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ON A NEW CLASS OF ABSTRACT IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this note we introduce a new class of abstract impulsive differential equations for which the impulses are not instantaneous. We introduce the concepts of mild and classical solution and we establish some results on the existence of these types of solutions. An example involving a partial differential equation is presented.

1. Introduction

In this note we introduce a class of abstract impulsive differential equations for which the impulses are not instantaneous. Specifically, we study the existence of solutions for an impulsive problem of the form

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in (s_i, t_{i+1}],\ i = 0, \dots, N, \tag{1.1}$$

$$u(t) = g_i(t, u(t)), \quad t \in (t_i, s_i],\ i = 1, \dots, N, \tag{1.2}$$

$$u(0) = x_0, \tag{1.3}$$

where $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$, $x_0 \in X$, $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_N \leq s_N \leq t_{N+1} = a$ are pre-fixed numbers, $g_i \in C((t_i, s_i] \times X; X)$ for all $i = 1, \dots, N$ and $f : [0, a] \times X \to X$ is a suitable function.

The literature on abstract impulsive differential equations considers basically problems for which the impulses are abrupt and instantaneous. The literature on this type of problem is vast, and different topics on the existence and qualitative properties of solutions are considered. Concerning the general motivations, relevant developments and the current status of the theory, we refer the reader to [1]-[16] and the references therein.

In this note we consider a class of problems for which the impulses are not instantaneous. In this paper the impulses start abruptly at the points t_i and their action continues on the interval $[t_i, s_i]$. We note that the considered problem, the technical approach, the results and applications presented in this work are totally new.

As a motivation for the study of systems such as (1.1)-(1.3), we consider the following simplified situation concerning the hemodynamical equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one

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can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes, we can interpret the above situation as an impulsive action which starts abruptly and stays active on a finite time interval.

Next, we introduce some notation and technical results. In this paper, $A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup of bounded linear operators $(T(t))_{t\geq 0}$ on $(X, \|\cdot\|)$ and $[D(A)]$ represents the domain of A endowed with the graph norm.

Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. In this paper, we denote by $\mathcal{L}(Z,W)$ the space of bounded linear operators from Z into W endowed with the norm of operators denoted by $\|\cdot\|_{\mathcal{L}(Z,W)}$ and we write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ when $Z = W$. In addition, $B_\gamma(z, Z)$ denotes the closed ball with center at $z \in Z$ and radius r in Z. As usual, $C(J, Z)$ (with $J \subset \mathbb{R}$) is the space formed by all the continuous bounded functions defined from J into Z, endowed with the uniform norm $\|u\|_{C(J,Z)} = \sup_{t\in J} \|u(t)\|_Z$.

To treat the impulsive conditions, we consider the space $\mathcal{PC}(X)$ which is formed by all the functions $u : [0, a] \to X$ such that $u(\cdot)$ is continuous at $t \neq t_i, u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for all $i = 1, \ldots, N$, endowed with the uniform norm on $[0, a]$ denoted by $\|u\|_{\mathcal{PC}(X)}$. It is easy to see that $\mathcal{PC}(X)$ is a Banach space. For a function $u \in \mathcal{PC}(X)$ and $i \in \{0, 1, \ldots, N\}$, we introduce the function $\tilde{u}_i \in C([t_i, t_{i+1}]; X)$ given by

$$\widetilde{u}_i(t) = \begin{cases} u(t), & \text{for } t \in (t_i, t_{i+1}], \\ u(t_i^+), & \text{for } t = t_i. \end{cases} \tag{1.4}$$

In addition, for $B \subseteq \mathcal{PC}(X)$ and $i \in \{0, 1, \ldots, N\}$, we use the notation $\widetilde{B}_i$ for the set $\widetilde{B}_i = \{\tilde{u}_i : u \in B\}$. We note the following Ascoli-Arzelà type criteria.

Lemma 1.1. *A set $B \subseteq \mathcal{PC}(X)$ is relatively compact in $\mathcal{PC}(X)$ if and only if each set $\widetilde{B}_i$ is relatively compact in $C([t_i, t_{i+1}], X)$.*

This paper has three sections. In section 2 we study the existence of solutions for the problem (1.1)-(1.3). In the last section, an application involving a partial differential equation is presented.

