

ON A MULTICLASS BATCH ARRIVAL RETRIAL QUEUE

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Abstract

Kulkarni (1986) derived expressions for the expected waiting times for customers of two types who arrive in batches at a single-channel repeated orders queueing system. We propose another method of solving this problem and extend Kulkarni's result to the case of $N \geq 2$ classes of customers.

RETRIALS; BATCH ARRIVALS

In the context of local area computer networks, Kulkarni (1986) considered the following queueing system. There is a single channel, and arriving customers belong to n different types. The arrival times of demands of the i th type (i -demands) form a Poisson process with rate λ_i ; at every arrival epoch with certain probability c_{ik} exactly k i -demands arrive. These demands we call primary calls. If an arriving batch of i -customers finds the channel free, one of the batch members immediately occupies the channel and the rest of the customers in that batch form the sources of repeated i -calls (i -sources). Every such source produces a Poisson process of repeated calls with intensity $\mu_i > 0$. If an incoming repeated call finds a free line it is served and leaves the system after service. Otherwise, if the channel is engaged, the system state does not change. Service times, for primary and for repeated i -calls, have the same distribution function $B_i(x)$. As usual we suppose that interarrival period, batch sizes, retrial times and service times are mutually independent.

Let $b_i(x) = B_i(x)/(1 - B_i(x))$ be the instantaneous service intensity of i -calls, $\beta_i(s) = \int_0^\infty \exp(-sx) dB_i(x)$ be the Laplace–Stieltjes transform of the service time distribution function $B_i(x)$, $\beta_{ik} = (-1)^k \beta_i^{(k)}(0)$ be the k th initial moment of the i -calls service time, $c_i(z) = \sum_{k=1}^\infty c_{ik} z^k$, \bar{c}_i , σ_i^2 be respectively the generating function, the mean and the variance of batch size of i -calls, $\lambda = \lambda_1 + \dots + \lambda_n$, $\rho_i = \lambda_i \bar{c}_i \beta_{i1}$ be the system load due to primary i -calls.

$$\beta(s) = \int_0^\infty \exp(-sx) dB(x), \quad \beta_k = (-1)^k \beta^{(k)}(0), \quad \rho = \sum_{i=1}^n \rho_i.$$

Let $c(t) = 0$ if at time t the channel is free; $c(t) = i$ if at time t the channel is occupied by some i -call; $N_i(t)$ is the number of i -sources at time t . If $c(t) \neq 0$ then $\xi(t)$ is the time during which the channel has been serving the call which occupies the channel at time t .

We shall consider the system in steady state, which exists if and only if $\rho < 1$, so the condition $\rho < 1$ is assumed to hold from now on. Our goal consists of finding mean queue lengths $N_i = EN_i(t)$, $1 \leq i \leq n$ as well as the variance–covariance matrix of the $(N_1(t), \dots, N_n(t))$. Kulkarni (1986) proposed a method of solving the problem and in the case of two types of customers obtained formulas for N_1 , N_2 . In this note we describe another approach to the problem and obtain a solution in the general case.

Theorem 1. The expected number of retrial customers of type i in steady state is

$$N_i = \frac{\lambda_i(\rho + \bar{c}_i - 1)}{\mu_i(1 - \rho)} + \frac{\lambda_i \bar{c}_i}{2} x_i,$$

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where the values x_i can be found as the solution of the system of linear equations

$$\sum_{j=1}^n \frac{\mu_j \rho_j}{\mu_i + \mu_j} (x_i + x_j) = x_i - \sum_{j=1}^n \lambda_j \bar{c}_j \beta_{j2} - \frac{\beta_{i1}}{\bar{c}_i} (\sigma_i^2 + \bar{c}_i^2 - \bar{c}_i).$$

Proof. Let $m = (m_1, \dots, m_n)$, $z = (z_1, \dots, z_n)$, $e_i = (0, \dots, 1, \dots, 0)$ be the n -dimensional vector with i th coordinate equal to 1 and the rest equal to 0, and $e = (1, \dots, 1)$ be the n -dimensional vector which has all coordinates equal to 1.

