varepsilon^2\Big)$$
$$\le C\delta^{-M}\Big(\int_{B_{1+\delta^{-M}\varepsilon}(0)} \frac{1}{\varepsilon}W'(u_\varepsilon)^2 + \varepsilon \int\limits_{B_{1+\delta^{-M}\varepsilon}(0)} v_\varepsilon^2\Big)$$

for M large enough, as R and ω are proportional to powers of δ. Combining with (3.36) and using (3.32)-(3.34) we obtain

$$\int\limits_{B_1(0)} \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\big)_+$$
$$\le C\delta \int\limits_{B_2(0)} \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\big) + C\delta^{-M}\varepsilon \int\limits_{B_{\max(2,1+\delta^{-M}\varepsilon)}(0)} v_\varepsilon^2$$
$$+ C\delta^{-M} \int\limits_{B_{\max(2,1+\delta^{-M}\varepsilon)}(0)} \frac{1}{\varepsilon}W'(u_\varepsilon)^2 + C\varepsilon\delta.$$

For arbitrary ϱ, we put $U_{\varepsilon/\varrho}(x) := u_\varepsilon(\varrho x)$, $v_{\varepsilon/\varrho}(x) := \varrho v_\varepsilon(\varrho x)$ and see that

$$-\frac{\varepsilon}{\varrho}\Delta U_{\varepsilon/\varrho} + \frac{\varrho}{\varepsilon}W'(U_{\varepsilon/\varrho}) = V_{\varepsilon/\varrho}$$

in $B_{\max(2,1+\delta^{-M}\varepsilon)}(0)$. Rescaling and applying the case $\varrho = 1$, we get

$$\begin{aligned}
&\varrho^{1-n} \int\limits_{B_\varrho(0)} \big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\big)_+ \\
&= \int\limits_{B_1(0)} \big(\frac{\varepsilon}{2\varrho}|\nabla U_{\varepsilon/\varrho}|^2 - \frac{\varrho}{\varepsilon}W(U_{\varepsilon/\varrho})\big)_+ \\
&\leq C\delta \int\limits_{B_2(0)} \left(\frac{\varepsilon}{2\varrho}|\nabla U_{\varepsilon/\varrho}|^2 + \frac{\varrho}{\varepsilon}W(U_{\varepsilon/\varrho})\right) + C\delta^{-M}\varepsilon\varrho^{-1} \int\limits_{B_{\max(2,1+\delta^{-M}\varepsilon)}(0)} V_{\varepsilon/\varrho}^2 \\
&\quad + C\delta^{-M} \int\limits_{B_{\max(2,1+\delta^{-M}\varepsilon)}(0)\cap[|U_{\varepsilon/\varrho}|\geq 1]} \frac{\varrho}{\varepsilon}W'\big(U_{\varepsilon/\varrho}\big)^2 + C\frac{\varepsilon}{\varrho}\delta \\
&= C\delta\varrho^{1-n} \int\limits_{B_{2\varrho}(0)} \left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right) + C\delta^{-M}\varepsilon\varrho^{1-n} \int\limits_{B_{\varrho_0}(0)} v_\varepsilon^2 \\
&\quad + C\delta^{-M}\varrho^{1-n} \int\limits_{B_{\varrho_0}(0)} \frac{1}{\varepsilon}W'(u_\varepsilon)^2 + C\frac{\varepsilon}{\varrho}\delta.
\end{aligned}$$

///

For further use we finally prove some bounds for local L^p norms of u_ε.

Proposition 3.6 *Consider $u_\varepsilon \in C^2(\Omega), v_\varepsilon \in C^0(\Omega)$ satisfying* (2.1)-(2.5). *Then for all $\Omega' \subset\subset \Omega$*

$$\| u_\varepsilon \|_{L^p(\Omega')} \leq C(\Omega', p) \quad \forall 1 \leq p < \infty,$$
$$\| u_\varepsilon \|_{L^\infty(\Omega')} \leq C(\Omega', \beta)\varepsilon^{-\beta} \quad \forall \beta > 0$$

holds.

Proof:

Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega, x_0 \in \Omega', r = \min(d(\Omega', \partial\Omega''), 1)$, hence $B_r(x_0) \subset\subset \Omega''$. For $t \geq 2$, we see

$$W'(t) = 4t(t^2 - 1) \geq 24,$$

hence

$$-\Delta(u_\varepsilon - 2)_+ \leq \varepsilon^{-1}(v_\varepsilon - \frac{1}{\varepsilon})_+ \quad \text{in } \Omega. \tag{3.37}$$

Clearly

$$\int\limits_{B_r(x_0)} (u_\varepsilon - 1)_+^2 \leq \int\limits_{\Omega''} (u_\varepsilon - 1)^2(u_\varepsilon + 1)^2 \leq C(\Omega')\varepsilon. \tag{3.38}$$

Further for $1 \leq q < 2$

$$\begin{aligned}
&\int\limits_{B_r(x_0)} (v_\varepsilon - \frac{1}{\varepsilon})_+^q \leq \mathcal{L}^n\Big(\Omega'' \cap [v_\varepsilon > \frac{1}{\varepsilon}]\Big)^{1-q/2}\Big(\int\limits_{\Omega''} |v_\varepsilon|^2\Big)^{q/2} \leq \\
&\leq \Big(\varepsilon^2 \int\limits_{\Omega''} |v_\varepsilon|^2\Big)^{1-q/2}\Big(\int\limits_{\Omega''} |v_\varepsilon|^2\Big)^{q/2} \leq \varepsilon^{2-q} \int\limits_{\Omega''} |v_\varepsilon|^2 \leq C(\Omega')\varepsilon^{3-q},
\end{aligned}$$

hence

$$\Big\|\varepsilon^{-1}(v_\varepsilon - \frac{1}{\varepsilon})_+\Big\|_{L^q(B_r(x_0))} \leq C(\Omega')\varepsilon^{3/q-2}. \tag{3.39}$$

We consider

$$-\Delta\psi = \varepsilon^{-1}(v_\varepsilon - \tfrac{1}{\varepsilon})_+ \quad \text{in } B_r(x_0),$$
$$\psi = 0 \quad \text{on } \partial B_r(x_0),$$

and see

$$-\Delta((u_\varepsilon - 2)_+ - \psi) \le 0 \quad \text{in } B_r(x_0),$$

hence

$$\sup_{B_{r/2}(x_0)} \Big((u_\varepsilon - 2)_+ - \psi\Big) \le C_n r^{-n}\Big(\parallel (u_\varepsilon - 2)_+ \parallel_{L^1(B_r(x_0))} + \parallel \psi \parallel_{L^1(B_r(x_0))} \Big). \tag{3.40}$$

By (3.37), we get for $1 < q < 2$

$$\parallel \psi \parallel_{W^{2,q}(B_r(x_0))} \le C(\Omega', q)\varepsilon^{3/q-2}.$$

For $q = 3/2$ and the Sobolev embedding $W^{2,3/2}(B_r(x_0)) \hookrightarrow L^p(B_r(x_0))$ for $1 \le p < \infty$, as $n \le 3$, we see

$$\parallel \psi \parallel_{L^p(B_r(x_0))} \le C(\Omega', p),$$

hence by (3.38) and (3.40)

$$\parallel u_{\varepsilon,+} \parallel_{L^p(B_{r/2}(x_0))} \le C(\Omega', p).$$

For $q = 3/(2-\beta) > 3/2$ and the Sobolev embedding $W^{2,q}(B_r(x_0)) \hookrightarrow L^\infty(B_r(x_0))$, we see

$$\parallel \psi \parallel_{L^\infty(B_r(x_0))} \le C(\Omega', \beta)\varepsilon^{-\beta},$$

hence again by (3.38) and (3.40)

$$\parallel u_{\varepsilon,+} \parallel_{L^\infty(B_{r/2}(x_0))} \le C(\Omega', \beta)\varepsilon^{-\beta}.$$

Covering Ω' appropriately and by symmetry, we obtain

$$\parallel u_\varepsilon \parallel_{L^p(\Omega')} \le C(\Omega', p) \quad \forall 1 \le p < \infty,$$
$$\parallel u_\varepsilon \parallel_{L^\infty(\Omega')} \le C(\Omega', \beta)\varepsilon^{-\beta} \quad \forall \beta > 0.$$

///

4 Rectifiability

In this section we prove that the limit μ of the energy measures is rectifiable. The line of arguments follows [HT00], [T02] but whereas their proofs are based on an L^∞-bound on the discrepancy measures we can only use an L^1-control over ξ_ε, which requires substantial changes in the proofs and additional arguments.

Theorem 4.1 *μ as in (2.6) is a rectifiable $(n-1)-$varifold with*

$$\theta_*^{n-1}(\mu) \ge \bar\theta \quad \textit{on spt } \mu$$

for some universal $\bar\theta > 0$ and weak mean curvature in $L^2_{loc}(\mu)$ with

$$|\mathbf{H}_\mu|^2\mu \le \alpha.$$

□

We prove this theorem in several steps and start with the important monotonicity formula.

Lemma 4.2 (Monotonicity formula, [T02] Lemma 3.1)

$$\frac{d}{d\varrho}\Big(\varrho^{1-n}\mu_\varepsilon(B_\varrho(0))\Big) =$$
$$= -\frac{1}{\varrho^n}\xi_\varepsilon(B_\varrho(0)) + \frac{1}{\varrho^{n+1}}\int\limits_{\partial B_\varrho(0)} \varepsilon(y\nabla u_\varepsilon(y))^2 \,\mathrm{d}\mathcal{H}^{n-1}(y) + \frac{1}{\varrho^n}\int\limits_{B_\varrho(0)} v_\varepsilon(y)(y\nabla u_\varepsilon(y))\,\mathrm{d}y$$

for $B_\varrho(0) \subset\subset \Omega$ *.*

Proof:
We multiply (2.1) by $\nabla u_\varepsilon \eta$ where $\eta \in C_0^1(\Omega,\mathbb{R}^n)$ and get

$$\int v_\varepsilon \nabla u_\varepsilon \eta = \int \Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big)\nabla u_\varepsilon \eta =$$
$$= \int \varepsilon\partial_i u_\varepsilon \partial_{ji} u_\varepsilon \eta_j + \int \varepsilon \partial_i u_\varepsilon \partial_j u_\varepsilon \partial_i \eta_j + \int \frac{1}{\varepsilon}\partial_j W(u_\varepsilon)\eta_j =$$
$$= -\int \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big) div\ \eta + \int \varepsilon\partial_i u_\varepsilon \partial_j u_\varepsilon \partial_i \eta_j. \tag{4.1}$$

Choosing $\eta_j(y) = y_j\phi_\tau(|y|)$ with $\phi_\tau = 1$ on $[0,\varrho], \phi_\tau = 0$ on $[\varrho+\tau,\infty[$ and $\phi_\tau' \le 0$, we get for $r(y) := |y|$

$$\int \Big(r\phi_\tau' + n\phi_\tau\Big)\,\mathrm{d}\mu_\varepsilon - \int \varepsilon\frac{\phi_\tau'}{r}(y\nabla u_\varepsilon)^2 - \int \varepsilon|\nabla u_\varepsilon|^2\phi_\tau = -\int v_\varepsilon(y\nabla u_\varepsilon)\phi_\tau.$$

Letting $\tau \to 0$ yields

$$-(n-1)\mu_\varepsilon(B_\varrho(0)) + \varrho\mu_\varepsilon(\partial B_\varrho(0)) =$$
$$= -\xi_\varepsilon(B_\varrho(0)) + \frac{1}{\varrho}\int\limits_{\partial B_\varrho(0)} \varepsilon(y\nabla u_\varepsilon)^2 + \int\limits_{B_\varrho(0)} v_\varepsilon(y\nabla u_\varepsilon).$$

Multiplying by ϱ^{-n} , the result follows.