2. Existence of solution

In this section we discuss the existence of mild and classical solutions for the impulsive system (1.1)-(1.3). To begin, we introduce the following concepts of solution.

Definition 2.1. A function $u \in \mathcal{PC}(X)$ is called a mild solution of the problem (1.1)-(1.3) if $u(0) = x_0$, $u(t) = g(t, u(t))$ for all $t \in (t_j, s_j]$ and each $j = 1, \ldots, N$, and

$$u(t) = T(t)x_0 + \int_0^t T(t-\tau)f(\tau, u(\tau))d\tau$$

for all $t \in [0, t_1]$ and

$$u(t) = T(t-s_i)g_i(s_i, u(s_i)) + \int_{s_i}^t T(t-\tau)f(\tau, u(\tau))d\tau$$

for all $t \in [s_i, t_{i+1}]$ and every $i = 1, \ldots, N$.

In the remainder of this work, for a function $u : [0, a] \to X$ and $J \subset [0, a]$, we use the notation $u_{|_J}$ to represent the restriction of $u(\cdot)$ to the set J.

Definition 2.2. A function $u \in \mathcal{PC}(X)$ is said to be a classical solution of the problem (1.1)-(1.3) if $u(0) = x_0$, $u(t) = g(t, u(t))$ for all $t \in (t_j, s_j]$ and each $j = 0, \dots, N$, $u_{|(s_i,t_{i+1}]} \in C((s_i, t_{i+1}]; [D(A)])$ for all $i = 1, \dots, N$ and $u(\cdot)$ satisfies (1.1).

For convenience, we state some well-known concepts concerning the Cauchy problem:

$$w'(t) = Aw(t) + \xi(t, u(t)), \qquad t \in [c, d], \tag{2.1}$$

$$w(c) = z \in X. \tag{2.2}$$

We note that a function $u \in C([c, d], X)$ is called a mild solution of (2.1)-(2.2) if

$$u(t) = T(t - c)z + \int_c^t T(t - \tau)\xi(\tau, u(\tau))d\tau, \qquad \forall\, t \in [c, d]. \tag{2.3}$$

A function $u \in C([c, d], X)$ is said to be a classical solution of (2.1)-(2.2) if $u(c) = x_0$, $u_{|_{(c,d]}} \in C((c, d]; [D(A)])$ and $u(\cdot)$ satisfies (2.1) on $(c, d]$.

The next proposition establishes the basic relation between the concepts of mild and classical solutions. We include the proof of this result for completeness.

Proposition 2.1. *If $u(\cdot)$ is a classical solution of* (1.1)-(1.3)*, then $u(\cdot)$ is a mild solution.*

Proof. It is easy to see that the function $u_{|_{[s_i,t_{i+1}]}}$ is a classical solution of the problem

$$w'(t) = Aw(t) + f(t, u(t)), \qquad t \in [s_i, t_{i+1}], \tag{2.4}$$

$$w(s_i) = u(s_i) \in X. \tag{2.5}$$

Now, from semigroup theory we obtain that $u_{|_{[s_i,t_{i+1}]}}$ is a mild solution of (2.4)-(2.5) and

$$u_{|_{[s_i,t_{i+1}]}}(t) = T(t - s_i)u(s_i) + \int_{s_i}^t T(t - s)f(s, u(s))ds, \qquad \forall\, t \in [s_i, t_{i+1}].$$

Finally, by noting that $u(s) = g(s, u(s))$ for all $s \in (t_i, s_i]$, $i \geq 1$, we infer that $u(\cdot)$ is a mild solution of (1.1)-(1.3). $\square$

There is a huge number of papers which consider conditions under which a mild solution of (2.1)-(2.2) is a classical solution. To shorten our developments, in the next result we use the notation $P(\xi, z)$ to represent a generic condition on $\xi(\cdot)$ and z which implies that a mild solution of (2.1)-(2.2) is a classical solution of (2.1)-(2.2).