Consider the system in steady state and write:

$$p_0(m) = P\{c(t) = 0, N_1(t) = m_1, \dots, N_n(t) = m_n\}$$

$$p_i(m, x) dx = P\{c(t) = i, x < \xi(t) < x + dx, N_1(t) = m_1, \dots, N_n(t) = m_n\}, \quad i = 1, \dots, n.$$

In a general way we obtain the equations of statistical equilibrium:

$$\begin{aligned} \left(\lambda + \sum_{i=1}^n \mu_i m_i\right) p_0(m) &= \sum_{i=1}^n \int_0^\infty p_i(m, x) b_i(x) dx, \\ \frac{d}{dx} p_j(m, x) &= -[\lambda + b_j(x)] p_j(x) \\ &\quad + \sum_{i=1}^n \lambda_i \sum_{k=1}^{m_i} c_{ik} p_j(m - k e_i, x), \\ p_j(m, 0) &= \lambda_j \sum_{k=1}^{m_j+1} c_{jk} p_0(m - (k-1) e_j) \\ &\quad + \mu_j (m_j + 1) p_0(m + e_j). \end{aligned}$$

For the generating functions

$$\begin{aligned} \varphi_0(z) &= \sum_{m_1=0}^\infty \dots \sum_{m_n=0}^\infty z_1^{m_1} \dots z_n^{m_n} p_0(m), \\ \varphi_i(z, x) &= \sum_{m_1=0}^\infty \dots \sum_{m_n=0}^\infty z_1^{m_1} \dots z_n^{m_n} p_i(m, x) \end{aligned}$$

these equations give

$$(1) \quad \lambda \varphi_0(z) + \sum_{i=1}^n \mu_i z_i \frac{\partial \varphi_0(z)}{\partial z_i} = \sum_{i=1}^n \int_0^\infty \varphi_i(z, x) b_i(x) dx,$$

$$(2) \quad \frac{\partial}{\partial x} \varphi_j(z, x) = -\left(\sum_{i=1}^n \lambda_i (1 - c_i(z_i)) + b_j(x)\right) \varphi_j(z, x),$$

$$(3) \quad \varphi_j(z, 0) = \lambda_j \frac{c_j(z_j)}{z_j} \varphi_0(z) + \mu_j \frac{\partial \varphi_0(z)}{\partial z_j}.$$

From (2) we find that $\varphi_j(z, x)$ depends upon x as follows:

$$(4) \quad \varphi_j(z, x) = \varphi_j(z, 0) [1 - B_j(x)] \exp(-sx),$$

where we have denoted $\sum_{i=1}^n \lambda_i (1 - c_i(z_i))$ by s .

From (4) it follows that

$$(5) \quad \varphi_j(z) = \int_0^\infty \varphi_j(z, x) dx = \varphi_j(z, 0) \frac{1 - \beta_j(s)}{s}.$$

Now with the help of (4) and (5), Equations (1) and (3) can be rewritten as follows:

$$(6) \quad \lambda \varphi_0(z) + \sum_{i=1}^n \mu_i z_i \frac{\partial \varphi_0(z)}{\partial z_i} = \sum_{i=1}^n \frac{s \beta_i(s)}{1 - \beta_i(s)} \varphi_i(z)$$

$$(7) \quad \lambda_j \frac{c_j(z_j)}{z_j} \varphi_0(z) + \mu_j \frac{\partial \varphi_0(z)}{\partial z_j} = \frac{s}{1 - \beta_j(s)} \varphi_j(z).$$

In order to find the distribution of the channel state we multiply (7) by z_j , then sum over $j = 1, \dots, n$ and subtract from (6); after some transformations we get

$$(8) \quad \varphi_0(z) = \sum_{i=1}^n \varphi_i(z) \frac{\beta_i(s) - z_i}{1 - \beta_i(s)}.$$

Fixing some j and putting $z_i = 1$ for all $i \neq j$,

$$\varphi_0(z) + \sum_{i=1}^n \varphi_i(z) = \frac{1 - z_j}{1 - \beta_j(\lambda_j - \lambda_j c_j(z_j))} \varphi_j(z).$$