///

In small dimensions $n = 2,3$, we can estimate the integral of the last two terms.

Proposition 4.3 *For* $n = 2,3, 0 < r \le 1, B_r(0) \subset\subset \Omega$ *, we get*

$$\int\limits_0^r \Big[\varrho^{-1-n}\int\limits_{\partial B_\varrho(0)} \varepsilon(y\nabla u_\varepsilon)^2 + \varrho^{-n}\int\limits_{B_\varrho(0)} v_\varepsilon(y\nabla u_\varepsilon)\Big]\,\mathrm{d}\varrho \ge -\frac{1}{4(n-1)^2}\alpha_\varepsilon(B_r(0)).$$

Proof:
We calculate

$$\int\limits_0^r \varrho^{-1-n}\Big(\int\limits_{\partial B_\varrho} \varepsilon(y\nabla u_\varepsilon)^2\,\mathrm{d}\mathcal{H}^{n-1}(y)\Big)\,\mathrm{d}\varrho = \int\limits_{B_r(0)} \varepsilon\frac{(y\nabla u_\varepsilon)^2}{|y|^{n+1}}\,\mathrm{d}y$$

and

$$\int\limits_0^r \varrho^{-n}\int\limits_{B_\varrho(0)} |v_\varepsilon(y\nabla u_\varepsilon)|\,\mathrm{d}\varrho = \int\limits_0^r \int\limits_{|y|<\varrho} \varrho^{-n}|v_\varepsilon(y\nabla u_\varepsilon)|\,\mathrm{d}y\,\mathrm{d}\varrho =$$

$$= \int\limits_{B_r(0)} \int\limits_{|y|}^{r} \varrho^{-n} |v_\varepsilon(y\nabla u_\varepsilon)| \,\mathrm{d}\varrho \,\mathrm{d}y \le \frac{1}{n-1} \int\limits_{B_r(0)} \frac{|v_\varepsilon(y\nabla u_\varepsilon)|}{|y|^{n-1}} \,\mathrm{d}y \le$$

$$\le \frac{1}{4(n-1)^2} \int\limits_{B_r(0)} \frac{1}{\varepsilon} v_\varepsilon^2 + \int\limits_{B_r(0)} \varepsilon \frac{(y\nabla u_\varepsilon)^2}{|y|^{2n-2}} \,\mathrm{d}y.$$

Observing $2n-2 \le n+1$, as $n \le 3$, the estimate follows.

///

Using results from section 3 we prove now that the positive part of the discrepancy measures vanishes in the limit $\varepsilon \to 0$.

Proposition 4.4

$$\xi_{\varepsilon,+} \to 0,$$

in particular $\xi \le 0$.

Proof:
For $B_{2\varrho} \subseteq \Omega' \subset\subset \Omega$ and $0 < \varepsilon \le \delta^M, 0 < \delta \le \delta_0$, we see by Lemma 3.1, Proposition 3.5 and 3.6

$$\xi_{\varepsilon_+}(B_\varrho) \le$$

$$\le C\delta\mu_\varepsilon(B_{2\varrho}) + C\delta^{-M}\varepsilon^2\alpha_\varepsilon(B_{2\varrho}) + C\delta^{-M} \int\limits_{B_{2\varrho}\cap[|u_\varepsilon|\ge 1]} \frac{1}{\varepsilon} W'(u_\varepsilon)^2 + C\varepsilon\delta\varrho^{n-2} \le$$

$$\le C(\Omega')\delta + C(\Omega')\delta^{-M}\varepsilon^2 + C\varepsilon\delta\varrho^{n-2},$$

hence

$$\limsup_{\varepsilon\to 0} \xi_{\varepsilon,+}(B_\varrho) \le C(\Omega')\delta$$

and $\xi_{\varepsilon,+}(B_\varrho) \to 0$.

///

An immediate consequence of the monotonicity formula and the last proposition is an upper bound on density ratios of μ.

Proposition 4.5 *For* $\Omega' \subset\subset \Omega, r_0(\Omega') := \min(1, d(\Omega', \partial\Omega)/2)$ *, we have*

$$r^{1-n}\mu(B_r(x)) \le C(\Omega') \quad \text{for } x \in \Omega', 0 < r \le r_0.$$

Proof:
For $x \in \Omega', 0 < r \le r_0$, we see by the monotonicity formula, Lemma 4.2, and Proposition 4.3

$$r_0^{1-n}\mu_\varepsilon(B_{r_0}(x)) \ge r^{1-n}\mu_\varepsilon(B_r(x)) - \int\limits_r^{r_0} \varrho^{-n}\xi_\varepsilon(B_\varrho(x)) \,\mathrm{d}\varrho - \frac{1}{4(n-1)^2}\alpha_\varepsilon(B_{r_0}(x)).$$

Letting $\varepsilon \to 0$ along our subsequence and observing $\xi \le 0$ by Proposition 4.4, we get

$$r^{1-n}\mu(B_r(x)) \le r_0^{1-n}\mu(\overline{B_{r_0}(x)}) + \frac{1}{4(n-1)^2}\alpha(\overline{B_{r_0}(x)})$$

and the proposition follows.

///

To obtain further estimates on density bounds we improve the monotonicity formula by combining Lemma 4.2 with the estimate proved in Lemma 3.1.

Proposition 4.6 *For* $B_{3r^{1-\beta}}(x) \subset\subset \Omega, \beta > 0, \varepsilon \le s \le \varrho \le r \le 1$ *, we get*

$$\varrho^{-n}\xi_{\varepsilon,+}(B_\varrho(x)) \le C\varrho^{-1+\gamma}\varrho^{1-n}\mu_\varepsilon(B_{2\varrho}(x)) +$$

$$+ C_\beta \varepsilon^2 \varrho^{-M\gamma-n}\alpha_\varepsilon(B_{3\varrho^{1-\beta}}(x)) + \varepsilon\varrho^{\gamma-2}\Big(C + C_\beta \int\limits_{B_{3r^{1-\beta}}(x)\cap[|u_\varepsilon|\ge 1]} W'(u_\varepsilon)^2\Big) \tag{4.2}$$

and

$$r^{1-n}\mu_\varepsilon(B_r(x)) \ge s^{1-n}\mu_\varepsilon(B_s(x)) - C\int\limits_s^r \varrho^{-1+\gamma}\varrho^{1-n}\mu_\varepsilon(B_{2\varrho}(x))\, \mathrm{d}\varrho$$

$$-C_\beta\varepsilon^2\int\limits_s^r \varrho^{-M\gamma-n}\alpha_\varepsilon(B_{3\varrho^{1-\beta}}(x))\, \mathrm{d}\varrho - \Big(C + C_\beta \int\limits_{B_{3r^{1-\beta}}(x)\cap[|u_\varepsilon|\ge 1]} W'(u_\varepsilon)^2\Big)\varepsilon^\gamma - C\alpha_\varepsilon(B_r(x)), \tag{4.3}$$

where $0 < \gamma < 1/M < 1/2$ *with* M *from Lemma 3.1.*

Proof:
First we apply Lemma 3.1 with $\delta = \varrho^\gamma, \varepsilon \le \varrho \le r$ for $0 < \gamma < 1/M < 1/2$. Observing that $\delta^{-M}\varepsilon \le \varrho^{1-M\gamma} \le 1$ holds we get

$$\varrho^{-n}\xi_{\varepsilon,+}(B_\varrho(x)) \le C\varrho^{-1+\gamma}\varrho^{1-n}\mu_\varepsilon(B_{2\varrho}(x)) +$$

$$+ C\varepsilon^2\varrho^{-M\gamma-n}\alpha_\varepsilon(B_{2\varrho}(x)) + C\varepsilon^{-1}\varrho^{-M\gamma-n}\int\limits_{B_{2\varrho}(x)\cap[|u_\varepsilon|\ge 1]} W'(u_\varepsilon)^2 + C\varepsilon\varrho^{\gamma-2}.$$

By Proposition 3.5 with $r := d(B_{2\varrho}(x), \partial B_{3\varrho^{1-\beta}}(x)) = 3\varrho^{1-\beta} - 2\varrho \ge \varrho^{1-\beta}$

$$\int\limits_{B_{2\varrho}(x)\cap[|u_\varepsilon|\ge 1]} W'(u_\varepsilon)^2 \le C_k\varepsilon^3\alpha_\varepsilon(B_{3\varrho^{1-\beta}}(x)) + C_k\varrho^{-2k(1-\beta)}\varepsilon^{2k}\int\limits_{B_{3\varrho^{1-\beta}}(x)\cap[|u_\varepsilon|\ge 1]} W'(u_\varepsilon)^2 \le$$

$$\le C_k\varepsilon^3\alpha_\varepsilon(B_{3\varrho^{1-\beta}}(x)) + C_k\varepsilon^{2k\beta}\int\limits_{B_{3\varrho^{1-\beta}}(0)\cap[|u_\varepsilon|\ge 1]} W'(u_\varepsilon)^2,$$

hence for $2k\beta \ge 5$ we obtain (4.2).

Plugging into the monotonicity formula, Lemma 4.2, we obtain

$$\frac{d}{d\varrho}\Big(\varrho^{1-n}\mu_\varepsilon(B_\varrho(x))\Big) \ge$$

$$\ge -C\varrho^{-1+\gamma}\varrho^{1-n}\mu_\varepsilon(B_{2\varrho}(x)) - C_\beta\varepsilon^2\varrho^{-M\gamma-n}\alpha_\varepsilon(B_{3\varrho^{1-\beta}}(x)) -$$

$$-C_\beta\varepsilon\varrho^{\gamma-2}\Big(1 + \int\limits_{B_{3\varrho^{1-\beta}}(0)\cap[|u_\varepsilon|\ge 1]} W'(u_\varepsilon)^2\Big) +$$

$$+\varrho^{-1-n}\int\limits_{\partial B_\varrho(x)} \varepsilon((y-x)\nabla u_\varepsilon)^2\, \mathrm{d}\mathcal{H}^{n-1}(y) + \varrho^{-n}\int\limits_{B_\varrho(x)} v_\varepsilon((y-x)\nabla u_\varepsilon)\, \mathrm{d}y.$$

Integration over $0 < \varepsilon \le s \le r$ yields (4.3) by Proposition 4.3

///

Under suitable assumptions on the error terms we can further simplify (4.3).