Proposition 2.2. *If $u(\cdot)$ is a mild solution of* (1.1)-(1.3) *and the conditions $P(f, x_0)$ and $P(f, g(s_i, u(s_i)))$, $i = 1, \dots, N$, are satisfied, then $u(\cdot)$ is a classical solution.*

Proof. For $i = 0, 1, \dots, N$, the function $u_{|_{[s_i,t_{i+1}]}}$ is a mild solution of the problem (2.4)-(2.5), which implies that $u_{|_{[s_i,t_{i+1}]}}$ is a classical solution of (2.4)-(2.5). By noting that $u(t) = g(t, u(t))$ for all $t \in (t_i, s_i]$, we conclude that $u(\cdot)$ is a classical solution of (1.1)-(1.3). $\square$

To prove our results on the existence of solutions we introduce the following conditions.

- $\mathbf{H_1}$ The functions g_i are continuous and there are positive constants L_{g_i} such that $\|g_i(t,x)-g_i(t,y)\| \le L_{g_i}\|x-y\|$ for all $x,y \in X$, $t \in (t_i,s_i]$ and each $i=0,1,\dots,N$.
- $\mathbf{H_2}$ For $x \in X$, the function $f(\cdot,x)$ is strongly measurable on $[0,a]$ and $f(t,\cdot) \in C(X,X)$ for $t \in [0,a]$. There are $m_f \in L^1([0,a];\mathbb{R}^+)$ and a nondecreasing function $W_f \in C([0,\infty);\mathbb{R}^+)$ such that $\|f(t,x)\| \le m_f(t)W_f(\|x\|)$ for all $(t,x) \in [0,a]\times X$.
- $\mathbf{H_3}$ The function $f(\cdot)$ belongs to $C([0,a]\times X;X)$, and there is a function $L_f \in L^1([0,a];\mathbb{R}^+)$ such that $\|f(t,x)-f(t,y)\| \le L_f(t)\|x-y\|$ for all $x,y \in X$ and every $t \in [0,a]$.

We can now establish our first result.

Theorem 2.1. *Assume the conditions* $\mathbf{H_1}$ *and* $\mathbf{H_3}$ *are satisfied and*

$$\Theta = C_0 \max\left\{L_{g_i} + \|L_f\|_{L^1([s_i,t_{i+1}])},\ \|L_f\|_{L^1([0,t_1])} : i=1,\dots,N\right\} < 1.$$

Then there exists a unique mild solution $u \in \mathcal{PC}(X)$ *of the problem* (1.1)-(1.3).

Proof. Let $\Gamma : \mathcal{PC}(X) \to \mathcal{PC}(X)$ be defined by $\Gamma u(0) = x_0$, $\Gamma u(t) = g_i(t,u(t))$ for $t \in (t_i,s_i]$ and

$$\Gamma u(t) = T(t-s_i)g_i(s_i,u(s_i)) + \int_{s_i}^{t} T(t-s)f(s,u(s))ds, \qquad t \in [s_i,t_{i+1}]. \tag{2.6}$$

From the assumption it is easy to see that Γ is well defined. Moreover, for $u,v \in \mathcal{PC}(X)$, $i \in \{1,\dots,N\}$ and $t \in [s_i,t_{i+1}]$ we get

$$\begin{aligned}\|\Gamma u(t)-\Gamma v(t)\| &\le \|T(t-s_i)g(s_i,u(s_i)) - T(t-s_i)g(s_i,v(s_i))\| \\ &\quad + C_0\int_{s_i}^{t}\|f(s,u(s))-f(s,v(s))\|ds \\ &\le C_0L_{g_i}\|u-v\|_{\mathcal{PC}(X)} + C_0\int_{s_i}^{t} L_f(s)\|u(s)-v(s)\|ds,\end{aligned}$$

and hence,

$$\|\Gamma u - \Gamma v\|_{C([s_i,t_{i+1}];X)} \le C_0\left(L_{g_i} + \|L_f\|_{L^1([s_i,t_{i+1}])}\right)\|u-v\|_{\mathcal{PC}(X)}.$$

Proceeding as above, we obtain that

$$\begin{aligned}\|\Gamma u - \Gamma v\|_{C([0,t_1];X)} &\le C_0\|L_f\|_{L^1([0,t_1])}\|u-v\|_{\mathcal{PC}(X)}, \\ \|\Gamma u - \Gamma v\|_{C((t_j,s_j];X)} &\le C_0L_{g_j}\|u-v\|_{\mathcal{PC}(X)}, \qquad j=1,\dots,N.\end{aligned}$$

From the above we have that $\|\Gamma u-\Gamma v\|_{\mathcal{PC}(X)} \le \Theta\|u-v\|_{\mathcal{PC}(X)}$, which implies that $\Gamma(\cdot)$ is a contraction and there exists a unique mild solution of (1.1)-(1.3). □

In the next result we establish the existence of a mild solution via a fixed point criterion for condensing operators.

