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ON A MONGE-AMPÈRE EQUATION ARISING IN GEOMETRIC OPTICS

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Abstract

In this paper we study a Monge-Ampère equation arising in geometric optics. We will establish the a priori estimates and derive the existence of solutions by the continuity method. We also give a Legendre-type transformation for this equation.

1. Introduction

We consider here an equation of Monge-Ampère type which arises in geometric optics. Suppose a point source of light is located at the origin $O \in I\!R^3$ and let Γ be a closed surface which is star-shaped with respect to the origin. If we identify each direction of the ray with a point on S^2, and the ray of the light reflects according to geometric optics, then the direction of the reflection defines a point on S^2. Hence we obtain a map from S^2 to S^2. In [26], as a part of Problem 21, Yau asked: "How much information does this map tell us about the surface?" Let Γ be represented as a graph over the unit sphere S^2, $\Gamma = \{x \cdot \rho(x); \;\; x \in S^2\}$. Let $\gamma(x)$ denote the unit outer normal of Γ at $x \cdot \rho(x)$, and $y = T(x) = T_\rho(x)$ the direction of the light reflected by Γ. Here we regard a unit vector as a point on S^2. By the reflection law we have

$$y = x - 2\langle x, \gamma\rangle\gamma.$$

Let $f(x)$ denote the intensity of the source O, and $g(y)$ the distribution of the directions of the reflected light on S^2. Both f and g are non-negative and measurable. Suppose no energy is lost in reflection, and

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T is a diffeomorphism from S^2 to itself. Then by the energy conservation, the Jocobian of $T(x)$ is equal to $f(x)/g(T(x))$, which leads to the equation

$$\frac{\det(\nabla_{ij}u + (u-\eta)e_{ij})}{\eta^2 \det(e_{ij})} = f(x)/g(T(x)), \tag{1.1}$$

where $u = 1/\rho$, ∇_{ij} denotes the covarient derivatives on S^2, e is the metric on S^2 and $\eta = (|\nabla u|^2 + u^2)/2u$. For the derivation of the equation we refer the reader to [22], and also [13] and [25].

The problem is closely related to reflector antenna design in engineering. In applications one is usually required to solve the equation (1.1) subject to the second boundary condition:

$$T(\Omega) = D, \tag{1.2}$$

where both Ω and D are domains on S^2 prescribed in advance. (1.2) means that the directions of the reflected light cover the domain D with energy distribution $g(y)$. For the background of the problem we refer the reader to [25].

The problem (1.1), (1.2) has been studied by both engineers and mathematicians; see [13] for some historical remarks. In the last two decades this problem has been studied by [25], [1], [16], [14], [15], [11], and [24] (and the references therein). A partial differential equation was derived in [1] by making use of complex analysis, and later rederived in [16] by means of geometric analysis. The existence and uniqueness of radial solutions to the equation were also obtained in [16], and were extended in [14] to non-radial f and g if they are small perturbations of radial ones. But compared with (1.1), the equation in [24] and [16] is indirect to the reflecting surface Γ. A transformation was made there and the phase function in their equation is actually defined on the domain D. The question of reconstructing the reflecting surface Γ from their phase function is by no means obvious, and it has been discussed by Oliker [15] under various assumptions; see also [13]. The uniqueness of smooth solutions has been obtained by Marder in [11].

The general existence and uniqueness of generalized solutions to (1.1) (1.2) were recently obtained by Wang [22]. It is proved in [22] that there exists a generalized solution of (1.1), unique up to positive constant multiples, so that,

$$\overline{D} \subset \overline{T(\Omega)}, \qquad |\{x \in \Omega, T(x) \not\subset D\}| = 0,$$

In [22] by example it is also shown that the solution may fail to be C^1 smooth even if both Ω and D are convex and all known data are C^∞ smooth. But if $\overline{T(\Omega)} = \overline{D}$, the solution turns out to be smooth.

We mention that in [3], Caffarelli and Oliker obtained the existence of generalized solutions to (1.1) for closed surface. As they noted, the problem has a remarkable resemblance with Minkowski problem. The approach in [3] is similar to that by Alexandrov and Pogorelov in solving the Minkowski problem. Instead of convex polyhedra approximations like in classical cases, Caffarelli and Oliker used approximations of co-focal paraboloids of revolution, which suits perfectly to the problem. But, the regularity part for the generalized solutions obtained in [3] was left open, since the a priori estimates of second derivatives for solutions were missing.

In this paper, we establish the existence, uniqueness and regularity of the solution to (1.1). Our approach is the continuity mothed, which was used by Nirenberg in [12] and Cheng-Yau in [4] for solving Minkowski and Weyl problems. The key step here is to establish a priori estimates for solutions to (1.1) up to the second order derivatives. In fact, we will work on equation (1.1) in any dimenssion great than 1 and will obtain LOCAL C^2 a priori estimates for general degenerate equations. There have been extensive works on degenerate Monge-Ampère equations recently (see [2], [10], [21], [21], [7] and references therein). In general, one cannot expect to obtain local C^2 a priori estimates for degenerate Monge-Ampère equations. Equation (1.1) is very special in this aspect. We note here that equation (1.1) is similar to the equation related to Alexandrov problem (see [9]). The main difference is that (1.1) involves η in the entries of the matrix. In order to get $C^{1,1}$ estimates for elliptic Monge-Ampère type equations, one has to impose some structure restriction, as Lewy-Heinz example indicates that in general it is false (see [19]). In [19], various structure conditions were discussed for two dimenssion Monge-Ampère type equations. Unfortunately, our equation (1.1) does not fall into that category. As for higher dimensions, local C^2 a priori estimates even fails for standard Monge-Ampère . Here, we follow similar ideas in [7] and [9], and use the special structure of the equation (1.1) to get the crucial LOCAL C^2 a priori estimates.

