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On a new class of fractional operators

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Abstract
This manuscript is based on the standard fractional calculus iteration procedure on conformable derivatives. We introduce new fractional integration and differentiation operators. We define spaces and present some theorems related to these operators.

Keywords: conformable derivatives; fractional conformable integrals; fractional conformable derivatives

1 Introduction

In the area of fractional calculus and its applications in many branches of science and engineering, several fractional derivatives were mainly utilized. The most common used were Caputo and Riemann-Liouville derivatives, which were successfully utilized in modeling complex dynamics appearing in physics, biology, engineering and many other fields [1–5]. As is well known, systems possessing a memory effect often appear in real world phenomena. However, for each type of data we always ask what is the optimal corresponding nonlocal model to be applied. Moreover, many authors studied new fractional operators with local, nonlocal, singular and non-singular kernels (see [6–13] and the references therein). The standard fractional calculus may not provide us the required kernel in order to extract important information from these types of systems. At this stage, we ask the following question. Can we generalize the standard fractional Riemann-Liouville integrals in a way such that we obtain unification to Riemann-Liouville, Hadamard and other fractional derivatives [14, 15]. The core of this procedure is to decide which differentiation operator should be used as a starting point for the iteration procedure. For the standard fractional calculus, we iterate the usual integral of a function and using the Cauchy formula we obtain the integral of higher integer orders and then replace this integer by any complex number. In [16], it was suggested that the conformable integral should be fractionalized properly. We recall that an integral type like the one from [16] has appeared already in [17]. The integral mentioned below in (2) appears in mathematical economics, namely they are used for describing discounting economical dynamics [17]. Also,this integral appears in describing the non-linear dissipative systems [17].

At this point we should say that the left and right conformable derivatives defined in [16], respectively, as

$$ {}_aT^{\alpha}f(x) = (x-a)^{1-\alpha}f'(x) \quad \text{and} \quad T_b^{\alpha}f(x) = (b-x)^{1-\alpha}f'(x), \tag{1} $$

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where f is a differentiable function, are local derivatives whose corresponding left and right integrals have the forms [16]

$$\left({}_aI^\alpha f\right)(x) = \int_a^x f(t)\frac{dt}{(t-a)^{1-\alpha}}, \quad 0 < \alpha < 1 \tag{2}$$

and

$$\left(I_b^\alpha f\right)(x) = \int_x^b f(t)\frac{dt}{(b-t)^{1-\alpha}}, \quad 0 < \alpha < 1, \tag{3}$$

respectively. We suggest that iterating the above integral will end up with new fractional operators with two parameters and kernels different from the usual kernels of usual fractional derivatives and integrals. From the data analysis point of view we suppose that this new type of calculus will provide better understanding of the complexity of the dynamics of the phenomena from porous media.

Depending on [2, 4, 5], in what follows, we recall some basic definitions and tools about classical fractional calculus.

For $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ the left Riemann-Liouville fractional integral of order α starting from a has the following form:

$$\left({}_a\mathbf{I}^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-y)^{\alpha-1} f(y)\,dy, \tag{4}$$

while the right Riemann-Liouville fractional integral of order $\alpha > 0$ ending at $b > a$ is defined by

$$\left(\mathbf{I}_b^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (y-t)^{\alpha-1} f(y)\,dy. \tag{5}$$

For $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geq 0$, the left Riemann-Liouville fractional derivative of order α starting at a is given below

$$\left({}_aD^\alpha f\right)(t) = \left(\frac{d}{dt}\right)^n \left({}_a\mathbf{I}^{n-\alpha} f\right)(t), \quad n = [\alpha] + 1. \tag{6}$$

Meanwhile, the right Riemann-Liouville fractional derivative of order α ending at b becomes

$$\left(D_b^\alpha f\right)(t) = \left(-\frac{d}{dt}\right)^n \left(\mathbf{I}_b^{n-\alpha} f\right)(t). \tag{7}$$