Proposition 4.7 *Let* $B_{3r^{1-\beta}}(x) \subset\subset \Omega, 0 < \beta_0, \beta \leq 1/2, 0 < \varepsilon \leq s \leq r \leq 1$ *, with*

$$\int\limits_{B_{3r^{1-\beta}}(x)\cap[|u_\varepsilon|\geq 1]} W'(u_\varepsilon)^2 \leq \Lambda,$$
$$\alpha_\varepsilon(B_\varrho(x)) \leq \Lambda \varrho^{\beta_0} \quad \textit{for } 3s^{1-\beta} \leq \varrho \leq 3r^{1-\beta}, n = 3,$$
$$\alpha_\varepsilon(B_{3r^{1-\beta}}(x)) \leq \Lambda \quad \textit{for } n = 2.$$

Then

$$s^{1-n}\mu_\varepsilon(\overline{B_s(x)}) \leq C_{\beta_0} r^{1-n}\mu_\varepsilon(\overline{B_r(x)}) + C_{\beta_0,\beta}(1+\Lambda).$$

Proof:
Putting

$$f(\varrho) := \varrho^{1-n}\mu_\varepsilon(B_\varrho(x)) \quad \text{for } s \leq \varrho \leq r,$$

we see

$$f(t) \leq 2^{n-1} f(r) \quad \text{for } r/2 \leq t \leq r$$

and get from Proposition 4.6 (4.3) with r replaced by $r/2$ for $\gamma = c_0\beta_0$ with c_0 small enough and any $s \leq t \leq r/2$

$$2^{n-1}f(r) - f(t) \geq (r/2)^{1-n}\mu_\varepsilon(B_{r/2}(x)) - t^{1-n}\mu_\varepsilon(B_t(x)) \geq$$

$$\geq -C\int\limits_t^{r/2} \varrho^{-1+\gamma} f(2\varrho)\ \mathrm{d}\varrho - C_\beta\varepsilon^2 \int\limits_\varepsilon^{r/2} \Lambda\varrho^{-M\gamma-n+\beta_0(1-\beta)}\ \mathrm{d}\varrho - C_\beta(1+\Lambda)\varepsilon^\gamma - C\Lambda \geq$$

$$\geq -C\int\limits_{2t}^{r} \varrho^{-1+c_0\beta_0} f(\varrho)\ \mathrm{d}\varrho - C_\beta(1+\Lambda).$$

Together

$$f(t) \leq 2^{n-1}f(r) + C_\beta(1+\Lambda) + \int\limits_{2t}^{r} C\varrho^{-1+c_0\beta_0} f(\varrho)\ \mathrm{d}\varrho \quad \text{for } s \leq t \leq \frac{r}{2}.$$

By Gronwall's lemma

$$f(t) \leq \exp(C_{\beta_0} r^{c_0\beta_0})\Big(2^{n-1}f(r) + C_\beta(1+\Lambda)\Big),$$

which is the assertion, when the closed balls are replaced by open balls. Approximating $t \searrow s$ gives the full assertion.

///

The next lemma gives an important lower estimate on density ratios.

Lemma 4.8 *There exists* $\bar{\theta} > 0$ *such that for any open set* $\Omega' \subset\subset \Omega$ *and appropriate* $r_1 = r_1(\Omega')$, $0 < r_1(\Omega') := \min(c_0, d(\Omega', \partial\Omega)/2) \leq r_0(\Omega') := \min(1, d(\Omega', \partial\Omega)/2)$ *for* $c_0 > 0$ *small enough, we have*

$$r^{1-n}\mu(B_r(x)) \geq \bar{\theta} - C\alpha(B_r(x)) \quad \text{for } x \in spt\ \mu \cap \Omega', 0 < r \leq r_1.$$

In particular

$$\theta_*^{n-1}(\mu) \geq \bar{\theta}\omega_{n-1}^{-1}$$

almost everywhere with respect to μ *in* Ω .

Proof:
We consider $0 \in spt\ \mu \cap \Omega'$ and choose $\beta = \beta(r_0) > 0$ such that

$$3\big(r_0/4\big)^{1-\beta} \leq r_0.$$

For $x \in B_{r/2}(0), 0 < r \leq r_0 \leq 1$, we see $B_r(x) \subseteq B_{3r_0/2}(0) \subset\subset \Omega$ and get by Proposition 3.6 and 4.6 (4.3) with r replaced by $r/4$ that

$$(r/4)^{1-n}\mu_\varepsilon(B_{r/4}(x)) \geq s^{1-n}\mu_\varepsilon(B_s(x)) - C\int_s^{r/4} \varrho^{-1+\gamma}\varrho^{1-n}\mu_\varepsilon(B_{2\varrho}(x))\ \mathrm{d}\varrho -$$
$$-C_\beta\varepsilon^2\int_s^{r/4} \varrho^{-M\gamma-n}\alpha_\varepsilon(B_{3\varrho^{1-\beta}}(x))\ \mathrm{d}\varrho - C_\beta(\Omega')\varepsilon^\gamma - C\alpha_\varepsilon(B_{r/4}(x)), \tag{4.4}$$

where $0 < \gamma < 1/M < 1/2$ with M from Lemma 3.1.

Next we seek a point $x \in B_{r/2}(0)$ satisfying

$$\varepsilon^{1-n}\mu_\varepsilon(B_\varepsilon(x)) \geq 2\bar{\theta_0} > \bar{\theta_0} \geq C_\beta\varepsilon^2\int_\varepsilon^{r/4} \varrho^{-M\gamma-n}\alpha_\varepsilon(B_{3\varrho^{1-\beta}}(x))\ \mathrm{d}\varrho \tag{4.5}$$

for some universal $\bar{\theta_0} > 0$. We consider $x \in B_{r/2}(0)$ with $|u_\varepsilon(x)| \leq 1 - \tau$ for some $0 < \tau < 1$. If $\varepsilon^{1-n}\mu_\varepsilon(B_\varepsilon(x)) \leq 1$, we see

$$\varepsilon^{-n}\int_{B_\varepsilon(x)} u_\varepsilon^4 \leq C\Big(1 + \int_{B_\varepsilon(x)} \varepsilon^{-n}W(u_\varepsilon)\Big) \leq C.$$

As for $n \leq 3$

$$\| \varepsilon v_\varepsilon(x + \varepsilon.) \|_{L^2(B_1(0))}^2 \leq C\varepsilon^{2-n}\int_{B_\varepsilon(x)} v_\varepsilon^2 \leq C\alpha_\varepsilon(B_\varepsilon(x)) \leq C,$$

we see by elliptic estimates

$$\| u_\varepsilon(x + \varepsilon.) \|_{C^{0,1/2}(B_{1/2}(0))} \leq C \| u_\varepsilon(x + \varepsilon.) \|_{W^{2,2}(B_{1/2}(0))} \leq C,$$

hence

$$|u_\varepsilon| \leq 1 - \tau/2 \quad \text{on } B_{c_0\tau^2\varepsilon}(x)$$

for $c_0 \ll 1$ small enough and

$$\varepsilon^{1-n}\mu_\varepsilon(B_\varepsilon(x)) \geq \varepsilon^{-n} \int\limits_{B_{c_0\tau^2\varepsilon}(x)} W(u_\varepsilon) \geq c_0\tau^{2n+2} := 2\bar{\theta}_0 > 0.$$

For $c_0 \ll 1$ this is also true in case $\varepsilon^{1-n}\mu_\varepsilon(B_\varepsilon(x)) \geq 1$, and we get

$$\varepsilon^{1-n}\mu_\varepsilon(B_\varepsilon(x)) \geq 2\bar{\theta}_0 \quad \text{for } x \in B_{r/2}(0) \cap [|u_\varepsilon| \leq 1-\tau]. \tag{4.6}$$

By Proposition 3.4 and 3.6, we get for τ small enough

$$\mu_\varepsilon(B_{r/4}(0)) = \mu_\varepsilon\Big(B_{r/4}(0) \cap [|u_\varepsilon| < 1-\tau]\Big) + \mu_\varepsilon\Big(B_{r/4}(0) \cap [|u_\varepsilon| \geq 1-\tau]\Big) \leq$$

$$\leq C\mu_\varepsilon\Big(B_{r/2}(0) \cap [|u_\varepsilon| \leq 1-\tau]\Big) + C\varepsilon^2\alpha_\varepsilon(B_{r/2}(0)) + C\varepsilon(\tau r^{n-1}\tau r^{n-2}) + Cr^{-2}\varepsilon,$$

hence, as $0 \in spt\ \mu$, for $\varepsilon \to 0$ along the subsequence

$$0 < \mu(B_{r/4}(0)) \leq \liminf_{\varepsilon\to 0} \mu_\varepsilon(B_{r/4}(0)) \leq \liminf_{\varepsilon\to 0} C\mu_\varepsilon\Big(B_{r/2}(0) \cap [|u_\varepsilon| \leq 1-\tau]\Big)$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon^{-1}\mathcal{L}^n\Big(B_{r/2}(0) \cap [|u_\varepsilon| \leq 1-\tau]\Big) \geq$$

$$\geq \liminf_{\varepsilon\to 0} c_0\tau^{-2} \int\limits_{B_{r/2}(0)\cap[|u_\varepsilon|\leq 1-\tau]} \frac{1}{\varepsilon}W(u_\varepsilon) =$$

$$= \liminf_{\varepsilon\to 0} c_0\tau^{-2}(\mu_\varepsilon - \xi_\varepsilon)\Big(B_{r/2}(0) \cap [|u_\varepsilon| \leq 1-\tau]\Big) >$$

$$> -\limsup_{\varepsilon\to 0} c_0\tau^{-2}\xi_\varepsilon\Big(B_{r/2}(0) \cap [|u_\varepsilon| \leq 1-\tau]\Big) \geq 0 \tag{4.7}$$

by Proposition 4.4.