To proceed further let us first recall the admissibility of closed surfaces with respect to (1.1) introduced in [22] and [3]. We say a closed surface Γ is admissible if for any point $p \in \Gamma$, there exists a paraboloid F with focus at the origin passing through this point p so that Γ lies on one side of the paraboloid. F will be called the supported paraboloid of Γ at p. The admissibility introduced above is analogous to that for the classical Monge-Ampère equation

$$\det D^2 u = f(x). \tag{1.3}$$

Actually paraboloids play the same role in equation (1.1) as planes in (1.3). In fact, let $F = \{x \cdot \psi(x);\ x \in S^2\}$ be a paraboloid with focus at the origin. Then $\psi(x)$ assumes the form

$$\psi(x) = \frac{C}{1 - \langle x, y \rangle}, \quad x \in S^2,\ x \neq y, \tag{1.4}$$

where y is the axial direction of F. Direct computations show that for $u = 1/\psi$, the matrix

$$\{\nabla_{ij} u + (u - \eta) e_{ij}\} \equiv 0.$$

From the above definition it follows that every closed admissible surface Γ is convex, and the matrix $\{\nabla_{ij} u + (u - \eta) e_{ij}\}$ is non-negative if Γ is C^2 smooth.

We would like to mention that if one considers the boundary problem (1.1) and (1.2), then the surface is not closed and in this case there are two classes of admissible surfaces, according to which side of the paraboloid the surface Γ lies on. Since both f and g are non-negative, equation (1.1) is elliptic when Γ is admissible. The notion of the support paraboloids and admissibility was introduced in [3] and [22].

For any admissible surface Γ, let $T(x) = T_\rho(x)$ denote the set of the axial directions of the support paraboloids of Γ at $x \cdot \rho(x)$. For any Borel set $E \subset S^2$, let $T(E) = \cup_{x \in E} T(x)$. It is proved in [22] that if Γ is admissible, the overlapped directions of the rays reflected by Γ have measure zero. Hence

$$\mu(E) = \int_{T(E)} g(y)dy \tag{1.5}$$

is a completely additive measure on S^2. We say Γ is a generalized solution of (1.1) if for every Borel set $E \subset S^2$,

$$\mu(E) = \int_E f(x)dx. \tag{1.6}$$

(1.6) is nothing but the energy conservation. It is easy to see that if Γ is C^2 smooth, then (1.6) is equivalent to (1.1). Taking $E = S^2$ we get the necessary condition for the solvability for equation (1.1):

$$\int_{S^2} f(x)dx = \int_{S^2} g(x)dx. \tag{1.7}$$

For simplicity in the following we will suppose that

$$\int_{S^2} f(x)dx = 1, \tag{1.8}$$

which means that the energy rediated from the origin is equal to 1.

We now state the main result of the paper.

Main Theorem. *Suppose f, g are C^∞ positive functions on S^2 and satisfy (1.7). Then there exists a C^∞ solution to (1.1), and the solution is unique up to multiplication of positive constants.*

The Main Theorem is a particular case of more general results in §3.

The paper is organized as follows: in §2, we treat a class of general degenerate Monge-Ampère equations modelled (1.1) on S^n, and derive the a priori estimates for the solutions to (1.1). In particular, local C^2 a priori estimates will be obtained for the degenerate Monge-Ampère equations. The existence and uniqueness of solutions to (1.1) will be proved in §3 by the continuity mothed. Here we employ a very simple argument to show that the linearized operator of the equations like (1.1) is surjective. In §4 we introduce a transformation for equation (1.1), which is the counterpart of the Legendre transformation for the classical Monge-Ampère equation (1.3). The transformation is actually related to the one in [24] and [16] mentioned above. We will show that the inverse of our transformation, which can be explicitly expressed, can recover the reflecting surfaces from the solutions obtained in [24] and [16].

2. A priori estimates

In this section, we prove a priori estimates for the solutions of equation (1.1). Here, we will deal with the problem on S^n for any $n \geq 2$. In this case we have the equation, instead of (1.1),

$$\frac{\det(\nabla_{ij}u + (u-\eta)e_{ij})}{\eta^n \det(e_{ij})} = f(x)/g(T(x)), \quad x \in S^n, \tag{2.1}$$

where f and g are as in §1, $\eta = (|\nabla u|^2 + u^2)/2u$. Equation (2.1) will be derived in the appendix. Observe that if u is a solution of (2.1), Cu is also a solution of (2.1) for any $C > 0$. For (2.1) one can introduce the support paraboloids and admissibility as in §1. The concept of generalized solutions can also be introduced in the same way.