The left Caputo fractional derivative of order $\alpha, \mathrm{Re}(\alpha) \geq 0$ starting from a has the following form:

$$\left({}_a^CD^\alpha f\right)(t) = \left({}_a\mathbf{I}^{n-\alpha} f^{(n)}\right)(t), \quad n = [\alpha] + 1, \tag{8}$$

while the right Caputo fractional derivative ending at b becomes

$$\left({}^CD_b^\alpha f\right)(t) = \left(\mathbf{I}_b^{n-\alpha} (-1)^n f^{(n)}\right)(t). \tag{9}$$

Hadamard-type fractional integrals and derivatives were introduced in [18] as:

The left Hadamard fractional integral of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ starting from a has the following form:

$$\big({}_a\mathfrak{J}^\alpha f\big)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\ln t - \ln y)^{\alpha-1} f(y) \frac{dy}{y} \tag{10}$$

and the right Hadamard fractional integral of order α ending at $b > a$ is defined by

$$\big(\mathfrak{J}_b^\alpha f\big)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\ln y - \ln t)^{\alpha-1} f(y) \frac{dy}{y}. \tag{11}$$

The left Hadamard fractional derivative of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geq 0$ starting at a is given as:

$$\big({}_a\mathfrak{D}^\alpha f\big)(t) = \left(t\frac{d}{dt}\right)^n \big({}_aI^{n-\alpha} f\big)(t), \quad n = [\alpha] + 1, \tag{12}$$

whereas the right Hadamard fractional derivative of order α ending at b becomes

$$\big(\mathfrak{D}_b^\alpha f\big)(t) = \left(-t\frac{d}{dt}\right)^n \big(I_b^{n-\alpha} f\big)(t). \tag{13}$$

In [19–21], the authors defined the left and right Caputo-Hadamard fractional derivatives of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geq 0$, respectively, as

$$\big({}_a^C\mathfrak{D}^\alpha f\big)(t) = {}_a\mathfrak{D}^\alpha \left[f(y) - \sum_{k=0}^{n-1} \frac{\delta^k f(a)}{k!} (\ln y - \ln a)^k \right](t), \quad \delta = t\frac{d}{dt}, \tag{14}$$

and in the space $AC_\delta^n[a,b] = \{g : [a,b] \to \mathbb{C} : \delta^{n-1}[g(x)] \in AC[a,b]\}$ equivalently by

$$\big({}_a^C\mathfrak{D}^\alpha f\big)(t) = \left({}_a\mathfrak{J}^{n-\alpha} \left(t\frac{d}{dt}\right)^n f\right)(t), \quad n = [\alpha] + 1, \tag{15}$$

and

$$\big({}^C\mathfrak{D}_b^\alpha f\big)(t) = \mathfrak{D}_b^\alpha \left[f(y) - \sum_{k=0}^{n-1} \frac{(-1)^k \delta^k f(b)}{k!} (\ln b - \ln y)^k \right](t); \tag{16}$$

and in the space $AC_\delta^n[a,b]$ equivalently by

$$\big({}^C\mathfrak{D}_b^\alpha f\big)(t) = \left(\mathfrak{J}_b^{n-\alpha} \left(-t\frac{d}{dt}\right)^n f\right)(t). \tag{17}$$

For $a < b, c \in \mathbb{R}$ and $1 \leq p < \infty$, define the function space

$$X_c^p(a,b) = \left\{ f : [a,b] \to \mathbb{R} : \|f\|_{X_c^p} = \left(\int_a^b \left|t^c f(t)\right|^p \frac{dt}{t} \right)^{1/p} < \infty \right\}.$$

For $p = \infty$, $\|f\|_{X_c^p} = \operatorname{ess\,sup}_{a \le t \le b}[t^c|f(t)|]$. In the frame of the above function space, the generalized left- and right-fractional integrals in the sense of Katugampola in [14] have the forms

$$\left({}_aI^{\alpha,\rho}f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - y^\rho}{\rho}\right)^{\alpha-1} f(y)\frac{dy}{y^{1-\rho}} \tag{18}$$

and

$$\left(I_b^{\alpha,\rho}f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\frac{y^\rho - t^\rho}{\rho}\right)^{\alpha-1} f(y)\frac{dy}{y^{1-\rho}}, \tag{19}$$

respectively.