To estimate the integral in (4.5), we define for $0 < \varrho \leq r_0$ the convolution

$$w_{\varepsilon,\varrho}(x) := \varrho^{-n}\Big(\chi_{B_\varrho(0)} * \frac{1}{\varepsilon}v_\varepsilon^2\Big)(x) = \varrho^{-n}\alpha_\varepsilon(B_\varrho(x))$$

and see $w_{\varepsilon,\varrho} \in L^1(B_{r_0/2}(0))$ with

$$\| w_{\varepsilon,\varrho} \|_{L^1(B_{r_0/2}(0))} \leq \int\limits_{B_{r_0/2+\varrho}(0)} \frac{1}{\varepsilon}v_\varepsilon^2 \leq \alpha_\varepsilon(B_{3r_0/2}(0)) < \infty.$$

Putting $w_\varepsilon := \int_0^{r_0} w_{\varepsilon,\varrho}\ \mathrm{d}\varrho$, we see

$$\| w_\varepsilon \|_{L^1(B_{r_0/2}(0))} \leq r_0\alpha_\varepsilon(B_{3r_0/2}(0)) < \infty$$

and calculate

$$\int\limits_\varepsilon^{r/4} \varrho^{-M\gamma-n}\alpha_\varepsilon(B_{3\varrho^{1-\beta}}(x))\ \mathrm{d}\varrho =$$

$$= \int\limits_{3\varepsilon^{1-\beta}}^{3(r/4)^{1-\beta}} (t/3)^{(-M\gamma-n)/(1-\beta)}\alpha_\varepsilon(B_t(x))t^{1/(1-\beta)-1}3^{-1/(1-\beta)}(1-\beta)^{-1}\ \mathrm{d}t \leq$$

$$\leq C \int_{3\varepsilon^{1-\beta}}^{3(r/4)^{1-\beta}} t^{(-M\gamma-n+\beta)/(1-\beta)}\alpha_\varepsilon(B_t(x))\ \mathrm{d}t \leq$$

$$\leq C\varepsilon^{(-M\gamma-(n-1)\beta)/(1-\beta)} \int_0^{r_0} w_{\varepsilon,\varrho}(x)\ \mathrm{d}\varrho = C\varepsilon^{(-M\gamma-(n-1)\beta)/(1-\beta)} w_\varepsilon(x).$$

Choosing $M\gamma < 1/2$ and $\beta < (2(n-1))^{-1}$, we get

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1}\mathcal{L}^n\Bigg(B_{r/2}(0) \cap \Big[C_\beta\varepsilon^2 \int_\varepsilon^{r/4} \varrho^{-M\gamma-n}\alpha_\varepsilon(B_{3\varrho^{1-\beta}}(x))\ \mathrm{d}\varrho \geq \bar\theta_0\Big]\Bigg) \leq$$

$$\leq \limsup_{\varepsilon\to 0} \varepsilon^{-1} C_\beta \varepsilon^{2-(M\gamma+(n-1)\beta)/(1-\beta)} \bar\theta_0^{-1} \parallel w_\varepsilon \parallel_{L^1(B_{r_0/2}(0))} \leq$$

$$\leq \limsup_{\varepsilon\to 0} \varepsilon^{1-(M\gamma+(n-1)\beta)/(1-\beta)} C_\beta \bar\theta_0^{-1} r_0 \alpha_\varepsilon(B_{3r_0/2}(0)) = 0.$$

Combining with (4.6) and (4.7), we see for ε small enough that $x \in B_{r/2}(0)$ satisfying (4.5) exists.

For such x , we claim

$$(r/2)^{1-n}\mu_\varepsilon(B_{r/2}(x)) \geq 2^{1-n}\bar\theta_0 - C_\gamma r^\gamma - C_\beta(\Omega')\varepsilon^\gamma - C\alpha_\varepsilon(B_{r/4}(x)). \tag{4.8}$$

If not, we put

$$s := \sup\{\varepsilon \leq \varrho \leq r/2 \mid \varrho^{1-n}\mu_\varepsilon(B_\varrho(x)) \geq 2\bar\theta_0\ \}.$$

Clearly $\varepsilon \leq s \leq r/4$, as we assume that (4.8) is not satisfied, and

$$s^{1-n}\mu_\varepsilon(B_s(x)) \geq 2\bar\theta_0,$$
$$\varrho^{1-n}\mu_\varepsilon(B_\varrho(x)) \leq 2\bar\theta_0 \quad \forall s \leq \varrho \leq r/2.$$

Then we obtain from (4.4) and (4.5)

$$2^{n-1}(r/2)^{1-n}\mu_\varepsilon(B_{r/2}(0)) \geq (r/4)^{1-n}\mu_\varepsilon(B_{r/4}(0)) \geq$$

$$\geq 2\bar\theta_0 - C\int_s^{r/4} 2\bar\theta_0\varrho^{-1+\gamma}\ \mathrm{d}\varrho - \bar\theta_0 - C_\beta(\Omega')\varepsilon^\gamma - C\alpha_\varepsilon(B_{r/4}(x)) \geq$$

$$\geq \bar\theta_0 - C_\gamma r^\gamma - C_\beta(\Omega')\varepsilon^\gamma - C\alpha_\varepsilon(B_{r/4}(x))$$

which yields (4.8).

As $B_{r/2}(x) \subseteq B_r(0)$, we get from (4.8) for $\varepsilon \to 0$ along the subsequence

$$r^{1-n}\mu(\overline{B_r(0)}) \geq \limsup_{\varepsilon\to 0} 2^{1-n}(r/2)^{1-n}\mu_\varepsilon(B_{r/2}(x)) \geq 4^{1-n}\bar\theta_0 - C_\gamma r^\gamma - C\alpha(\overline{B_r(0)}).$$

Approximating $r' \nearrow r$, we get for $0 < r \leq r_1(\Omega') \leq r_0(\Omega')$ that

$$r^{1-n}\mu(B_r(0)) \geq c_0\bar\theta_0 - C\alpha(B_r(0)),$$

which yields the first estimate of the proposition.

This implies $\theta_*^{n-1}(\mu,x) \geq \bar\theta\omega_{n-1}^{-1}$ for $x \in spt\ \mu \cap \Omega$ with $\alpha(\{x\}) = 0$. As $\{x \in \Omega \mid \alpha(\{x\}) > 0\ \}$ is countable and $\mu(\{x\}) = 0$ for any $x \in \Omega$ by Proposition 4.5, as $n \geq 2$, we see

$$\mu(\{x \in \Omega \mid \alpha(\{x\}) > 0\ \}) = 0$$

and

$$\theta_*^{n-1}(\mu) \geq \bar\theta\omega_{n-1}^{-1}$$

almost everywhere with respect to μ in Ω .

///

Revisiting the monotonicity formula Lemma 4.2 we can prove that the full discrepancy vanishes in the limit.

Proposition 4.9 ([T02] Proposition 4.3)

$$|\xi_\varepsilon| \to 0 \quad \textit{and} \quad \xi = 0.$$

Proof:
We recall $\xi \leq 0$ by Proposition 4.4. First we show

$$\theta_*^{n-1}(|\xi|) = 0 \quad \text{in } \Omega. \tag{4.9}$$

If not, there exists $0 < \varrho_0, \delta < 1$ such that $B_\varrho(x) \subset\subset \Omega$ and

$$\varrho^{1-n}|\xi|(B_\varrho(x)) \geq \delta \quad \forall 0 < \varrho \leq \varrho_0.$$

By monotonicity formula, Lemma 4.2, we get

$$\frac{d}{d\varrho}\Big(\varrho^{1-n}\mu_\varepsilon(B_\varrho(x))\Big) \geq$$

$$\geq -\varrho^{-n}\xi_\varepsilon(B_\varrho(x)) + \varrho^{1-n}\int\limits_{B_\varrho(x)} \varepsilon(y\nabla u_\varepsilon)^2\ \mathrm{d}y + \varrho^{-n}\int\limits_{\partial B_\varrho(x)} v_\varepsilon(y\nabla u_\varepsilon)\ \mathrm{d}\mathcal{H}^{n-1}(y).$$

Integrating from r to ϱ_0 , we get by Proposition 4.3

$$\int\limits_r^{\varrho_0} \varrho^{-n}\xi_{\varepsilon,-}(B_\varrho(x))\ \mathrm{d}\varrho \leq \varrho_0^{1-n}\mu_\varepsilon(B_{\varrho_0}(x)) + \frac{1}{4(n-1)^2}\alpha_\varepsilon(B_{\varrho_0}(x))$$

$$+ \int\limits_r^{\varrho_0} \varrho^{-n}\xi_{\varepsilon,+}(B_\varrho(x))\ \mathrm{d}\varrho.$$

Letting $\varepsilon \to 0$ along the subsequence recalling $\xi_{\varepsilon,+} \to 0$ by Proposition 4.4 and 4.5 we get

$$\infty > \varrho_0^{1-n}\mu(\overline{B_{\varrho_0}(x)}) + \frac{1}{4(n-1)^2}\alpha(\overline{B_{\varrho_0}(x)}) \geq \int\limits_r^{\varrho_0} \varrho^{-n}|\xi|(B_\varrho(x))\ \mathrm{d}\varrho \geq \delta\log(\varrho_0/r).$$

Letting $r \to 0$, we get a contradiction to $\delta > 0$ and conclude (4.9).

By Lemma 4.8, we know $\theta_*^{n-1}(\mu,x) \geq \bar\theta\omega_{n-1}^{-1}$ for μ almost all $x \in \Omega$, hence for such x

$$\underline{D}_\mu|\xi|(x) = \liminf_{\varrho\to 0} \frac{|\xi|(B_\varrho(x))}{\mu(B_\varrho(x))} \leq$$

$$\leq \liminf_{\varrho\to 0} \varrho^{1-n}|\xi|(B_\varrho(x))\Big(\liminf_{\varrho\to 0} \varrho^{1-n}\mu(B_\varrho(x))\Big)^{-1} \leq \theta_*^{n-1}(|\xi|,x)\bar\theta^{-1}\omega_{n-1} = 0.$$

As clearly $|\xi_\varepsilon| \leq \mu_\varepsilon$, hence $|\xi| \leq \mu$, we get by differentiation theorem for measures, see [Sim] Theorem 4.7, that $|\xi| = D_\mu|\xi|\cdot\mu = 0$, in particular $\xi_\varepsilon \to \xi = 0$. Finally Proposition 4.4 implies

$$|\xi_\varepsilon| = \xi_{\varepsilon,+} + \xi_{\varepsilon,-} = -\xi_\varepsilon + 2\xi_{\varepsilon,+} \to 0.$$

///

We expect the measures μ_ε roughly to describe the position of the *transition layers* of u_ε. We incorporate more detailed geometric information by assigning a normal direction and a *generalized varifold* V_ε to μ_ε. The first variation of V_ε is determined by v_ε and the discrepancy measures.