Theorem 2.2. *Assume the conditions* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *are satisfied, the semigroup* $(T(t))_{t\geq 0}$ *is compact, the functions* $g_i(\cdot,0)$ *are bounded and*

$$\begin{aligned} C_0\|m_f\|_{L^1([0,t_1])}\limsup_{r\to\infty}\frac{W(r)}{r} &< 1,\\ C_0\|m_f\|_{L^1([s_i,t_{i+1}])}\limsup_{r\to\infty}\frac{W(r)}{r}+(C_0+1)L_{g_i} &< 1, \qquad \forall\, i=1,\dots,N. \end{aligned}$$

Then there exists a mild solution $u\in\mathcal{PC}(X)$ *of the problem* (1.1)-(1.3).

Proof. Let $r>1$ and $0<\theta<1$ be such that

$$\begin{aligned} C_0\|x_0\|+(C_0+1)\max_{i=1,\dots,N}\|g_i(\cdot,0)\|_{C((t_i,s_i];X)} &< (1-\theta)r,\\ C_0\max_{i=1,\dots,N}\left\{\|m_f\|_{L^1([s_i,t_{i+1}])}\frac{W(s)}{s}+(C_0+1)L_{g_i}\right\} &< \theta, \qquad \forall\, s\geq r,\\ C_0\|m_f\|_{L^1([0,t_1])}\frac{W(s)}{s} &< \theta, \qquad \forall\, s\geq r. \end{aligned} \tag{2.7}$$

Next, we prove that the map Γ introduced in the proof of Theorem 2.1 is a condensing map from $B_r(0,\mathcal{PC}(X))$ into $B_r(0,\mathcal{PC}(X))$. To this end, we introduce the decomposition $\Gamma=\sum_{i=0}^N\Gamma_i^1+\sum_{i=0}^N\Gamma_i^2$, where $\Gamma_i^j:\mathcal{PC}(X)\to\mathcal{PC}(X)$, $i=0,\dots,N$, $j=1,2$, are given by

$$\begin{aligned} \Gamma_i^1u(t) &= \begin{cases} g_i(t,u(t)), & \text{for } t\in(t_i,s_i], i\geq 1,\\ T(t-s_i)g_i(s_i,u(s_i)), & \text{for } t\in(s_i,t_{i+1}], i\geq 1,\\ 0 & \text{for } t\notin(t_i,t_{i+1}],\ i\geq 0,\\ T(t)x_0 & \text{for } t\in[0,t_1],\ i=0,\end{cases}\\ \Gamma_i^2u(t) &= \begin{cases} \displaystyle\int_{s_i}^t T(t-s)f(s,u(s))ds, & \text{for } t\in(s_i,t_{i+1}],\ i\geq 0,\\ 0 & \text{for } t\notin(s_i,t_{i+1}],\ i\geq 0.\end{cases} \end{aligned}$$

We divide the remainder of the proof into five steps.

Step 1. $\Gamma B_r(0,\mathcal{PC}(X))\subset B_r(0,\mathcal{PC}(X))$.

Let $u\in B_r(0,\mathcal{PC}(X))$. For $i\geq 1$ and $t\in(t_i,t_{i+1}]$, we get

$$\begin{aligned} \|\Gamma u(t)\| &\leq L_{g_i}\|u(t)\|+\|g_i(t,0)\|+\ C_0(L_{g_i}\|u(t)\|+\|g_i(t,0)\|)\\ &\quad + C_0\int_{s_i}^{t_{i+1}} m_f(s)W(\|u(s)\|)\\ &\leq (C_0+1)L_{g_i}\|u\|_{\mathcal{PC}(X)}+\ (C_0+1)\|g_i(\cdot,0)\|_{C((t_i,s_i];X)}\\ &\quad + C_0W(r)\|m_f\|_{L^1([s_i,t_{i+1}])}\\ &\leq (C_0+1)L_{g_i}r+(1-\theta)r+C_0W(r)\|m_f\|_{L^1([s_i,t_{i+1}])}\\ &\leq (1-\theta)r+\theta r, \end{aligned}$$