We establish the a priori estimates in two steps. First is C^0 bounds for the solutions, the proof is similar to the one produced in [3]. Then, C^1 bounds follow by the convexity. The second step, which consists of the main part of the section, is C^2 bounds for the solutions. The a priori estimates hold for more general degenerate Monge-Ampère equations. We will explore that under great generality. The techniques we employ here are similar to that used in [7] and [9]. Our estimates here are purely LOCAL, while the estimates in [7] and [9] are global (in fact, local C^2 estimates fail for equations treated in [7] and [9]). Once C^2 estimates are achieved, (2.1) becomes uniformly elliptic if f and g are strictly positive, and the higher order regularity of solutions to (2.1) follows from the elliptic regularity theory (see, e.g., [6]). As for higher regularity of degenerate Monge-Ampère equations, we refer to the discussion in [8]. The existence and uniqueness of solutions will be treated in the next section.

2.1. C^0 estimate. Let u be a solution of (2.1) and $\Gamma = \{x \cdot \rho(x)\}$ the corresponding reflecting surface. Then $\rho > 0$. By multiplying a positive constant we may suppose $\inf_{S^n} \rho(x) = 1$.

For any nonnegative function $g(x)$ defined on S^n, since $\int_{B_\tau(y)} g(x)dx \to 0$ uniformly for $y \in S^n$ as $\tau \to 0$, there exists $r > 0$ so that for any $y \in S^n$,

$$2\int_{B_r(y)} g(x)dx \leq \int_{S^n} g(x)dx \leq 2\int_{B_{2\pi-r}(y)} g(x)dx, \tag{2.2}$$

where $B_r(y)$ denotes the ball on S^n centered at y with geodesic radius r. We also suppose $f(x)$ satisfies (2.2) with the same r. We claim that there exists $C > 0$ depending only on r so that

$$\sup_{x\in S^n} \rho(x) \leq C. \tag{2.3}$$

Indeed, suppose $\inf_{S^n} \rho(x)$ is attained at x_0. Without loss of generality we may suppose x_0 is the south pole, i.e., $x_0 = -e_{n+1}$. Let $\psi_0(x) = \frac{C_0}{1-\langle x,y_0\rangle}$ be a support paraboloid of Γ at $x_0 \cdot \rho(x_0)$, where y_0 is the axis of the support paraboloid. Since $\rho(x)$ attains its minimum at x_0, we see that $-e_{n+1}$ is the unit normal of Γ at $x_0 \cdot \rho(x_0)$, and $y_0 = e_{n+1}$. Moreover by $\inf_{S^n} \rho(x) = 1$ we have $C_0 = 2$.

To prove (2.3) it suffices to show that $\Gamma \subset \{x \in I\!R^{n+1};\ x_{n+1} < \widetilde{C}\}$ for some $\widetilde{C} > 1$ depending only on r. For any point $p \in \Gamma \cap \{x_{n+1} \le \widetilde{C}\}$, let $\psi_1(x) = \frac{C_1}{1-\langle x, y_1\rangle}$ be a support paraboloid of Γ at p. If there exists a point $p \in \Gamma \cap \{x_{n+1} \le \widetilde{C}\}$ such that $y_1 \notin B_r(e_{n+1})$, then C_1 is bounded by a constant depending only on r and $\widetilde{C}$, and Γ is bounded by the two paraboloids ψ_0 and ψ_1, and so (2.3) follows. Hence we may suppose that for any point $p \in \Gamma \cap \{x_{n+1} \le \widetilde{C}\}$, $y_1 \in B_r(e_{n+1})$. Then for $\widetilde{C}$ large enough, we have $T(x) \in B_r(e_{n+1})$ for any $x \in S^n \backslash B_r(e_{n+1})$. Hence by (2.2) we obtain

$$\begin{aligned}\frac{1}{2} > \int_{B_r(e_{n+1})} g(x)dx &\ge \int_{S^n \backslash B_r(e_{n+1})} f(x)dx \\ &> \frac{1}{2}\int_{S^n} f(x)dx = \frac{1}{2},\end{aligned}$$

a contradiction, from which (2.3) follows.

2.2. C^2 estimate. By (2.3) and the convexity we have $|\nabla u| \le C$. Next we derive the a priori bound for the second derivatives of solutions to (2.1). Let $\{w_{ij}\} = \{\nabla_{ij} u + (u-\eta)e_{ij}\}$. We consider the equation

$$\det(w_{ij}) = k(x, u, \nabla u)\det(e_{ij}) \quad \text{on} \quad S^n, \tag{2.4}$$

where

$$k(x, u, \nabla u) = f(x)g(x, u, Du).$$

Defintion 2.1. Let h be a bounded function defined in a domain Ω in S^n. We say h is Pseudo-subharmonic in Ω if there is a positive constant $A > 0$ such that

$$\Delta h \ge -A, \quad \forall\ x \in \Omega. \tag{2.5}$$

We remark here that the class of pseudo-harmonic functions is quite large. It includes all bounded subharmonic functions and all $C^{1,1}$ functions. We refer [7] for some description of the pseudo-subharmonic functions. We now derive the following key local estimate

Lemma 2.1. *Let $\lambda \ge 0$, $\lambda \in C^2(S^n)$ be a cut-off function. Let $\Omega_\lambda = \{x \in S^n;\ \lambda > 0\}$. Suppose g is positive and $C^{1,1}$ smooth in $(x, u, \nabla u)$, and f is nonnegative. Suppose $u \in C^4(\Omega_\lambda)$ is a positive solution of (2.4). If either* *(i)* $f^{1/(n-1)}$ *is Lipschitz and pseudo-subharmonic in* Ω_λ*; or*