Proposition 4.10 *We choose borel-measurable functions* $\nu_\varepsilon : \Omega \to \partial B_1(0)$ *extending* $\nabla u_\varepsilon/|\nabla u_\varepsilon|$ *on* $\nabla u_\varepsilon = 0$ *and consider the generalized varifold* $V_\varepsilon := \mu_\varepsilon \otimes \nu_\varepsilon$ *that is*

$$\int_{\Omega\times G(n,n-1)} \phi(x,S)\ \mathrm{d}V_\varepsilon(x,S) = \int_\Omega \Phi(x,\nu_\varepsilon(x))\ \mathrm{d}\mu_\varepsilon(x) \quad \text{for } \phi \in C_0^0(\Omega\times G(n,n-1)).$$

Then the first variation of V_ε *is given by*

$$(\delta V_\varepsilon)(\eta) = -\int v_\varepsilon \nabla u_\varepsilon \eta\ \mathrm{d}\mathcal{L}^n + \int \nu_\varepsilon^T D\eta\ \nu_\varepsilon\ \mathrm{d}\xi_\varepsilon \quad \text{for } \eta \in C_0^1(\Omega,\mathbb{R}^n).$$

Proof:

By definition

$$(\delta V_\varepsilon)(\eta) = \int_{\Omega\times G(n,n-1)} div_S\eta(x)\ \mathrm{d}V_\varepsilon(x,S) = \int_\Omega \Big(div\ \eta - \nu_\varepsilon^T D\eta\ \nu_\varepsilon\Big)\ \mathrm{d}\mu_\varepsilon =$$

$$= \int_\Omega \Big(div\ \eta - \nu_\varepsilon^T D\eta\ \nu_\varepsilon\Big)\Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\ \mathrm{d}\mathcal{L}^n.$$

Integration by parts yields

$$(\delta V_\varepsilon)(\eta) = -\int_\Omega \varepsilon\nabla u_\varepsilon\Big(D^2u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big)\eta\ \mathrm{d}\mathcal{L}^n - \int_\Omega \varepsilon\nabla u_\varepsilon D\eta\ \nabla u_\varepsilon^T\ \mathrm{d}\mathcal{L}^n + \int_\Omega \nu_\varepsilon^T D\eta\ \nu_\varepsilon\ \mathrm{d}\xi_\varepsilon =$$

$$= -\int_\Omega \varepsilon\nabla u_\varepsilon\Big(D^2u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big)\eta\ \mathrm{d}\mathcal{L}^n + \int_\Omega \varepsilon\nabla u_\varepsilon\eta\Delta u_\varepsilon\ \mathrm{d}\mathcal{L}^n +$$

$$+ \int_\Omega \varepsilon\nabla u_\varepsilon D^2u_\varepsilon\eta\ \mathrm{d}\mathcal{L}^n + \int_\Omega \nu_\varepsilon^T D\eta\ \nu_\varepsilon\ \mathrm{d}\xi_\varepsilon =$$

$$= \int_\Omega \Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big)\nabla u_\varepsilon\eta\ \mathrm{d}\mathcal{L}^n + \int_\Omega \nu_\varepsilon^T D\eta\ \nu_\varepsilon\ \mathrm{d}\xi_\varepsilon,$$

and the assertion follows, as $v_\varepsilon = -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)$.

///

We are now prepared to proof the rectifiability of μ.

Proof of Theorem 4.1:

As $\parallel V_\varepsilon \parallel = \mu_\varepsilon$ is locally uniformly bounded in Ω , we may assume after passing to a further subsequence

$$V_\varepsilon \to V,$$

where V is a generalized varifold in Ω . Clearly

$$\mu_V \leftarrow \mu_{V_\varepsilon} = \mu_\varepsilon \to \mu. \tag{4.10}$$

By Proposition 4.10, we conclude for $\eta \in C_0^1(\Omega, \mathbb{R}^n)$ with *supp* $\eta \subseteq U \subset\subset \Omega$ that

$$|(\delta V)(\eta)| \leftarrow |(\delta V_\varepsilon)(\eta)| \leq \Big(\int\limits_{supp\ \eta} \frac{1}{\varepsilon} v_\varepsilon^2 \Big)^{1/2} \Big(\int \eta^2 \varepsilon |\nabla u_\varepsilon|^2 \Big)^{1/2} + \int |D\eta| \, \mathrm{d}|\xi_\varepsilon|.$$

By Proposition 4.9, we see $\varepsilon |\nabla u_\varepsilon|^2 = \mu_\varepsilon - \xi_\varepsilon \to \mu$, hence

$$|(\delta V)(\eta)| \leq \alpha(U)^{1/2} \Big(\int \eta^2 \, d\mu_V \Big)^{1/2}.$$

We conclude that V has generalized mean curvature $\mathbf{H}_V \in L^2_{loc}(\mu_V)$ and

$$|\mathbf{H}_V|^2 \mu_V \leq \alpha. \tag{4.11}$$

By (4.10) and Lemma 4.8, we see $\theta_*^{n-1}(\mu_V) = \theta_*^{n-1}(\mu) > 0$ almost everywhere with respect to $\mu = \mu_V$. Then by standard rectifiability theorem, see [Sim] Theorem 42.4, V respectively $\mu = \mu_V$ are rectifiable, hence $\mathbf{H}_V = \mathbf{H}_\mu$ and by (4.10) and (4.11)

$$|\mathbf{H}_\mu|^2 \mu \leq \alpha.$$

///

5 Integrality

The task of this section is to prove that $\sigma^{-1}\mu$ is integral.

Theorem 5.1 *$\sigma^{-1}\mu$ with μ as in (2.6) is an integral $(n-1)-$varifold.*

Proof:

We have to prove that $\theta^{n-1}(\mu, x_0) \in \mathbb{N}$ for $\mu-$almost all $x_0 \in \Omega$. As we already know by Theorem 4.1 that μ is rectifiable, we may assume that $T_{x_0}\mu = \theta T$ exists with $\theta > 0$ and $T \in G(n, n-1)$. Writing $x_0 = 0$ for simplicity this means

$$\zeta_{\varrho,\#}\mu \to \theta \mathcal{H}^{n-1} \lfloor T$$

weakly as varifolds and where $\zeta_\varrho(x) := \varrho^{-1}x$. Choosing a subsequence $\varrho_k \to 0$ and $\varepsilon_k \to 0$ appropriate and small enough, we get

$$\zeta_{\varrho_k,\#}\mu_{\varepsilon_k} \to \theta \mathcal{H}^{n-1} \lfloor T$$

weakly* as Radon measures,

$$\begin{gathered}\tilde\varepsilon_k := \varepsilon_k/\varrho_k \to 0,\\ \alpha_{\varepsilon_k}(B_\varrho(0)) \le \alpha(B_{2\varrho}(0)) + \varrho_k^{n-2} \quad \text{for } \varrho_k \le \varrho \le \varrho_0,\end{gathered}$$

with $B_{2\varrho_0}(0) \subset\subset \Omega$, as $\limsup_{\varepsilon\to 0} \alpha_\varepsilon(B_\varrho(0)) \le \alpha(\overline{B_\varrho(0)})$ for fixed $0 < \varrho \le \varrho_0$.

Putting $\tilde u_{\tilde\varepsilon_k}(x) := u_{\varepsilon_k}(\varrho_k x), \tilde v_{\tilde\varepsilon_k}(x) := \varrho_k v_{\varepsilon_k}(\varrho_k x)$ for $x \in B_{\varrho_0/\varrho_k}(0)$, we see

$$\begin{gathered}-\tilde\varepsilon_k \Delta \tilde u_{\tilde\varepsilon_k} + \tfrac{1}{\tilde\varepsilon_k} W'(\tilde u_{\tilde\varepsilon_k}) =: \tilde v_{\tilde\varepsilon_k} \quad \text{in } B_{\varrho_0/\varrho_k}(0),\\ \tilde\mu_{\tilde\varepsilon_k} := \Big(\tfrac{\tilde\varepsilon_k}{2}|\nabla \tilde u_{\tilde\varepsilon_k}|^2 + \tfrac{1}{\tilde\varepsilon_k} W(u_{\tilde\varepsilon_k})\Big)\mathcal{L}^n = \zeta_{\varrho_k,\#}\mu_{\varepsilon_k} \to \theta\mathcal{H}^{n-1}\lfloor T,\\ \tilde\xi_{\tilde\varepsilon_k} := \Big(\tfrac{\tilde\varepsilon_k}{2}|\nabla \tilde u_{\tilde\varepsilon_k}|^2 - \tfrac{1}{\tilde\varepsilon_k} W(u_{\tilde\varepsilon_k})\Big)\mathcal{L}^n,\\ \tilde\alpha_{\tilde\varepsilon_k} := \tfrac{1}{\tilde\varepsilon_k}\tilde v_{\tilde\varepsilon_k}^2 \mathcal{L}^n.\end{gathered}$$

By the above assumptions we get further for $1 \le R \le \varrho_0/\varrho_k$

$$\tilde\alpha_{\tilde\varepsilon_k}(B_R(0)) = \varrho_k^{3-n}\alpha_{\varepsilon_k}(B_{R\varrho_k}(0)) \le \varrho_k^{3-n}\Big(\alpha(B_{2R\varrho_k}(0)) + \varrho_k^{n-2}\Big).$$

Assuming $\alpha(\{x_0\}) = 0$ for $n = 3$, which is true on a co-countable set of Ω , and hence for $\mu-$almost all $x_0 \in \Omega$ by Proposition 4.5 as $n \ge 2$, we get $\limsup_{k\to\infty} \tilde\alpha_{\tilde\varepsilon_k}(B_R(0)) = 0$ and

$$\alpha_{\tilde\varepsilon_k} \to 0.$$

Therefore we have reduced the theorem to the special situation of the following proposition.

///

Proposition 5.2 *Assume in (2.1) - (2.6) with* $B_4(0) \subset\subset \Omega$ *additionally that*

$$\begin{gathered}\mu = \theta\mathcal{H}^{n-1}\lfloor T \quad \textit{for some } \theta > 0, T \in G(n, n-1),\\ \alpha = 0.\end{gathered}$$

Then

$$\sigma^{-1}\theta \in \mathbb{N}.$$

□

We proceed as in [HT00] and [T02], carefully adapting their proofs to our situation with less control on the discrepancy measures. Proposition 5.4 states a kind of multilayer monotonicity, as it was already used in the proof of Allard's Integral Compactness Theorem in [All72]. Proposition 5.4 will follow by induction from the following result.

Proposition 5.3 ([HT00] Lemma 5.4)

Assume (2.1) - (2.5) and consider $X \subseteq \{0\}\times]t_1 + d, t_2 - d[\subseteq \mathbb{R}^n$ *consisting of no more than* $N \in \mathbb{N}$ *elements with* $\cup_{x\in X} B_{3R^{1-\beta}}(x) \subseteq \Omega, -\infty \le t_1 < t_2 \le \infty, 0 < \varepsilon \le d \le R \le 1/2, 0 < \beta \le 1/2$, *satisfying*

$$(\Gamma + 1)\textit{diam } X < R \quad \textit{for some } \Gamma \ge 1, \tag{5.1}$$

$$|x - y| > 3d \quad \textit{for } x \ne y \in X, \tag{5.2}$$

$$\int\limits_{B_{3R^{1-\beta}}(x)\cap[|u_\varepsilon|\geq 1]} W'(u_\varepsilon)^2 \leq \Lambda \quad \textit{for some } \Lambda < \infty, \tag{5.3}$$

$$\int\limits_d^R \varrho^{-n} \Big| \int\limits_{B_\varrho(x)\cap[y_n=t_j]} \Big((y_n - x_n)(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)) - \varepsilon\partial_n u_\varepsilon((y-x)\nabla u_\varepsilon) \Big) \, \mathrm{d}\mathcal{H}^{n-1}(y) \Big| \, \mathrm{d}\varrho$$

$$\leq \omega \quad \textit{for } j = 1, 2, \tag{5.4}$$

for some $\omega > 0$,

$$|\xi_\varepsilon|(B_\varrho(x)) + \int\limits_{B_\varrho(x)} \varepsilon|\nabla u_\varepsilon|^2 \sqrt{1 - \nu_{\varepsilon,n}^2} \leq \omega\varrho^{n-1} \quad \textit{for } d \leq \varrho \leq R, \tag{5.5}$$

where $\nu_\varepsilon = \nabla u_\varepsilon / |\nabla u_\varepsilon|$ *for* $\nabla u_\varepsilon \neq 0$,

$$\begin{gathered} \alpha_\varepsilon(B_\varrho(x)) \leq \Lambda\varrho^{\beta_0} \quad \textit{for } 3d^{1-\beta} \leq \varrho \leq 3R^{1-\beta}, n = 3, \\ \alpha_\varepsilon(B_{3R^{1-\beta}}(x)) \leq \Lambda \quad \textit{for } n = 2, \end{gathered} \tag{5.6}$$

for some $0 < \beta_0 \leq 1/2$,

$$R^{1-n}\mu_\varepsilon(B_{2R}(x)) \leq \Lambda \tag{5.7}$$

for all $x \in X$.