The left and right generalized fractional derivatives of order $\alpha > 0$ are defined by [15]

$$\left({}_aD^{\alpha,\rho}f\right)(t) = \gamma^n\left({}_aI^{n-\alpha,\rho}f\right)(t) = \frac{\gamma^n}{\Gamma(n-\alpha)}\int_a^t \left(\frac{t^\rho - y^\rho}{\rho}\right)^{n-\alpha-1} f(y)\frac{dy}{y^{1-\rho}} \tag{20}$$

and

$$\left(D_b^{\alpha,\rho}f\right)(t) = (-\gamma)^n\left({}_aI^{n-\alpha,\rho}f\right)(t) = \frac{(-\gamma)^n}{\Gamma(n-\alpha)}\int_t^b \left(\frac{y^\rho - t^\rho}{\rho}\right)^{n-\alpha-1} f(y)\frac{dy}{y^{1-\rho}}, \tag{21}$$

respectively, where $\rho > 0$ and where $\gamma = t^{1-\rho}\frac{d}{dt}$.

Depending on [15], the authors in [22] presented the Caputo modification of the left and right generalized fractional derivatives, respectively, by

$$\left({}_a^CD^{\alpha,\rho}f\right)(t) = \left({}_aI^{n-\alpha,\rho}\gamma^nf\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \left(\frac{t^\rho - u^\rho}{\rho}\right)^{n-\alpha-1} \gamma^nf(y)\frac{dy}{y^{1-\rho}} \tag{22}$$

and

$$\left({}^CD_b^{\alpha,\rho}f\right)(t) = \left({}_aI^{n-\alpha,\rho}(-\gamma)^nf\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_t^b \left(\frac{y^\rho - t^\rho}{\rho}\right)^{n-\alpha-1} (-\gamma)^nf(y)\frac{dy}{y^{1-\rho}}. \tag{23}$$

This article is organized as follows. In Section 2, we define the left- and right-fractional conformable integrals and derivatives. In Section 3, we define the fractional conformable derivatives of functions belonging to certain spaces and state their properties. In Section 4 we present the fractional conformable derivatives in the Caputo setting and state their properties. Finally, the last section is devoted to our conclusion.

2 The fractional conformable integrals and derivatives

The left and right conformable integrals were defined in [16] as can be seen in (1) and (2). Moreover, left and right conformable integrals were extended to higher order in [16] so that for $\alpha = n+1$ we have $({}_aI^\alpha f)(x) = ({}_a\mathbf{I}^\alpha f)(x)$ and $(I_b^\alpha f)(x) = (\mathbf{I}_b^\alpha f)(x)$.

Now, iterating the integral in (2) n times and by interchanging the order of integrals will result in the following:

$$
\begin{aligned}
{}_aI^{n,\alpha}f(x) &= \int_a^x \frac{dt_1}{(t_1-a)^{1-\alpha}} \int_a^{t_1} \frac{dt_2}{(t_2-a)^{1-\alpha}} \cdots \int_a^{t_{n-1}} \frac{f(t_n)\,dt_n}{(t_n-a)^{1-\alpha}} \\
&= \frac{1}{\Gamma(n)} \int_a^x \left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{n-1} f(t)\frac{dt}{(t-a)^{1-\alpha}}. \qquad (24)
\end{aligned}
$$

Definition 2.1 Replacing the integer n by any number $\beta \in \mathbb{C}$, $\operatorname{Re}(\beta) > 0$, we define the left-fractional conformable integral operator by

$$
{}^{\beta}_{a}\mathfrak{I}^{\alpha}f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1} f(t)\frac{dt}{(t-a)^{1-\alpha}}. \qquad (25)
$$

The fractional integral in (25) coincides with the Riemann-Liouville fractional integral (4) when $a = 0$ and $\alpha = 1$. It also coincides with the Hadamard fractional integral (10) once $a = 0$ and $\alpha \to 0$ and with the generalized fractional integral (18) when $a = 0$. Similarly, we can state the following.