(ii) f^α *pseudo-harmonic in* Ω_λ*, for some* $0<\alpha<1/(n-1)$*,*

then there is a constant $C>0$ *such that,*

$$|\nabla^2 u(x)| \le \frac{C}{\lambda(x)} \quad x\in\Omega_\lambda.$$

Proof. Let $H(x)=tr(u_{ij})$. Since $(w_{ij})\ge 0$, we only need to get an upper bound for $E(x) =: \lambda(x)H(x)$. Suppose $E(x_0)=\max_{x\in\Omega_\lambda} E(x)$. We may assume $x_0\in\Omega_\lambda$ and $H(x_0)\ge 1+C\sup_{S^n}(u-\eta)$. Let us pick up an orthonormal coordinate system at x_0, and assume $\{u_{ij}(x_0)\}$ is diagonal at x_0 (so is $\{w_{ij}(x_0)\}$). Then at x_0, $\nabla_i E=0$ and $\{\nabla_{ij}E\le 0\}$. That is,

$$H_i = -\frac{\lambda_i}{\lambda}H, \tag{2.6}$$

$$\lambda H_{ij} \le -(\lambda_{ij} - 2\frac{\lambda_i\lambda_j}{\lambda})H. \tag{2.7}$$

Let $\{w^{ij}\}=\{w_{ij}\}^{-1}$. Then, at x_0,

$$\begin{aligned}
-w^{ij}(\lambda_{ij}-2\frac{\lambda_i\lambda_j}{\lambda})H &\ge \lambda w^{ij}H_{ij} = \lambda w^{ii}(\Delta u)_{ii}\\
&= \lambda w^{ii}\{\Delta(u_{ii}) + 2\Delta u - 2nu_{ii}\}\\
&= \lambda\{w^{ii}\Delta(u_{ii}) + 2H(\Sigma w^{ii})\\
&\qquad - 2nw^{ii}w_{ii} + 2n(u-\eta)\Sigma w^{ii}\}\\
&= \lambda w^{ii}\Delta(u_{ii}) + \lambda O(1+H\Sigma w^{ii}).
\end{aligned} \tag{2.8}$$

We now compute $w^{ii}\Delta(u_{ii})$. Applying Δ to the equation $\det(w_{ij})^{1/(n-1)} = k^{1/(n-1)}$ yields, at x_0,

$$\begin{aligned}
w^{ii}\Delta(u_{ii}) &= w^{ij}\Delta(u_{ij}) = w^{ij}\{\Delta(w_{ij}) - \Delta[(u-\eta)e_{ij}]\}\\
&= w^{ij}\Delta(w_{ij}) + w^{ii}\Delta\eta + O(Hw^{ii})\\
&= \Big\{ w^{ik}w^{jl}(\Sigma_\beta \nabla_\beta w_{ij}\nabla_\beta w_{kl})
\end{aligned} \tag{2.9}$$

$$
\begin{aligned}
&- \frac{1}{n-1} w^{ij} w^{kl} (\Sigma_\beta \nabla_\beta w_{ij} \nabla_\beta w_{kl}) \Big\} \\
&+ k^{-2} \Big\{ k \Delta k - \frac{n-2}{n-1} |\nabla k|^2 \Big\} + O(H \Sigma w^{ii}) + \Delta \eta \Sigma w^{ii} \\
=:\ & I + I\!I + O(H\Sigma w^{ii}) + \Delta \eta \Sigma w^{ii}.
\end{aligned}
$$

From (2.8) it follows

$$
\begin{aligned}
-w^{ii}(\lambda_{ij} - \frac{2\lambda_i \lambda_j}{\lambda}) H \geq & \lambda I + \lambda I\!I + \lambda \Delta \eta \Sigma w^{ii} \\
& - C\lambda(1 + H\Sigma w^{ii}).
\end{aligned} \tag{2.10}
$$

While by (2.6) we have

$$
\begin{aligned}
\Delta \eta &= \frac{1}{u} \sum_{j=1}^n u_{jj}^2 + \frac{1}{u} \nabla u \cdot \nabla(\Delta u) + O(H) \\
&= \frac{1}{u} \sum_{j=1}^n u_{jj}^2 - H \frac{\nabla \lambda}{\lambda} \cdot \frac{\nabla u}{u} + O(H) \\
&\geq \overline{c} H^2 - C(1 + \frac{|\nabla \lambda|}{\lambda}) H
\end{aligned}
$$

for some $\overline{c} > 0$ depending only on $\inf u$. Hence (2.10) becomes

$$
-w^{ii}(\lambda_{ij} - \frac{2\lambda_i\lambda_j}{\lambda}) H \geq \lambda I + \lambda I\!I + + \overline{c} \lambda H^2 \Sigma w^{ii} - CH\Sigma w^{ii}. \tag{2.11}
$$

By definition,

$$
\begin{aligned}
I\!I =& f^{-2} \Big\{ f \Delta f - \frac{n-2}{n-1} |\nabla f|^2 \Big\} + \frac{2}{n-1} f^{-1} g^{-1} \nabla f \cdot \nabla g \\
&+ g^{-2} \Big\{ g \Delta g - \frac{n-2}{n-1} |\nabla g|^2 \Big\}.
\end{aligned}
$$