which implies that $\|\Gamma u\|_{C((t_i,t_{i+1}];X)}\leq r$ for all $i\geq 1$. Arguing as above, we find that

$$\|\Gamma u\|_{C([0,t_1],X)}\leq C_0\|x_0\|+\ C_0W(r)\|m_f\|_{L^1([0,t_1])}\leq r,$$

from which we infer $\|\Gamma u\|_{\mathcal{PC}(X)}\leq r$ and Γ has values in $B_r(0,\mathcal{PC}(X))$.

Step 2. The map $\Gamma_1 = \sum_{i=0}^{N} \Gamma_i^1$ is a contraction on $B_r(0, \mathcal{PC}(X))$.

For $u, v \in B_r(0, \mathcal{PC}(X))$, $i \in \{1, \ldots, N\}$ and $t \in (t_i, t_{i+1}]$, it is easy to see that

$$\|\Gamma_i^1 u(t) - \Gamma_i^1 v(t)\|_{\mathcal{PC}(X)} \leq (C_0 + 1) L_{g_i} \|u - v\|_{C((t_i, t_{i+1}]; X)},$$

which implies that $\| \sum_{i=0}^{N} \Gamma_i^1 u - \sum_{i=0}^{N} \Gamma_i^1 v\|_{\mathcal{PC}(X)} \leq \Theta \|u - v\|_{\mathcal{PC}(X)}$ and Γ^1 is a contraction on $B_r(0, \mathcal{PC}(X))$.

Next, we use the notation $\Gamma_i^2 B_r(0, \mathcal{PC}(X))(t) = \{\Gamma_i^2 u(t) : B_r(0, \mathcal{PC}(X))\}$.

Step 3. For $i = 0, \ldots, N$ and $s_i < s < t \leq t_{i+1}$, the set $\bigcup_{\tau \in [s,t]} \Gamma_i^2 B_r(0, \mathcal{PC}(X))(\tau)$ is relatively compact in X.

Let $s_i < \mu < s$. For $\varepsilon > 0$ we select $0 < \delta < \frac{s-\mu}{2}$ such that

$$C_0 \|m_f\|_{L^1(I)} W(r) \leq \varepsilon$$

for all intervals $I \subset [0, a]$ with $\mathrm{Diam}(I) \leq \delta$. Then, for $\tau \in [s, t]$ and $u \in B_r(0, \mathcal{PC}(X))$ we get

$$\begin{aligned} \Gamma_i^2 u(\tau) &= T(\delta) \int_{s_i}^{\tau - \delta} T(\tau - \theta - \delta) f(\theta, u(\theta)) d\theta + \int_{\tau - \delta}^{\tau} T(\tau - \theta) f(\theta, u(\theta)) d\theta \\ &\in T(\delta) B_{r_1}(0, X) + B_{r_{1,\varepsilon}}(0, X), \end{aligned}$$

where $r_1 = C_0 \|m_f\|_{L^1([0,a])} W(r)$ and $r_{1,\varepsilon} = C_0 \|m_f\|_{L^1([\tau-\delta,\tau])} W(r)$, which implies that $\bigcup_{\theta \in [s,t]} \Gamma_i^2 B_r(0, \mathcal{PC}(X))(\theta) \subset T(\delta) B_{r_1}(0, X) + B_\varepsilon(0, X)$. Since $T(\delta) B_{r_1}(0, X)$ is relatively compact and $\mathrm{Diam}(B_\varepsilon(0, X)) \to 0$ as $\varepsilon \to 0$, it follows that $\bigcup_{\theta \in [s,t]} \Gamma_i^2 B_r(0, \mathcal{PC}(X))(\tau)$ is relatively compact in X.

In the next step we use the notation introduced in (1.4).

Step 4. The set of functions $[\Gamma_i^2 B_r \widetilde{(0, \mathcal{PC}(X))}]_i$, $i = 0, \ldots, N$, is an equicontinuous subset of $C([t_i, t_{i+1}]; X)$.