Definition 2.2 The right-fractional conformable integral of order $\beta \in \mathbb{C}$, $\operatorname{Re}(\beta) > 0$ is defined by

$$
{}^{\beta}\mathfrak{I}^{\alpha}_{b}f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \left(\frac{(b-x)^\alpha-(b-t)^\alpha}{\alpha}\right)^{\beta-1} f(t)\frac{dt}{(b-t)^{1-\alpha}}. \qquad (26)
$$

Notice that, if $(Qf)(t) = f(a+b-t)$, then we have $({}^{\beta}_{a}\mathfrak{I}^{\alpha}Qf)(x) = Q({}^{\beta}\mathfrak{I}^{\alpha}_{b}f)(x)$. Moreover, (26) coincides with the Riemann-Liouville fractional integral (5) when $b = 0$ and $\alpha = 1$. It also coincides with the Hadamard fractional integral (11) once $b = 0$ and $\alpha \to 0$ and with the generalized fractional integral (19) when $b = 0$.

We now state the definition of fractional conformable derivatives.

Definition 2.3 We define the left- and right-fractional conformable derivatives of order $\beta \in \mathbb{C}$, $\operatorname{Re}(\beta) \geq 0$ in Riemann-Liouville setting, respectively, by

$$
\begin{aligned}
{}^{\beta}_{a}\mathfrak{D}^{\alpha}f(x) &= {}^{n}_{a}T^{\alpha}\left({}^{n-\beta}_{a}\mathfrak{I}^{\alpha}\right)f(x) \\
&= \frac{{}^{n}_{a}T^{\alpha}}{\Gamma(n-\beta)} \int_a^x \left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{n-\beta-1} f(t)\frac{dt}{(t-a)^{1-\alpha}}, \qquad (27) \\
{}^{\beta}\mathfrak{D}^{\alpha}_{b}f(x) &= {}^{n}T^{\alpha}_{b}\left({}^{n-\beta}\mathfrak{I}^{\alpha}_{b}\right)f(x) \\
&= \frac{(-1)^{n}\,{}^{n}T^{\alpha}_{b}}{\Gamma(n-\beta)} \int_x^b \left(\frac{(b-x)^\alpha-(b-t)^\alpha}{\alpha}\right)^{n-\beta-1} f(t)\frac{dt}{(b-t)^{1-\alpha}}, \qquad (28)
\end{aligned}
$$

where

$$
n = \left[\operatorname{Re}(\beta)\right]+1, \quad {}^{n}_{a}T^{\alpha} = \underbrace{{}_aT^{\alpha}\,{}_aT^{\alpha}\cdots{}_aT^{\alpha}}_{n \text{ times}}, \qquad {}^{n}T^{\alpha}_{b} = \underbrace{T^{\alpha}_{b}T^{\alpha}_{b}\cdots T^{\alpha}_{b}}_{n \text{ times}}, \qquad (29)
$$

and ${}_aT^{\alpha}$ and T^{α}_{b} are the left and right conformable differential operators presented in (1).

The fractional derivative in (27) coincides with the Riemann-Liouville fractional derivative (6) when $a = 0$ and $\alpha = 1$, the Hadamard fractional derivative (12) once $a = 0$ and $\alpha \to 0$ and with the generalized fractional integral (20) when $a = 0$. Whereas the fractional derivative in (28) coincides with the Riemann-Liouville fractional derivative (7) when $b = 0$ and $\alpha = 1$, it coincides with the Hadamard fractional integral (13) once $b = 0$ and $\alpha \to 0$ and with the generalized fractional integral (21) when $b = 0$.

Now we consider some properties of the fractional conformable integrals and derivatives.

Theorem 2.1 *Let* $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > 0$. *Then*

$$ {}_a^\beta\mathfrak{I}^\alpha \left({}_a^\gamma\mathfrak{I}^\alpha\right) f(x) = {}_a^{\beta+\gamma}\mathfrak{I}^\alpha f(x), \qquad {}^\beta\mathfrak{I}_b^\alpha \left({}^\gamma\mathfrak{I}_b^\alpha\right) f(x) = {}^{\beta+\gamma}\mathfrak{I}_b^\alpha f(x). \tag{30} $$