If f satisfies assumption (i) in the lemma, by (2.6) we have

$$
I\!I \geq -A f^{-1/(n-1)} - CH f^{-1/(n-1)} - CH^2 - C \frac{|\nabla \lambda|}{\lambda} H. \tag{2.12}
$$

On the other hand, $\forall \epsilon > 0$,

$$
I\!I \geq f^{-2} \{ f\Delta f - \frac{n-2+\epsilon}{n-1} |\nabla f|^2 \} + g^{-2} \{ g \Delta g - \frac{n-2+\frac{1}{\epsilon}}{n-1} |\nabla g|^2 \}.
$$

If f satisfies assumption (ii) in the lemma, set $\epsilon = 1-(n-1)\alpha$. Again using of (2.6) gives

$$
\begin{aligned}
I\!I &\geq -Af^{-\alpha} - CH^2 - C\frac{|\nabla\lambda|}{\lambda}H \\
&\geq -Cf^{-1/(n-1)} - CH^2 - C\frac{|\nabla\lambda|}{\lambda}H.
\end{aligned}
\tag{2.13}
$$

As for I, for each fixed β, we have either (i): there exist $J_1 \neq \emptyset$ and $J_2 \neq \emptyset$ so that $\nabla_\beta w_{jj} \geq 0$ for $j \in J_1$ and $\nabla_\beta w_{jj} < 0$ for $j \in J_2$; or (ii): $\nabla_\beta w_{ii}, i = 1, 2, \cdots, n$, has the same sign. If (i) is the case, we have at x_0,

$$
\begin{aligned}
I_\beta =: &\sum_{i=1}^{n}(w^{ii})^2(\nabla_\beta w_{ii})^2 - \frac{1}{n-1}(\sum_{i=1}^{n} w^{ii}\nabla_\beta w_{ii})^2 \\
\geq &[\sum_{i\in J_1}(w^{ii})^2(\nabla_\beta w_{ii})^2 - \frac{1}{n-1}(\sum_{j\in J_1} w^{ii}\nabla_\beta w_{ii})^2] \\
&+ [\sum_{i\in J_2}(w^{ii})^2(\nabla_\beta w_{ii})^2 - \frac{1}{n-1}(\sum_{j\in J_2} w^{ii}\nabla_\beta w_{ii})^2] \geq 0
\end{aligned}
\tag{2.14}
$$

In case(ii), we may suppose at x_0, $u_{11} \leq \cdots \leq u_{nn}$ (and so $0 \leq w_{11} \leq \cdots \leq w_{nn}$). Thus at x_0,

$$
w^{nn} = \frac{1}{w_{nn}} = \frac{1}{u_{nn} + (u-\eta)} \leq \frac{1}{H/n + (u-\eta)} \leq \frac{2n}{H},
$$

where by assumption, $H > 2n(u-\eta)$. By (2.6) and (ii), $|\nabla_\beta w_{ii}| \leq H\frac{|\nabla\lambda|}{\lambda}$. Therefore,

$$
\begin{aligned}
I_\beta \geq &\sum_{i=1}^{n-1}(w^{ii}\nabla_\beta w_{ii})^2 - \frac{1}{n-1}\Big(\sum_{i=1}^{n-1} w^{ii}\nabla_\beta w_{ii}\Big)^2 \\
&+ \frac{n-2}{n-1}(w^{nn}\nabla_\beta w_{nn})^2 \\
&- \frac{2}{n-1}\Big(\sum_{i=1}^{n-1} w^{ii}\nabla_\beta w_{ii}\Big) w^{nn}\nabla_\beta w_{nn} \\
\geq &-\frac{2}{n-1}\Big(\sum_{i=1}^{n} w^{ii}\Big) w^{nn}|\frac{\nabla\lambda}{\lambda}|^2 H^2 \\
\geq &-\frac{2}{n-1}\Big(\sum_{i=1}^{n} w^{ii}\Big)|\frac{\nabla\lambda}{\lambda}|^2 H.
\end{aligned}
\tag{2.15}
$$

Combining (2.14) and (2.15) we obtain

$$I \geq -\frac{2n}{n-1}\left(\sum_{i=1}^{n} w^{ii}\right)\left|\frac{\nabla\lambda}{\lambda}\right|^2 H. \tag{2.16}$$

By (2.12), (2.13) and (2.16), we deduce from (2.11) that

$$\begin{aligned} -w^{ij}(\lambda_{ij} - \frac{2\lambda_i\lambda_j}{\lambda})H \geq \bar{c}\lambda H^2(\Sigma w^{ii}) &- A\lambda f^{-1/(n-1)}(1+\widetilde{C}H) \\ &- C\lambda H^2 - C\lambda(1+\frac{|\nabla\lambda|^2}{\lambda^2})H\Sigma w^{ii}). \end{aligned}$$

Since $\lambda \in C^2(S^n)$, we have $|\nabla\lambda|^2 \leq C\lambda$. It follows that

$$\bar{c}\lambda H^2 \Sigma w^{ii} \leq A\lambda f^{-1/(n-1)}(1+CH) + C\lambda H^2 + CH\Sigma w^{ii}. \tag{2.17}$$

Since $\Sigma w^{ii} \geq CH^{1/(n-1)} f^{-1/(n-1)}$, we conclude from (2.17) that at x_0, $\lambda H \leq C$. This completes the proof. q.e.d.