Then putting $S_t^{t'} := [t < y_n < t']$

$$d^{1-n}\mu_\varepsilon(B_d(x)) \leq$$

$$\leq R^{1-n}\mu_\varepsilon(B_R(x) \cap S_{t_1}^{t_2}) + C_{\beta_0,\beta}\Big((1+\Lambda)R^{c_0\beta_0} + \omega\Big) \quad \textit{for all } x \in X. \tag{5.8}$$

Further if X *consists of more than one point, there exists* $t_3 \in]t_1, t_2[$ *such that*

$$|x_n - t_3| > d \quad \textit{for all } x \in X, \tag{5.9}$$

$$\int\limits_d^{\tilde{R}} \varrho^{-n} \int\limits_{B_\varrho(x)\cap[y_n=t_3]} \Big| (y_n - x_n)(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)) - \varepsilon\partial_n u_\varepsilon((y-x)\nabla u_\varepsilon) \Big| \, \mathrm{d}\mathcal{H}^{n-1}(y) \, \mathrm{d}\varrho$$

$$\leq 6N^2\Gamma\omega \quad \textit{for all } x \in X, \tag{5.10}$$

where $\tilde{R} = \Gamma$ *diam* X *, and denoting* $X_+ := X \cap S_{t_1}^{t_3}, X_- := X \cap S_{t_3}^{t_2}$ *both are non-empty and*

$$\tilde{R}^{1-n}\Big(\mu_\varepsilon(\cup_{x\in X_-} B_{\tilde{R}}(x) \cap S_{t_1}^{t_3}) + \mu_\varepsilon(\cup_{x\in X_+} B_{\tilde{R}}(x) \cap S_{t_3}^{t_2})\Big) \leq$$

$$\leq (1 + 1/\Gamma)^{n-1} R^{1-n}\mu_\varepsilon(\cup_{x\in X} B_R(x) \cap S_{t_1}^{t_2}) + C_{\beta_0,\beta}\Big((1+\Lambda)R^{c_0\beta_0} + \omega\Big). \tag{5.11}$$

Proof:

We derive a weighted monotonicity formula from (4.1) choosing $\eta(y) := (y-x)\phi_\delta(|y-x|)\chi(y_n)$, where $\phi'_\delta \leq 0$,

$$\phi_\delta = \begin{cases} 1 & \text{on } [0,\varrho], \\ 0 & \text{on } [\varrho+\delta,\infty[\end{cases} \qquad \text{and} \qquad \begin{aligned} \chi_\delta &= 1 \quad \text{on } [t_1+\delta, t_2-\delta], \\ \chi_\delta &= 0 \quad \text{on }]-\infty,t_1] \cup [t_2,\infty[, \\ \chi'_\delta &\geq 0 \quad \text{on } [t_1,t_1+\delta], \\ \chi'_\delta &\leq 0 \quad \text{on } [t_2-\delta,t_2]. \end{aligned}$$

As in Lemma 4.2 we get then for $x \in X, 0 < \varrho \leq R$ that

$$\int \Big(|y-x|\phi'_\delta\chi_\delta + n\phi_\delta\chi_\delta \big) \, \mathrm{d}\mu_\varepsilon - \int \varepsilon \frac{\phi'_\delta\chi_\delta}{|y-x|}((y-x)\nabla u_\varepsilon)^2 - \int \varepsilon|\nabla u_\varepsilon|^2\phi_\delta\chi_\delta =$$

$$= -\int v_\varepsilon((y-x)\nabla u_\varepsilon)\phi_\delta\chi_\delta - \int (y_n-x_n)\phi_\delta\chi'_\delta \, \mathrm{d}\mu_\varepsilon + \int \varepsilon\partial_n u_\varepsilon((y-x)\nabla u_\varepsilon)\phi_\delta\chi'_\delta.$$

Letting $\delta \to 0$ and multiplying by ϱ^{-n} yields

$$\frac{d}{d\varrho}\Big(\varrho^{1-n}\mu_\varepsilon(B_\varrho(x)\cap S_{t_1}^{t_2})\Big) = -\frac{1}{\varrho^n}\xi_\varepsilon(B_\varrho(x)\cap S_{t_1}^{t_2}) +$$

$$+\frac{1}{\varrho^{n+1}} \int\limits_{\partial B_\varrho(x)\cap S_{t_1}^{t_2}} \varepsilon((y-x)\nabla u_\varepsilon)^2 \, \mathrm{d}\mathcal{H}^{n-1}(y) + \frac{1}{\varrho^n} \int\limits_{B_\varrho(x)\cap S_{t_1}^{t_2}} v_\varepsilon((y-x)\nabla u_\varepsilon) \, \mathrm{d}y +$$

$$+\sum_{j=1}^{2}(-1)^j\frac{1}{\varrho^n} \int\limits_{B_\varrho(x)\cap[y_n=t_j]} \Big((y_n-x_n)(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)) - \varepsilon\partial_n u_\varepsilon((y-x)\nabla u_\varepsilon) \Big) \, \mathrm{d}\mathcal{H}^{n-1}(y) \, \mathrm{d}\varrho.$$

From Proposition 4.7 and (5.3), (5.6), we get

$$\varrho^{1-n}\mu_\varepsilon(B_\varrho(x)) \leq C_{\beta_0,\beta}(1+\Lambda) \quad \text{for } d \leq \varrho \leq 2R. \tag{5.12}$$

Applying Lemma 3.1, Proposition 3.5 as in Proposition 4.6 (4.2) with $\delta = \varrho^\gamma, \varepsilon \leq \varrho \leq R \leq 1$ for $0 < \gamma < 1/M < 1/2$ and observing $t_1 < x_n < t_2$, we get using (5.3), (5.6), (5.12)

$$\varrho^{-n}\xi_{\varepsilon,+}(B_\varrho(x)) \leq C_{\beta_0,\beta}(1+\Lambda)\varrho^{-1+\gamma} + C_\beta\Lambda\varepsilon^2\varrho^{-M\gamma-3+\beta_0(1-\beta)} + \varepsilon\varrho^{\gamma-2}(1+\Lambda).$$

Integrating from $\varrho \geq d \geq \varepsilon$ to $R \leq 1$, we obtain using Proposition 4.3 with $(y\nabla u_\varepsilon)$ replaced by $(y\nabla u_\varepsilon)\chi_{S_{t_1}^{t_2}}$ for $\gamma = c_0\beta_0$,

$$R^{1-n}\mu_\varepsilon(B_R(x)\cap S_{t_1}^{t_2}) \geq \varrho^{1-n}\mu_\varepsilon(B_\varrho(x)\cap S_{t_1}^{t_2}) +$$

$$-C_{\beta_0,\beta}(1+\Lambda)R^{c_0\beta_0} - C_\beta\Lambda\varepsilon^{\beta_0(1-\beta)-M\gamma} - C(1+\Lambda)\varepsilon^\gamma - C\alpha(B_R(x)) - 2\omega.$$

For $c_0 = 1/(4M) < 1/8$ and $\gamma = c_0\beta_0 = \beta_0/(4M) < 1/M$, hence $\beta_0(1-\beta) - M\gamma \geq \beta_0/4$, this yields by (5.6)

$$R^{1-n}\mu_\varepsilon(B_R(x)\cap S_{t_1}^{t_2}) \geq \varrho^{1-n}\mu_\varepsilon(B_\varrho(x)\cap S_{t_1}^{t_2})$$

$$-C_{\beta_0,\beta}(1+\Lambda)R^{c_0\beta_0} - C_\beta(1+\Lambda)\varepsilon^{c_0\beta_0} - C\Lambda R^{\beta_0} - 2\omega \quad \text{for } d \leq \varrho \leq R, x \in X. \tag{5.13}$$

Observing that $B_d(x) \subseteq S_{t_1}^{t_2}$, we obtain (5.8).

If X consists of more than one point, we can choose $x_\pm \in X$ such that $x_{+,n} - x_{-,n} > diam\ X/N$ and that there is no element of X in $\{0\}\times]x_{-,n}, x_{+,n}[$. Let $\tilde{t}_1 := x_{-,n} + (x_{+,n} - x_{-,n})/3$ and $\tilde{t}_2 := x_{+,n} - (x_{+,n} - x_{-,n})/3$. For $x \in X, y \in B_\varrho(x), d \le \varrho \le \tilde{R}$ we calculate

$$|(y_n - x_n)(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)) - \varepsilon\partial_n u_\varepsilon((y-x)\nabla u_\varepsilon)| \le$$

$$\le \left|\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\right|\varrho + \varepsilon|\nabla u_\varepsilon|^2\ |(y_n - x_n) - \nu_{\varepsilon,n}(y-x)\cdot\nu_\varepsilon|$$

$$\le \left|\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\right|\varrho + \varepsilon|\nabla u_\varepsilon|^2\varrho(1 - \nu_{\varepsilon,n}^2 + \sqrt{1-\nu_{\varepsilon,n}^2})$$

and estimate by (5.5)

$$\int_{\tilde{t}_1}^{\tilde{t}_2}\int_d^{\tilde{R}} \varrho^{-n} \int_{B_\varrho(x)\cap[y_n=t]} \left|(y_n - x_n)(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)) - \varepsilon\partial_n u_\varepsilon((y-x)\nabla u_\varepsilon)\right| \mathrm{d}\mathcal{H}^{n-1}(y)\ \mathrm{d}\varrho\ \mathrm{d}t =$$

$$= \int_d^{\tilde{R}} \varrho^{-n} \int_{B_\varrho(x)\cap S_{\tilde{t}_1}^{\tilde{t}_2}} \left|(y_n - x_n)(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)) - \varepsilon\partial_n u_\varepsilon((y-x)\nabla u_\varepsilon)\right| \mathrm{d}y\ \mathrm{d}\varrho \le$$

$$\le \int_d^{\tilde{R}} \varrho^{1-n} \int_{B_\varrho(x)} \Big(\left|\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\right| + \varepsilon|\nabla u_\varepsilon|^2(1 - \nu_{\varepsilon,n}^2 + \sqrt{1-\nu_{\varepsilon,n}^2})\Big)\ \mathrm{d}y\ \mathrm{d}\varrho \le 2\tilde{R}\omega.$$

Therefore there exists $t_3 \in]\tilde{t}_1, \tilde{t}_2[$ satisfying

$$\int_d^{\tilde{R}} \varrho^{-n} \int_{B_\varrho(x)\cap[y_n=t_3]} \left|(y_n - x_n)(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)) - \varepsilon\partial_n u_\varepsilon((y-x)\nabla u_\varepsilon)\right| \mathrm{d}\mathcal{H}^{n-1}(y)\ \mathrm{d}\varrho \le$$

$$\le 2N\tilde{R}\omega/(\tilde{t}_2 - \tilde{t}_1) \le 6N^2\varGamma\omega \quad \text{for all } x \in X,$$

as $\tilde{t}_2 - \tilde{t}_1 \ge diam\ X/(3N) = \tilde{R}/(3N\varGamma)$, hence conclude (5.10). As

$$t_3 - x_{-,n} \ge \tilde{t}_1 - x_{-,n} = (x_{+,n} - x_{-,n})/3 > d$$

and likewise

$$x_{+,n} - t_3 \ge x_{+,n} - \tilde{t}_2 = (x_{+,n} - x_{-,n})/3 > d$$

by (5.2), we get (5.9).