Setting $z_j = 1$ and taking into account the normalization condition $\sum_{i=0}^n \varphi_i(e) = 1$ we get

$$(9) \quad \begin{aligned} \varphi_j(e) &= \rho_j \\ \varphi_0(e) &= 1 - \sum_{j=1}^n \varphi_j(e) = 1 - \rho. \end{aligned}$$

Also, with $z = e$ (7) and (9) yield

$$(10) \quad \frac{\partial \varphi_0(e)}{\partial z_j} = \frac{\lambda_j(\rho - 1 + \bar{c}_j)}{\mu_j}.$$

Summing up (7) over $j = 1, \dots, n$ and subtracting from (6) we have

$$(11) \quad \sum_{i=1}^n \lambda_i \left[c_i(z_i) - \frac{c_i(z_i)}{z_i} \right] \varphi_0(z) + \sum_{i=1}^n \mu_i (z_i - 1) \frac{\partial \varphi_0(z)}{\partial z_i} = \sum_{i=1}^n \lambda_i (c_i(z_i) - 1) N(z),$$

where $N(z) = \sum_{i=0}^n \varphi_i(z)$.

Differentiating (11) with respect to $z_i z_j$ at the point $z = e$ we obtain, after some algebra,

$$(12) \quad \begin{aligned} (\mu_i + \mu_j) \frac{\partial^2 \varphi_0(e)}{\partial z_i \partial z_j} &= \lambda_i \bar{c}_i N_j + \lambda_j \bar{c}_j N_i - \lambda_i \lambda_j \left(\frac{\rho - 1 + \bar{c}_j}{\mu_j} + \frac{\rho - 1 + \bar{c}_i}{\mu_i} \right) \\ &\quad + \delta_{i,j} \lambda_i [\sigma_i^2 + \bar{c}_i^2 - 2(\bar{c}_i - 1)(1 - \rho) - \bar{c}_i]. \end{aligned}$$

Now differentiate (7) with respect to z_i at the point $z = e$:

$$(13) \quad \frac{1}{\beta_{j1}} \frac{\partial \varphi_j(e)}{\partial z_i} = \mu_j \frac{\partial^2 \varphi_0(e)}{\partial z_i \partial z_j} + \frac{\lambda_j \lambda_i (\rho - 1 + \bar{c}_i)}{\mu_i} + \lambda_i \lambda_j \bar{c}_i \bar{c}_j \frac{\beta_{j2}}{2\beta_{j1}} + \delta_{i,j} \lambda_j [c_j - 1)(1 - \rho).$$

Then multiply (13) by β_{j1} and sum up over $j = 1, \dots, n$:

$$\begin{aligned} \sum_{j=1}^n \mu_j \beta_{j1} \frac{\partial^2 \varphi_0(e)}{\partial z_i \partial z_j} &= N_i - \frac{\lambda_i (\rho - 1 + \bar{c}_i)}{\mu_i} (1 + \lambda \beta_1) - \frac{\lambda_i \bar{c}_i}{2} \sum_{j=1}^n \lambda_j \bar{c}_j \beta_{j2} \\ &\quad - \lambda_i \beta_{i1} (c_i - 1)(1 - \rho). \end{aligned}$$

Using (12) we obtain from this equality:

$$\begin{aligned}
 \sum_{j=1}^n \frac{\mu_j \beta_{j1}}{\mu_i + \mu_j} (\lambda_i \bar{c}_i N_j + \lambda_j \bar{c}_j N_i) &= N_i - \frac{\lambda_i \bar{c}_i}{2} \sum_{j=1}^n \lambda_j \bar{c}_j \beta_{j2} \\
 &\quad - \frac{\lambda_i \beta_{i1}}{2} (\sigma_i^2 + \bar{c}_i^2 - \bar{c}_i) - \frac{\lambda_i}{\mu_i} (\rho - 1 + \bar{c}_i + \lambda \beta_1 \bar{c}_i) \\
 &\quad + \sum_{j=1}^n \mu_j \beta_{j1} \frac{\lambda_i \lambda_j}{\mu_i + \mu_j} \left(\frac{\bar{c}_j}{\mu_j} + \frac{\bar{c}_i}{\mu_i} \right).
 \end{aligned}
 \tag{14}$$

Introducing the new variables x_i by the formula

$$N_i = \frac{\lambda_i(\rho + \bar{c}_i - 1)}{\mu_i(1 - \rho)} + \frac{\lambda_i \bar{c}_i}{2} x_i$$

completes the proof.