Remark. In $n = 2, 3$ cases, every C^∞ nonnegative function satisfies condition (i) in Lemma 2.1. For $n = 2$, $f \in C^{1,1}$ is suffice; as for $n = 3$, $f \in C^{3,1}$ is suffice by a result of C. Fefferman (see [7]).

By the elliptic regularity we therefore obtain

Theorem 2.1. *Let $u \in C^4(S^n)$ be a solution of (2.1). Suppose $f, g \in C^2(S^n)$ and $f, g \geq C_0 > 0$. Then $\|u\|_{C^{3,\alpha}(S^n)} \leq C$, where $\alpha \in (0,1)$.*

3. Existence and uniqueness

Lemma 3.1. *Suppose $f, g \in C^{2,\alpha}(S^n)$ are positive and satisfy the energy conservation (1.7). Then (1.1) has a solution $u \in C^{4,\alpha}(S^n)$.*

Proof. We use the continuity method to prove the existence. To do so let $f_0(x) = g(-x)$. For any $t \in [0,1]$ let $f_t(x) = tf(x) + (1-t)f_0(x)$. Then f_t satisfies (1.7). We consider the equation

$$M(u) = g(T(x))\frac{\det(\nabla_{ij}u + (u-\eta)e_{ij})}{\eta^n \det(e_{ij})} = f_t(x), \quad x \in S^n. \tag{3.1}$$

When $t = 0$, (3.1) has a solution $u \equiv 1$. Let S denote the set of $t \in [0,1]$ in which (3.1) has a solution. By the a priori estimates in Section 2, we see that S is closed. We need only to show that S is also open.

For any given $t_0 \in S$, let u be a solution to (3.1). Without loss of generality we suppose $t_0 = 1$. Let L be the linearized operator of M at u. By the a priori estimates in Section 2, L is uniformly elliptic. For any $v(x) \in C^2(S^n)$, let $h_t = M(u + tv)$. For t small enough we see that $u + tv$ is admissible. By (1.7) and (1.8) we obtain

$$\int_{S^n} h_t(x)dx = \int_{S^n} g(x)dx = 1. \tag{3.2}$$

Hence for any $v \in C^2(S^n)$,

$$\int_{S^n} Lvdx = \lim_{t\to 0} \frac{1}{t} \int_{S^n} (h_t(x) - h_0(x))dx = 0. \tag{3.3}$$

Let E denote the set of all $C^{4,\alpha}$-functions on S^n, and F the set of all $C^{2,\alpha}$-functions on S^n such that $\int_{S^n} f(x)dx = 0$, where $\alpha \in (0,1)$. Then L is a mapping from E to F. To prove the openness of S it suffices to show L is surjective, or equivalently to show the kernel of L^*, the adjoint of L, is the null set $\{0\}$.

Suppose $L^*w = a_{ij}(x)w_{ij} + b_i(x)w_i(x) + c(x)w$. To show the kernel of L^* is the null set, it suffices to show $c(x) \equiv 0$ by the maximum principle. Notice that by (3.3),

$$\int_{S^n} v(x)c(x)dx = \int_{S^n} vL^*(1)dx = \int_{S^n} Lvdx = 0 \quad \forall \quad v \in E.$$

Hence $c(x) \equiv 0$. q.e.d.

By the a priori estimates in §2 and using approximation we thus obtain

Theorem 3.1.

(i) Suppose f, g are nonnegative functions on S^n and satisfy (1.7). Then there exists a generalized solution to (2.1).

(ii) If in addition $g \in C^{1,1}(S^n)$, $g > 0$, and f satisfies one of the conditions in Lemma 2.1, then (2.1) has a solution $u \in C^{1,1}(S^n)$. In particular, if $n = 2$, f is nonegative and $C^{1,1}$ smooth, then equation (2.1) has a solution $u \in C^{1,1}(S^2)$.

(iii) If furthermore $f > 0$, and $f \in C^{1,1}(S^n)$, then (2.1) has a solution $u \in C^{3,\alpha}(S^n)$ for any $\alpha \in (0,1)$.

Next we prove the uniqueness of solutions to (2.1).

Theorem 3.2. *Let $f(x)$ and $g(x)$ satisfy (1.7). Suppose*

$$\int_O g(x)dx > 0 \tag{3.4}$$

for any open $O \subset S^n$. Then the generalized solutions of (2.1) is unique up to a positive constant multiple.

Proof. We follow the idea in [17] (see §7.2 in [17]). Suppose ρ_1 and ρ_2 are two solutions to (2.1), and $\rho_1/\rho_2 \not\equiv const.$ We may suppose $\rho_1 = \rho_2$ at some point $x_0 \in \Omega$, and both sets $\Omega_1 = \{\rho_1(x)/\rho_2(x) > 1\}$ and $\Omega_2 = \{\rho_1(x)/\rho_2(x) < 1\}$ are nonempty. We claim that

$$T_{\rho_1}(\Omega_1) \supset T_{\rho_2}(\Omega_1). \tag{3.5}$$

Indeed, for any $y \in T_{\rho_2}(\Omega_1)$, let $\{\psi_C = \frac{C}{1-\langle x,y\rangle}\}$ be a family of paraboloids with focus at the origin and axes equal to y. One decreases the value of C and finds that the graph of ψ_C will touch the graph of ρ_1 before that of ρ_2, which implies (3.5). Similarly we have $T_{\rho_1}(\Omega_2) \subset T_{\rho_2}(\Omega_2)$.