We put $X_+ := \{x \in X : x_n \ge t_3$, $X_- := \{x \in X : x_n < t_3$. Clearly $X_\pm \ne \emptyset$, as $x_\pm \in X_\pm$, and

$$(\cup_{x\in X_-} B_{\tilde{R}}(x) \cap S_{t_1}^{t_3}) + (\cup_{x\in X_+} B_{\tilde{R}}(x) \cap S_{t_3}^{t_2}) \subseteq B_{\tilde{R}+diam\ X}(x_0) \cap S_{t_1}^{t_2}$$

for any $x_0 \in X$. Observing $6d \le \tilde{R} + diam\ X = (\varGamma+1) diam\ X < R$ by (5.1), (5.2), as X has at least two elements, this yields (5.13)

$$\tilde{R}^{1-n}\Big(\mu_\varepsilon(\cup_{x\in X_-} B_{\tilde{R}}(x) \cap S_{t_1}^{t_3}) + \mu_\varepsilon(\cup_{x\in X_+} B_{\tilde{R}}(x) \cap S_{t_3}^{t_2})\Big) \le$$

$$\le \tilde{R}^{1-n}\mu_\varepsilon(B_{\tilde{R}+diam\ X}(x_0) \cap S_{t_1}^{t_2}) =$$

$$= (1+1/\Gamma)^{n-1}(\tilde{R} + diam\ X)^{1-n}\mu_\varepsilon(B_{\tilde{R}+diam\ X}(x_0) \cap S_{t_1}^{t_2}) \leq$$

$$\leq (1+1/\Gamma)^{n-1}\Big(R^{1-n}\mu_\varepsilon(B_R(x_0) \cap S_{t_1}^{t_2}) + C_{\beta_0,\beta}\Big((1+\Lambda)R^{c_0\beta_0} + \omega\Big)\Big) \leq$$

$$\leq (1+1/\Gamma)^{n-1}R^{1-n}\mu_\varepsilon(\cup_{x\in X} B_R(x) \cap S_{t_1}^{t_2}) + C_{\beta_0,\beta}\Big((1+\Lambda)R^{c_0\beta_0} + \omega\Big),$$

which is (5.11).

///

Starting with $t_1 = -\infty, t_2 = \infty$, and choosing Γ large and ω, ε small, we inductively use Proposition 5.3 to separate each element of X in a horizontal strip and get the following *multilayer monotonicity.*

Proposition 5.4 ([HT00] Lemma 5.5)

For $N \in \mathbb{N}, \delta > 0, 0 < \beta_0, \beta \leq 1/2, \Lambda < \infty$ *there exists* $\omega = \omega(N, \delta, \beta_0, \beta, \Lambda) > 0$ *satisfying:*

Assume (2.1) - (2.5) and consider $X \subseteq \{0\} \times \mathbb{R} \subseteq \mathbb{R}^n$ *consisting of no more than* $N \in \mathbb{N}$ *elements with* $\cup_{x\in X} B_{3R^{1-\beta}}(x) \subseteq \Omega, 0 < \varepsilon \leq d \leq R \leq \omega$ *, satisfying*

$$diam\ X < \omega R \tag{5.14}$$

$$|x-y| > 3d \quad \text{for } x \neq y \in X, \tag{5.15}$$

$$\int_{B_{3R^{1-\beta}}(x)\cap[|u_\varepsilon|\geq 1]} W'(u_\varepsilon)^2 \leq \Lambda, \tag{5.16}$$

$$|\xi_\varepsilon|(B_\varrho(x)) + \int_{B_\varrho(x)} \varepsilon|\nabla u_\varepsilon|^2\sqrt{1-\nu_{\varepsilon,n}^2} \leq \omega\varrho^{n-1} \quad \text{for } d \leq \varrho \leq R, \tag{5.17}$$

where $\nu_\varepsilon = \nabla u_\varepsilon/|\nabla u_\varepsilon|$ *for* $\nabla u_\varepsilon \neq 0$ *,*

$$\begin{gathered} \alpha_\varepsilon(B_\varrho(x)) \leq \Lambda\varrho^{\beta_0} \quad \text{for } 3d^{1-\beta} \leq \varrho \leq 3R^{1-\beta}, n=3, \\ \alpha_\varepsilon(B_{3R^{1-\beta}}(x)) \leq \Lambda \quad \text{for } n=2. \end{gathered} \tag{5.18}$$

$$\mu_\varepsilon(B_{2R}(x)) \leq \Lambda. \tag{5.19}$$

for $c_0 \ll 1$ *universal small enough, and all* $x \in X$ *.*

Then

$$\sum_{x\in X} d^{1-n}\mu_\varepsilon(B_d(x)) \leq (1+\delta)R^{1-n}\mu_\varepsilon(\cup_{x\in X} B_R(x)) + \delta. \tag{5.20}$$

□

The next proposition allows, under the assumption of small discrepancy and tilt-excess, to identify transition layers and a definite amount of energy within such layers.

Proposition 5.5 ([HT00] Lemma 5.6)

For $\tau, \delta > 0, \Lambda < \infty$ *there exists* $\omega = \omega(\delta, \tau, \Lambda) > 0, 1 < L = L(\delta, \tau) < \infty$ *satisfying:*
Assume (2.1) - (2.5) with $\Omega = B_{4L\varepsilon}(0)$ *and*

$$|u_\varepsilon(0)| \leq 1 - \tau, \tag{5.21}$$

$$|\xi_\varepsilon|(B_{4L\varepsilon}(0)) + \int_{B_{4L\varepsilon}(0)} \varepsilon |\nabla u_\varepsilon|^2 \sqrt{1 - \nu_{\varepsilon,n}^2} \leq \omega (4L\varepsilon)^{n-1}. \tag{5.22}$$

where $\nu_\varepsilon = \nabla u_\varepsilon / |\nabla u_\varepsilon|$ *for* $\nabla u_\varepsilon \neq 0$ *,*

$$\alpha_\varepsilon(B_{4L\varepsilon}(0)) \leq \Lambda (4L\varepsilon)^{n-3} \tag{5.23}$$

$$\mu_\varepsilon(B_{4L\varepsilon}(0)) \leq \Lambda (4L\varepsilon)^{n-1}. \tag{5.24}$$

Then

$$|u(0,t)| \geq 1 - \tau/2 \quad \textit{for all } L\varepsilon \leq |t| \leq 3L\varepsilon, \tag{5.25}$$

$$\Big| \frac{1}{\omega_{n-1}(L\varepsilon)^{n-1}} \mu_\varepsilon(B_{L\varepsilon}(0)) - \sigma \Big| \leq \delta. \tag{5.26}$$

$$\Big| \int_{-L\varepsilon}^{L\varepsilon} \frac{1}{\varepsilon} W(u_\varepsilon(0,t)) \, \mathrm{d}t - \sigma/2 \Big| \leq \delta. \tag{5.27}$$

Proof:
We may assume $\varepsilon = 1$ after rescaling. By (5.23), we see $\int_{B_{4L}(0)} v^2 \leq \Lambda (4L)^{n-3}$.

We consider the solution of the ordinary differential equation

$$q_0'(t) = \sqrt{2W(q_0(t))} \quad \text{for } t \in \mathbb{R},$$
$$q_0(0) = u(0).$$

We note that $\lim_{t \to \pm\infty} q_0(t) = \pm 1$ and

$$\int_{-\infty}^{\infty} \frac{1}{2} |q_0'(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{\infty} W(q_0(t)) \, \mathrm{d}t = \frac{1}{2} \int_{-\infty}^{\infty} q_0'(t) \sqrt{2W(q_0(t))} \, \mathrm{d}t = \frac{1}{2} \int_{-1}^{1} \sqrt{2W} = \sigma/2.$$

On $\mathbb{R}^n$, we write $q(x) := q_0(x_n)$ and choose $L > 1$ large enough depending on $\tau, \delta > 0$ such that

$$\begin{gathered} |q(0,t)| \geq 1 - \tau/3 \quad \text{for all } L \leq |t| \leq 3L, \\ \Big| \frac{1}{\omega_{n-1} L^{n-1}} \int_{B_L(0)} \Big(\frac{1}{2} |\nabla q|^2 + W(q) \Big) - \sigma \Big| \leq \delta/2, \\ \Big| \int_{-L}^{L} W(q(0,t)) \, \mathrm{d}t - \sigma/2 \Big| \leq \delta/2, \end{gathered} \tag{5.28}$$

whenever $|q(0)| \leq 1 - \tau$.

For any $x \in B_{3L}(0)$, we get from the monotonicity formula, Lemma 4.2, Lemma 4.3, (5.22) and (5.23) that

$$L^{1-n} \mu(B_L(x)) - \mu(B_1(x)) \geq$$

$$\geq -\int_1^L \varrho^{-n}\omega(4L)^{n-1}\ \mathrm{d}\varrho - \alpha(B_L(x)) \geq -C\omega L^{n-1} - CL^{n-3} \geq -C,$$

if $\omega L^{n-1} \leq 1$, and by (5.24)

$$\mu(B_1(x)) \leq C(1+\Lambda).$$

By Proposition 3.3, we get

$$\| u \|_{L^\infty(B_{1/2}(x))} \leq 1 + C \int_{B_1(x)} (|u|-1)_+ + C \int_{B_1(x)} |v|^2 \leq$$

$$\leq C(1+\mu(B_1(x)) + \alpha(B_1(x)) \leq C(1+\Lambda).$$

As $-\Delta u + W'(u) = v$, we get by standard elliptic estimates and covering

$$\| u \|_{W^{2,2}(B_{3L}(0))} \leq C(\Lambda, L). \tag{5.29}$$

If there is no $\omega > 0$ such that (5.25) and (5.26) are satisfied, there are $\omega_j \to 0$ and u_j, v_j as above, but not satisfying all (5.25) - (5.27). By (5.29), we get after passing to a suitable subsequence that $u_j \to u$ weakly in $W^{2,2}(B_{3L}(0))$ and $v_j \to v$ weakly in $L^2(B_{3L}(0))$. By the Sobolev embedding $W^{2,2}(B_L(0)) \hookrightarrow C^0(B_L(0))$, as $n \leq 3$, we see $u_j \to u$ uniformly on $B_{3L}(0)$.