It is obvious that $[\Gamma_i^2 B_r \widetilde{(0, \mathcal{PC}(X))}]_i$ is right equicontinuous on $[t_i, s_i)$ and left equicontinuous on $(t_i, s_i]$. Assume $t \in (s_i, t_{i+1})$. Since the set $\Gamma_i^2 B_r(0, \mathcal{PC}(X))(t)$ is relatively compact in X and $(T(t))_{t \geq 0}$ is a C_0-semigroup, for given $\varepsilon > 0$ there exists $0 < \delta < t_{i+1} - t$ such that $\|(T(s) - I)x\| \leq \varepsilon$ for all $0 < s < \delta$ and each $x \in \Gamma_i^2 B_r(0, \mathcal{PC}(X))(t)$. Then, for $u \in B_r(0, \mathcal{PC}(X))$ and $0 < h < \delta$ we get

$$\begin{aligned} &\|\widetilde{\Gamma_i^2} u(t+h) - \widetilde{\Gamma_i^2} u(t)\| = \|\Gamma_i^2 u(t+h) - \Gamma_i^2 u(t)\| \\ &\quad \leq \quad \| \int_t^{t+h} T(t+h-s) f(s, u(s)) ds\| + \|(T(h) - I) \int_{s_i}^{t} T(t-s) f(s, u(s)) ds\| \\ &\quad \leq \quad C_0 \|m_f\|_{L^1([t,t+h])} W(r) + \sup\{ \|(T(h) - I)x\| \, : x \in \Gamma_i^2 B_r(0, \mathcal{PC}(X))(t) \} \\ &\quad \leq \quad C_0 \|m_f\|_{L^1([t,t+h])} W(r) + \varepsilon, \end{aligned}$$

which proves that $[\Gamma_i^2 B_r \widetilde{(0, \mathcal{PC}(X))}]_i$ is right equicontinuous at t.

Proceeding as above, for $t = s_i$ and $h > 0$ with $s_i + h < t_{i+1}$ we have that

$$\begin{aligned} \|\widetilde{\Gamma_i^2} u(s_i + h) - \widetilde{\Gamma_i^2} u(s_i)\| &= \| \int_{s_i}^{s_i + h} T(t+h-s) f(s, u(s)) ds\| \\ &\leq C_0 \|m_f\|_{L^1([s_i, t+s_i])} W(r), \end{aligned}$$

which implies that $[\Gamma_i^2 B_r \widetilde{(0, \mathcal{PC}(X))}]_i$ is right equicontinuous at s_i.

Suppose now that $t \in (s_i, t_{i+1}]$. Let $\mu \in (s_i, t]$. Since $\bigcup_{s\in[\mu,t]} \Gamma_i^2 B_r(0,\mathcal{PC}(X))(s)$ is relatively compact in X (see Step 3), for $\varepsilon > 0$ given we select $0 < \delta < \frac{t-\mu}{2}$ such that $\|(I - T(h))x\| \leq \varepsilon$ for all $0 < h \leq \delta$ and each $x \in \bigcup_{s\in[\mu,t]} \Gamma_i^2 B_r(0,\mathcal{PC}(X))(s)$. Under these conditions, for $0 < h \leq \delta$ and $u \in B_r(0,\mathcal{PC}(X))$ we see that

$$
\begin{aligned}
\|\widetilde{\Gamma_i^2}u(t-h) - \widetilde{\Gamma_i^2}u(t)\| = \|\Gamma_i^2 u(t-h) - \Gamma_i^2 u(t)\| \\
&\leq \int_{t-h}^{t} \|T(t-s)f(s,u(s))\|ds \\
&\quad + \|(I - T(h))\int_{s_i}^{t-h} T(t-h-s)f(s,u(s))ds\| \\
&\leq C_0\|m_f\|_{L^1([t-h,t])}W(r) + \|(I-T(h))\Gamma_i^2 u(t-h)\| \\
&\leq C_0\|m_f\|_{L^1([t-h,t])}W(r) \\
&\quad + \sup\{\ (I-T(h))x\| \ : x \in \bigcup_{s\in[\mu,t]} \Gamma_i^2 B_r(0,\mathcal{PC}(X))(s)\ \} \\
&\leq C_0\|m_f\|_{L^1([t,t+h])}W(r) + \varepsilon,
\end{aligned}
$$

which shows that $[\Gamma_i^2 B_r\widetilde{(0,\mathcal{PC}}(X))]_i$ is left equicontinuity at $t \in (s_i, t_{i+1}]$. This completes the proof that the set $[\Gamma_i^2 B_r\widetilde{(0,\mathcal{PC}}(X))]_i$ is equicontinuous.