Proof

$$ \begin{aligned} {}_a^\beta\mathfrak{I}^\alpha \left({}_a^\gamma\mathfrak{I}^\alpha\right) f(x) &= \frac{1}{\Gamma(\beta)\Gamma(\gamma)} \int_a^x \int_a^t \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{\beta-1} \\ &\quad \times \left(\frac{(t-a)^\alpha - (u-a)^\alpha}{\alpha}\right)^{\gamma-1} f(u) \frac{du}{(u-a)^{1-\alpha}} \frac{dt}{(t-a)^{1-\alpha}} \\ &= \frac{1}{\Gamma(\beta)\Gamma(\gamma)\alpha^{\beta+\gamma-2}} \int_a^x \int_u^x \left((x-a)^\alpha - (t-a)^\alpha\right)^{\beta-1} \\ &\quad \times \left((t-a)^\alpha - (u-a)^\alpha\right)^{\gamma-1} f(u) \frac{dt}{(t-a)^{1-\alpha}} \frac{du}{(u-a)^{1-\alpha}} \\ &= \frac{1}{\Gamma(\beta)\Gamma(\gamma)\alpha^{\beta+\gamma-1}} \int_a^x \left((x-a)^\alpha - (u-a)\right)^{\beta+\gamma-1} f(u) \frac{du}{(u-a)^{1-\alpha}} \\ &\quad \times \int_0^1 (1-z)^{\beta-1} z^{\gamma-1}\, dy \\ &= \frac{1}{\Gamma(\beta+\gamma)} \int_a^x \left(\frac{(x-a)^\alpha - (u-a)}{\alpha}\right)^{\beta+\gamma-1} f(u) \frac{du}{(1-u)^{1-\alpha}} \\ &= {}_a^{\beta+\gamma}\mathfrak{I}^\alpha f(x). \end{aligned} $$

Here we have used the change of variable

$$ z = \frac{(t-a)^\alpha - (u-a)^\alpha}{(x-a)^\alpha - (u-a)^\alpha}. $$

The second formula can be proved in a similar way or by using the action of the Q-operator. □

Lemma 2.2 *For* $\mathrm{Re}(\nu) > 0$, *we have*

$$ \left({}_a^\beta\mathfrak{I}^\alpha (t-a)^{\alpha\nu-\alpha}\right)(x) = \frac{1}{\alpha^\beta} \frac{\Gamma(\nu)}{\Gamma(\beta+\nu)} (x-a)^{\alpha(\beta+\nu-1)}, \tag{31} $$

$$ \left({}^\beta\mathfrak{I}_b^\alpha (b-t)^{\alpha\nu-\alpha}\right)(x) = \frac{1}{\alpha^\beta} \frac{\Gamma(\nu)}{\Gamma(\beta+\nu)} (b-x)^{\alpha(\beta+\nu-1)}. \tag{32} $$

Proof We have

$$\left({}_a^\beta\mathfrak{J}^\alpha(t-a)^{\nu-1}\right)(x)=\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}(t-a)^{\alpha\nu-\alpha}\frac{dt}{(t-a)^{1-\alpha}}.$$

Letting $u=(\frac{t-a}{x-a})^\alpha$, we obtain

$$\left({}_a^\beta\mathfrak{J}^\alpha(t-a)^{\nu-1}\right)(x)=\frac{(x-a)^{\alpha(\beta+\nu-1)}}{\Gamma(\beta)\alpha^\beta}\int_0^1(1-u)^{\beta-1}u^{\nu-1}\,du=\frac{\Gamma(\nu)}{\alpha^\beta\Gamma(\beta+\nu)}(x-a)^{\alpha(\beta+\nu-1)}.$$

Equation (32) can be proved in a similar way or by using the action of the Q-operator. □

Lemma 2.3 *For* $\mathrm{Re}(n-\alpha)>0$, *we have*

$$\left[{}_a^\beta\mathfrak{D}^\alpha(t-a)^{\alpha\nu-\alpha}\right](x)=\alpha^\beta\frac{\Gamma(\nu)}{\Gamma(\nu-\beta)}(x-a)^{\alpha(\nu-\beta-1)},\tag{33}$$

$$\left[{}^\beta\mathfrak{D}_b^\alpha(b-t)^{\alpha\nu-\alpha}\right](x)=\alpha^\beta\frac{\Gamma(\nu)}{\Gamma(\nu-\beta)}(b-x)^{\alpha(\nu-\beta-1)}.\tag{34}$$