Writing $\nabla = (\nabla', \partial_n)$ for the gradient, we get from (5.22)

$$\int_{B_{3L}(0)} \left|\frac{1}{2}|\nabla u|^2 - W(u)\right| \leq \liminf_{j\to\infty} \int_{B_{3L}(0)} \left|\frac{1}{2}|\nabla u_j|^2 - W(u_j)\right| \leq \liminf_{j\to\infty} |\xi_j|(B_{3L}(0))| = 0$$

and

$$\int_{B_{3L}(0)} |\nabla' u| \leq \liminf_{j\to\infty} \int_{B_{3L}(0)} |\nabla' u_j| \leq \liminf_{j\to\infty} \int_{B_{3L}(0)} |\nabla u_j|^2 \sqrt{1-\nu_{j,n}^2} = 0,$$

where $\nu_j = \nabla u_j / |\nabla u_j|$ for $\nabla u_j \neq 0$. Therefore $|\nabla u|^2 = 2W(u)$ and $u(y,t) = u_0(t)$ for some $u_0 \in W^{2,2}(]-3L, 3L[) \hookrightarrow C^{1,1/2}(]-3L,3L[)$ and $|u_0'| = \sqrt{2W(u_0)}$. As $|u_0(0)| \leq 1-\tau$ by uniform convergence, we see $|u_0| < 1$ and $|u_0'| > 0$. After reflection $(y, x_n) \mapsto (y, -x_n)$, if necessary, which does neither affect the assumptions nor the conclusions of the proposition, we may assume $u_0' > 0$, hence $u_0' = \sqrt{2W(u_0)}$. This yields $u_0 = q_0$ and $u = q$. By $u_j \to u = q$ uniform and strong in $W^{1,2}(B_{3L}(0))$, we conclude by (5.28) that u_j satisfies (5.25) - (5.27) for j large enough which is a contradiction, and the proposition follows.

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We are now ready to finish the proof of the integrality of μ.

Proof of Proposition 5.2:

Let $N \in \mathbb{N}$ be the smallest integer with $N > \sigma^{-1}\theta$ and $0 < \delta \leq 1$ be small. We assume $T = \mathbb{R}^{n-1} \times \{0\}$ and let $\pi : \mathbb{R}^n \to T$ be the orthogonal projection. In the proof of Theorem 4.1, we have seen that the limit of $V_\varepsilon = \mu_\varepsilon \otimes \nu_\varepsilon \to V$ is rectifiable and $\mu_V = \mu$. Therefore $V = \mathcal{H}^{n-1} \lfloor T \otimes \delta_T$ and

$$\lim_{\varepsilon\to 0} \int_{B_4(0)} \varepsilon |\nabla u_\varepsilon|^2 \sqrt{1-\nu_{\varepsilon,n}^2} = 0. \tag{5.30}$$

By Proposition 3.4 and 3.6, we can fix $\tau > 0$ such that for $\varepsilon > 0$ small enough

$$\int\limits_{[|u_\varepsilon| \geq 1-\tau] \cap B_4(0)} \frac{1}{\varepsilon} W'(u_\varepsilon)^2 \leq \delta, \tag{5.31}$$

in particular by Proposition 4.9 for $\varepsilon \leq \varepsilon(\delta)$ small enough

$$\mu_\varepsilon\Big([|u_\varepsilon| \geq 1-\tau] \cap B_4(0)\Big) \leq |\xi_\varepsilon|(B_4(0)) + 2 \int\limits_{[|u_\varepsilon| \geq 1-\tau] \cap B_4(0)} \frac{1}{\varepsilon} W(u_\varepsilon) \leq 3\delta. \tag{5.32}$$

Next we choose $0 < \omega \leq \omega(N, \delta, 1/2, 1/2, C), \omega(\delta, \tau, C) \leq 1, L = L(\delta, \tau), \beta_0 = \beta = 1/2$ as in Proposition 5.4 and 5.5, where C is the constant in (2.5) corresponding to $\Omega = B_4(0)$. We define

$$\begin{aligned} A_\varepsilon := \{ x \in B_1(0) \mid & |u_\varepsilon(x)| \leq 1 - \tau, \\ & \forall \varepsilon \leq \varrho \leq 3 : |\xi_\varepsilon|(B_\varrho(x)) + \textstyle\int_{B_\varrho(x)} \varepsilon |\nabla u_\varepsilon|^2 \sqrt{1 - \nu_{\varepsilon,n}^2} \leq \omega \varrho^{n-1}, \\ & \alpha_\varepsilon(B_\varrho(x)) \leq \omega \varrho^{\beta_0}. \ \} \end{aligned}$$

By Besicovitch's covering theorem, we can cover $[|u_\varepsilon| \leq 1-\tau] - A_\varepsilon$ with countably many closed balls with bounded overlap

$$[|u_\varepsilon| \leq 1 - \tau] - A_\varepsilon \subseteq \cup_{i=1}^\infty \overline{B_{\varrho_i}(x_i)}$$

with $\varepsilon \leq \varrho_i \leq 3$

$$|\xi_\varepsilon|(B_{\varrho_i}(x_i)) + \int_{B_{\varrho_i}(x_i)} \varepsilon |\nabla u_\varepsilon|^2 \sqrt{1 - \nu_{\varepsilon,n}^2} \geq \omega \varrho_i^{n-1}$$

or

$$\alpha_\varepsilon(B_{\varrho_i}(x_i)) \geq \omega \varrho_i^{\beta_0}.$$

When ε is such small that $\alpha_\varepsilon(B_4(0)) \ll \omega$, we may additionally assume

$$\alpha_\varepsilon(B_\varrho(x_i)) \leq \omega \varrho^{\beta_0} \quad \text{for } \varrho_i \leq \varrho \leq 3.$$

From Proposition 4.7 and (5.31), we get

$$\mu_\varepsilon(\overline{B_{\varrho_i}(x_i)}) \leq C \varrho_i^{n-1}.$$

As the overlap is bounded and by (5.32), we obtain

$$\mu_\varepsilon(B_1(0) - A_\varepsilon) \leq 3\delta + \sum_{i=1}^\infty C \varrho_i^{n-1} \leq$$

$$\leq 3\delta + C\omega^{-1}\Big(|\xi_\varepsilon|(B_4(0)) + \int\limits_{B_4(0)} \varepsilon |\nabla u_\varepsilon|^2 \sqrt{1 - \nu_{\varepsilon,n}^2} + \alpha_\varepsilon(B_4(0))\Big) \leq 4\delta \tag{5.33}$$

for $\varepsilon \leq \varepsilon(\delta)$ small enough using Proposition 4.9, (5.30) and $\alpha = 0$.

We see by Proposition 5.4 for $N = 1$ and Proposition 5.5

$$R^{1-n} \mu_\varepsilon(B_R(x)) \geq \sigma \omega_{n-1} - C\delta \quad \text{for } L\varepsilon \leq R \leq \omega, x \in A_\varepsilon. \tag{5.34}$$

As $\mu_\varepsilon(\Omega - \{|x_n| \leq \zeta\}) \to 0$ for $\zeta > 0$, we see for δ small enough

$$A_\varepsilon \subseteq \{|x_n| \leq \zeta_\varepsilon\} \tag{5.35}$$

with $\zeta(\varepsilon) \to 0$ for $\varepsilon \to 0$.

Next consider $y \in B_1(0) \cap T$ and a maximal subset $X = \{y\} \times \{t_1 < \ldots < t_K\} \subseteq A_\varepsilon \cap \pi^{-1}(y)$ with $|x - x'| \geq 3L\varepsilon$ for $x \neq x' \in X$. If $K \geq N$, we conclude by Proposition 5.4 with $d = L\varepsilon, R = \omega$ and (5.34) observing $diam\ X \leq 2\zeta(\varepsilon)$ that

$$N\sigma\omega_{n-1} - CN\delta \leq (1+\delta)R^{1-n}\mu_\varepsilon(B_{R+\zeta_\varepsilon}(y)) + \delta.$$

As

$$\limsup_{\varepsilon\to 0} R^{1-n}\mu_\varepsilon(B_{R+\zeta_\varepsilon}(y)) \leq R^{1-n}\mu(\overline{B_R(y)}) = \theta\omega_{n-1}$$

and $N\sigma > \theta$, this leads to a contadiction for ε, δ small enough, and we conclude

$$K \leq N - 1. \tag{5.36}$$

As X is maximal, we infer from Proposition 5.5 (5.25)

$$A_\varepsilon \cap \pi^{-1}(y) \subseteq \{y\} \times \cup_{k=1}^K]t_k - L\varepsilon, t_k + L\varepsilon[.$$

and from (5.27)

$$\int_{t_k - L\varepsilon}^{t_k + L\varepsilon} \frac{1}{\varepsilon} W(u_\varepsilon(y,t))\ \mathrm{d}t \leq \sigma/2 + \delta \quad \text{for } k = 1, \ldots, K,$$

hence by (5.36)

$$\int_{\pi^{-1}(y)\cap A_\varepsilon} \frac{1}{\varepsilon} W(u_\varepsilon)\ \mathrm{d}\mathcal{H}^1 \leq (N-1)\sigma/2 + (N-1)\delta. \tag{5.37}$$

This yields

$$\int_{B_1(0)\cap A_\varepsilon} \frac{1}{\varepsilon} W(u_\varepsilon)\ \mathrm{d}\mathcal{L}^n \leq \int_{B_1(0)\cap T} \int_{\pi^{-1}(y)\cap A_\varepsilon} \frac{1}{\varepsilon} W(u_\varepsilon)\ \mathrm{d}\mathcal{H}^1\ \mathrm{d}\mathcal{L}^{n-1} \leq (N-1)\sigma\omega_{n-1}/2 + C\delta,$$

hence by (5.33)

$$\mu_\varepsilon(B_1(0)) \leq 2 \int_{B_1(0)\cap A_\varepsilon} \frac{1}{\varepsilon} W(u_\varepsilon)\ \mathrm{d}\mathcal{L}^n + |\xi_\varepsilon(B_1(0))| + \mu_\varepsilon(B_1(0) - A_\varepsilon) \leq$$

$$\leq (N-1)\sigma\omega_{n-1} + C\delta.$$

As $\liminf_{\varepsilon\to 0} \mu_\varepsilon(B_1(0)) \to \mu(B_1(0)) = \theta\omega_{n-1}$, we conclude

$$\theta \leq (N-1)\sigma + C\delta.$$

As $\delta > 0$ was arbitrary and $N - 1 \leq \sigma^{-1}\theta, \theta > 0$, we arrive at

$$\sigma^{-1}\theta = N - 1 \in \mathbb{N},$$

and the proposition is proved.

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Acknowledgements The second author expresses his thorough thanks to Giovanni Bellettini for a discussion during his stay at Centro De Giorgi in Pisa in November 2003 which encouraged and stimulated him a lot to work on this De Giorgi conjecture.

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