The proof of the next assertion is obvious.

Step 5. For $i \neq j$, the set $[\Gamma_i^2 B_r\widetilde{(0,\mathcal{PC}}(X))]_j$ is an equicontinuous subset of $C([t_j, t_{j+1}]; X)$.

From the above steps and Lemma 1.1 it follows that Γ_1 is a contraction, Γ_2 is completely continuous and $\Gamma = \Gamma_1 + \Gamma_2$ is a condensing operator from $B_r(0,\mathcal{PC}(X))$ into $B_r(0,\mathcal{PC}(X))$. Finally, from [17, Theorem 4.3.2] we infer there exists a mild solution of (1.1)-(1.3). $\square$

We complete this section with a result on the existence of a classical solution. From semigroup theory and Proposition 2.2, we establish without proof the following result.

Proposition 2.3. *Assume $u(\cdot)$ is a mild solution of* (1.1)-(1.3), *$x_0 \in D(A)$, $g(s_i, u(s_i) \in D(A)$ for all $i = 1, \ldots, N$ and $f \in C^1([0,a]; X)$. Then $u(\cdot)$ is a classical solution.*

3. Example

In this section, $X = L^2([0,\pi])$ and $A : D(A) \subset X \to X$ is the operator given by $Ax = x''$ on $D(A) := \{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a compact semigroup $(T(t))_{t\geq 0}$ on X and that $\|T(t)\| \leq e^{-t}$ for all $t \geq 0$.

Consider the impulsive problem

$$\frac{\partial}{\partial t}w(t,\xi) = \frac{\partial^2}{\partial \xi^2}w(t,\xi) + F(t,w(t,\xi)),\ (t,\xi) \in \bigcup_{i=1}^{N}[s_i,t_{i+1}] \times [0,\pi], \tag{3.1}$$

$$w(t,0) = w(t,\pi) = 0, \qquad t \in [0,a], \tag{3.2}$$

$$w(0,\xi) = z(\xi), \qquad \xi \in [0,\pi], \tag{3.3}$$

$$w(t,\xi) = G_i(t,w(t,\xi)), \qquad \xi \in [0,\pi], t \in (t_i,s_i], \tag{3.4}$$

where $0 = t_0 = s_0 < t_1 \leq s_1 < ... < t_N \leq s_N < t_{N+1} = a$ are fixed real numbers, $z \in X$, $F \in C([0,a] \times \mathbb{R};\mathbb{R})$ and $G_i \in C((t_i,s_i] \times \mathbb{R};\mathbb{R})$ for all $i = 1,\ldots,N$.

To represent the impulsive problem (3.1)-(3.4) in the abstract form (1.1)-(1.3) we introduce the functions $f : [0,a] \times X \to X$ and $g_i : (t_i,s_i] \times X \to X$ defined by $f(t,x)(\xi) = F(t,x(\xi))$ and $g_i(t,x)(\xi) = G_i(t,x(\xi))$. Next, we say that $u \in \mathcal{PC}(X)$ is a mild solution of (3.1)-(3.4) if $u(\cdot)$ is a mild solution of the associated abstract problem (1.1)-(1.3). The next result follows from Theorem 2.1 and Theorem 2.2.

Proposition 3.1. *If any of the following conditions is satisfied, then there exists a mild solution $u \in \mathcal{PC}(X)$ of (3.1)-(3.3).*

- **(a)** *The functions F and G_i are Lipschitz with Lipschitz constants L_F and L_{G_i} respectively and* $\max\{L_{G_i} + L_F(t_{i+1} - s_i), L_F t_1 : i = 1,\ldots,N\} < 1$.
- **(b)** *The functions $G_i(\cdot)$ are Lipschitz with Lipschitz constants L_{G_i}, the function $F(\cdot)$ is bounded and $2L_{G_i} < 1$ for all $i = 1,\ldots,N$.*

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