Let G denote the set of the points at which both ρ_1 and ρ_2 are differentiable. We claim that $T_{\rho_1}(x) = T_{\rho_2}(x)\ \forall\ x \in G$, from which it follows $\rho_1 \equiv \rho_2$. If it is not true, there exists $x_0 \in G$ such that $T_{\rho_1}(x_0) \neq T_{\rho_2}(x_0)$. Multiplying ρ_2 by a positive constant (which doesn't change $T_{\rho_2}(x_0)$) we may suppose $\rho_1(x_0) = \rho_2(x_0)$ and $\Omega_1 = \{x \in \Omega,\ \ \rho_1(x)/\rho_2(x) > 1\}$ is nonempty. Let $y_0 = T_{\rho_2}(x_0)$ and $\psi_{y_0} = \frac{C_0}{1-\langle x,y_0\rangle}$ be the support paraboloid of ρ_2 at x_0. Since $T_{\rho_1}(x_0) \neq T_{\rho_2}(x_0)$, there exists a paraboloid $\psi_{y_\varepsilon} = \frac{C_\varepsilon}{1-\langle x,y_\varepsilon\rangle}$ which is a small perturbation of ψ_{y_0}, such that ψ_{y_ε} is a support paraboloid of ρ_2 at some point $x_\varepsilon \notin \Omega_1$, and ψ_{y_ε} cuts off a cap from the graph of ρ_1. This means that y_ε is an interior point of $T_{\rho_1}(\Omega_1)$ and $y_\varepsilon \notin T_{\rho_2}(\Omega_1)$. Therefore by (3.5) $T_{\rho_1}(\Omega_1) \underset{\neq}{\supset} T_{\rho_2}(\Omega_2)$. From (3.4) we therefore reach a contradiction since

$$\int_{T_{\rho_1}(\Omega_1)} g(x)dx = \int_{\Omega_1} f(x)dx = \int_{T_{\rho_2}(\Omega_1)} g(x)dx.$$

This completes the proof. q.e.d.

Now, our Main Theorem follows directly from Theorems 3.1 and 3.2.

Remark. In [3], it is proved that there is a weak solution to the global reflection problem (1.1) when $f, g \in L^1(S^n)$ and satisfy (1.7).

Here, our existence results can produce an unique weak solution (in the sense of [3]) for local reflection problem (1.1) with boundary condition (1.2) for f and g satisfy the similar compatibility condition. To see this, we may extend f and g to be defined in whole S^n (vanishing out side Ω and D respectively). Then by smoothing f and g, we obtain approximate solutions. Our C^0 estimates give that a subsequence converges to a weak solution. The uniqueness follows the same line of the proof to Theorem 3.2. In addition, if f and g are smooth and positive respectively in D and Ω, and $\overline{T}(\Omega) = \overline{D}$, then the solution is smooth by elliptic theory and our local C^2 estimates. These results were proved by different motheds in [22] in $n = 2$ case.

4. Legendre type transformation

Let Ω and D be two domains on S^n. Suppose $\Gamma = \{x \cdot \rho(x), x \in \Omega\}$ be an admissible C^3 surface so that $\{\nabla_{ij} u + (u - \eta) e_{ij}\}$ is strictly positive, where u and η are as in (1.1). Then T_ρ is a diffeomorphism from Ω to $D = T_\rho(\Omega)$. For any $y \in D$, let

$$\rho^*(y) = \inf\{\frac{1}{\rho(x)} \frac{1}{1 - \langle x, y \rangle}, \quad x \in \Omega\}. \tag{4.1}$$

Obviously $\rho^*(y)$ is a positive function. By the admissibility it is easy to see that the infimum is attained at the point $x \in \Omega$ with $T_\rho(x) = y$. Hence

$$\rho^*(T_\rho(x)) = \frac{1}{\rho(x)} \frac{1}{1 - \langle x, T_\rho(x) \rangle}. \tag{4.2}$$

Since T_ρ is a diffeomorphism we see that ρ^* is C^2 smooth.

Lemma 4.1. *$\rho^*(y)$ is admissible.*

Proof. For any $y_0 \in D$, let $x_0 \in \Omega$ so that $T_\rho(x_0) = y_0$. Let $\psi(y) = \frac{C}{1 - \langle y, x_0 \rangle}$ be a paraboloid with axis x_0 so that $\psi(y_0) = \rho^*(y_0)$, namely

$$\frac{C}{1 - \langle y_0, x_0 \rangle} = \frac{1}{\rho(x_0)} \frac{1}{1 - \langle x_0, T_\rho(x_0) \rangle},$$

we have $C = \frac{1}{\rho(x_0)}$. Hence

$$\psi(y) = \frac{1}{\rho(x_0)} \frac{1}{1 - \langle x_0, y \rangle}. \tag{4.3}$$

For any $y \in D$, by definition we see that

$$\rho^*(y) \leq \psi(y).$$

Namely, ψ is a support paraboloid of Γ^* at $y_0 \cdot \psi(y_0)$. q.e.d.