For every concrete n it is easy to obtain the solution in explicit form. For example, if $n = 2$ then we have the system of two linear equations with two unknown variables and so after some algebra we get the main results of Kulkarni (1986):

$$\begin{aligned}
 N_1 &= \frac{\lambda_1(\rho + \bar{c}_1 - 1)}{\mu_1(1 - \rho)} \\
 &\quad + \frac{\lambda_1 \bar{c}_1 [\mu_2 + (1 - \rho)\mu_1]A + [(1 - \rho)\mu_1 + (1 - \rho_2)\mu_2]B_1 + \mu_2 \rho_2 B_2}{2(1 - \rho)[(1 - \rho_1)\mu_1 + (1 - \rho_2)\mu_2]}, \\
 N_2 &= \frac{\lambda_2(\rho + \bar{c}_2 - 1)}{\mu_2(1 - \rho)} \\
 &\quad + \frac{\lambda_2 \bar{c}_2 [\mu_1 + (1 - \rho)\mu_2]A + [(1 - \rho)\mu_2 + (1 - \rho_1)\mu_1]B_2 + \mu_1 \rho_1 B_1}{2(1 - \rho)[(1 - \rho_1)\mu_1 + (1 - \rho_2)\mu_2]}
 \end{aligned}$$

where

$$A = \sum_{j=1}^n \lambda_j \bar{c}_j \beta_{j2}, \quad B_i = \frac{\beta_{i1}}{\bar{c}_i} (\sigma_i^2 + \bar{c}_i^2 - \bar{c}_i).$$

It is, of course, generally convenient to use a computer to carry out the calculations.

Our method allows us to obtain second moments of queue lengths

$$N_{ij} = \frac{\partial N(e)}{\partial z_i \partial z_j} = EN_i(t)N_j(t) - \delta_{i,j} \cdot EN_i(t).$$

For lack of space we consider only the case $c_i(z) \equiv z, i = 1, \dots, n$, i.e. singleton arrivals.

Theorem 2. The second moments of queue lengths in the steady state are

$$N_{ij} = \lambda_i \lambda_j x_{ij} + \frac{\lambda_i \lambda_j}{2} \frac{x_i + x_j}{\mu_i + \mu_j} + \frac{\lambda_i \lambda_j}{\mu_i \mu_j} \frac{\rho^2}{(1 - \rho)^2},$$

where the values x_i were defined in Theorem 1 and the values x_{ij} can be found as the solution

of the system of linear equations:

$$\sum_{k=1}^n \mu_k \rho_k \frac{x_{ij} + x_{ik} + x_{kj}}{\mu_i + \mu_j + \mu_k} = x_{ij} - \frac{\rho}{2} \frac{x_i + x_j}{\mu_i + \mu_j} - \frac{\lambda \beta_3}{3} - \frac{\lambda \beta_2}{2} \cdot \frac{\rho}{1 - \rho} \cdot \frac{\mu_i + \mu_j}{\mu_i \cdot \mu_j} - \frac{1}{4} \sum_{k=1}^n \lambda_k \mu_k \beta_{k2} \left(\frac{x_i + x_k}{\mu_i + \mu_k} + \frac{x_j + x_k}{\mu_j + \mu_k} \right).$$

The proof is along the lines of Theorem 1, but now we have to differentiate (11) with respect to z_i, z_j, z_k (instead of differentiating it with respect to z_i, z_j as we did earlier in obtaining (12)) and differentiate (7) with respect to z_i, z_k (instead of differentiating it with respect to z_i as we did earlier in obtaining (14)).

References

KULKARNI, V. G. (1986) Expected waiting times in a multiclass batch arrival retrial queue. *J. Appl. Prob.* **23**, 144–154.