From the proof above we see that $(x_0 + y_0)/|x_0 + y_0|$ is the normal of Γ at $x_0 \cdot \rho(x_0)$ and the normal of Γ^* at $y_0 \cdot \rho^*(y_0)$, where $y_0 = T_\rho(x_0)$. In particular we have

Corollary 4.2. *The inverse of T_ρ is T_{ρ^*}.*

Hence T_{ρ^*} is also a diffeomorphism from D to Ω. Now we suppose ρ is a solution of (1.1) and (1.2). Suppose f, g are positive and C^2 smooth. By the a priori estimates in §2 we see that T_ρ is a diffeomorphism from Ω to D. We can now consider the problem conversely. Let us take the graph Γ^* as the reflecting surface, and suppose the intensity of the light rediated from the origin is $g(y), y \in D$. Then the directions of the light reflected by Γ^* cover Ω with intensity $f(x)$. Hence similar to (1.1) we have

Lemma 4.3. *Let $u^*(y) = 1/\rho^*(y)$. Then u^* satisfies the equation*

$$\frac{det(\nabla_{ij}u^* + (u^* - \eta^*)e_{ij})}{\eta^{*2} det(e_{ij})} = \frac{g(y)}{f(T_{\rho^*}(y))}, \tag{4.4}$$

where $\eta^* = \frac{1}{2u^*}(|\nabla u^*|^2 + u^{*2})$.

We can define similarly to (4.1) that

$$\rho^{**}(x) = \inf\{\frac{1}{\rho^*(y)} \frac{1}{1 - \langle x, y\rangle}, \quad y \in D\}, \quad x \in \Omega. \tag{4.5}$$

For any $x_0 \in \Omega$, the infimum is attained at the point y_0 so that $T_{\rho^*}(y_0) = x_0$. By (4.2) we have

$$\rho^{**}(x_0) = \frac{1}{\rho^*(y_0)} \frac{1}{1 - \langle x_0, y_0\rangle} = \rho(x_0). \tag{4.6}$$

Hence we may regard Γ^* as the dual of Γ.

Remark. Finally we point out that the phase function $p(y)$ in the equations in [16] and [14] is actually $1/\rho^*(y)$. Indeed, we have $1/\rho^*(y) = \rho(x)(1 - \langle x, y\rangle)$, where $T_\rho(x) = y$. By the definition in [16] (see Remark 1.3.1 in [16]), $p(y) = 2\rho(x)\langle x, \gamma\rangle^2$, where γ is the unit normal of Γ. It is easy to see $2\langle x, \gamma\rangle^2 = 1 - \langle x, y\rangle$. Hence $1/\rho^*(y) = p(y)$. This answers the question of reconstructing the reflecting surfaces from the solutions obtained in [16] and [14].

Appendix

In this appendix we derive equation (2.1). Let Ω and D be two domains on S^n, and f and g be two nonnegative function defined on Ω and D respectively. Suppose the rays are originated from the origin with intensity $f(x)$. Let $\Gamma = \{x \cdot \rho(x), \text{ and } x \in \Omega\}$ be a C^2 reflecting surface so that the directions of the reflected rays cover the domain D and the distribution is equal to g. Here we identify a direction x with a point x on S^2.

Let $x \in \Omega$ be a ray from the origin which goes to $y = T(x) \in D$ after reflection. Let $(e_1, \cdots, e_n)$ be an orthonormal basis of S^n near x. Let γ denote the unit normal of Γ and let $u = 1/\rho$. Then $\gamma = -(\nabla u + ux)/\sqrt{u^2 + |\nabla u|^2}$. We have

$$T(x) = x - 2\langle x, \gamma\rangle\gamma = -\frac{1}{\eta}[\nabla u + (u - \eta)x].$$

where $\eta = (|\nabla u|^2 + u^2)/2u$. Direct computation shows that (see [16] and [22])

$$\partial_i T(x) = \frac{-1}{\eta} q_{ij}(\partial_j x - \frac{u_j}{u}\beta),$$

where $\beta = (\nabla u + ux)/\eta$, $\partial_i x = e_i$, and $q_{ij} = \nabla_{ij} u + (u - \eta)e_{ij}$.

For admissible surface Γ, it is known (see [22]) that the overlaped directions of the reflected light has measure zero. Namely, the measure of the set of axes of the support paraboloids of Γ whose intresection with Γ has more than one point is zero. Suppose there is no loss of energy in reflection. Then we have the energy conservation:

$$\int_E f(x)dx = \int_{T(E)} g(x)dx, \quad \forall \text{ Borel set } E \subset \Omega.$$

Let $E = B_\tau(x)$ be the ball on S^n certered at x with geodesic radius τ, and let τ goes to zero. Then we get the equation

$$|\frac{dT(x)}{dx}| = f(x)/g(T(x)).$$

The left-hand side equals the determinent of the matrix

$$[\partial_1 T(x), \cdots, \partial_n T(x), T(x)],$$

i.e.,

$$\begin{aligned}\left|\frac{dT(x)}{dx}\right| &= \frac{1}{\eta^n}\det[q_{1j}(e_j - \frac{u_j}{u}\beta); \cdots ; q_{nj}(e_j - \frac{u_j}{u}\beta); \beta - x] \\ &= \frac{1}{\eta^n}\det[q_{1j}(e_j - \frac{u_j}{u}x); \cdots ; q_{nj}(e_j - \frac{u_j}{u}x); \beta - x] \\ &= \frac{1}{\eta^n}\det[q_{1j}e_j; \cdots ; q_{nj}e_j; x] \\ &= \frac{1}{\eta^n}\det(q_{ij}).\end{aligned}$$

Hence we obtain (2.1).

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