Proof The proof can be obtained by a straightforward calculation. □

Remark 2.1 It can be shown that

$${}_a^\beta\mathfrak{D}^\alpha f={}_a^\beta\mathfrak{J}^{-\alpha}f,\qquad {}^\beta\mathfrak{D}_b^\alpha={}^\beta\mathfrak{J}_b^{-\alpha}f.\tag{35}$$

3 Fractional derivatives on the spaces $C^n_{\alpha,a}$ and $C^n_{\alpha,b}$

In this section, we consider the fractional conformable derivatives of functions belonging to spaces stated in the following definitions.

Definition 3.1 For $\alpha\in(0,1]$ and $n=1,2,3,\ldots$, define

$$C^n_{\alpha,a}\left([a,b]\right)=\left\{f:[a,b]\to\mathbb{R}\text{ such that }{}_a^{n-1}T^\alpha f\in I_\alpha\left([a,b]\right)\right\},\tag{36}$$

$$C^n_{\alpha,b}\left([a,b]\right)=\left\{f:[a,b]\to\mathbb{R}\text{ such that }{}^{n-1}T_b^\alpha f\in{}_\alpha I\left([a,b]\right)\right\},\tag{37}$$

where $I_\alpha([a,b])$ and ${}_\alpha I([a,b])$ are the spaces defined in Definition 3.1 in [16].

Lemma 3.1 *Let* $\alpha>0$. *A function* $f\in C^n_{\alpha,a}([a,b])$ *if and only if* f *is presented in the form*

$$f(x)=\frac{1}{(n-1)!}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{n-1}\frac{\psi(t)}{(t-a)^{1-\alpha}}\,dt+\sum_{k=0}^{n-1}\frac{{}_a^nT^\alpha f(a)}{k!}\frac{(x-a)^{\alpha k}}{\alpha^k},\tag{38}$$

where $\psi(t)={}_a^nT^\alpha f(t)$.

Proof Let $f\in C^n_{\alpha,a}([a,b])$. Then ${}_a^{n-1}T^\alpha f\in I_\alpha([a,b])$ and thus

$${}_a^{n-1}T^\alpha f(x)=\int_a^x\psi(t)\frac{dt}{(t-a)^{1-\alpha}}+{}_a^{n-1}T^\alpha f(a),\tag{39}$$

where ψ is a continuous function. Then

$$(x-a)^{1-\alpha}\frac{d}{dx}\left({}_a^{n-2}T^\alpha f(x)\right)=\int_a^x \psi(t)\frac{dt}{(t-a)^{1-\alpha}}+{}_a^{n-1}T^\alpha f(a) \tag{40}$$

and

$$\frac{d}{dx}\left({}_a^{n-2}T^\alpha f(x)\right)=\frac{1}{(x-a)^{1-\alpha}}\int_a^x \psi(t)\frac{dt}{(t-a)^{1-\alpha}}+\frac{{}_a^{n-1}T^\alpha f(a)}{(x-a)^{1-\alpha}}. \tag{41}$$

Integrating we get

$$\begin{aligned}{}_a^{n-2}T^\alpha f(x)&=\int_a^x \frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\psi(t)\frac{dt}{(t-a)^{1-\alpha}}\\&\quad+{}_a^{n-1}T^\alpha f(a)\frac{(x-a)^\alpha}{\alpha}+{}_a^{n-2}T^\alpha f(a).\end{aligned} \tag{42}$$

Dividing by $(x-a)^{1-\alpha}$ and integrating once more we get

$$\begin{aligned}{}_a^{n-3}T^\alpha f(x)&=\frac{1}{2!}\int_a^x \left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^2\psi(t)\frac{dt}{(t-a)^{1-\alpha}}+{}_a^{n-2}T^\alpha f(a)\frac{(x-a)^{2\alpha}}{2\alpha^2}\\&\quad+{}_a^{n-2}T^\alpha f(a)\frac{(x-a)^\alpha}{\alpha}+{}_a^{n-3}T^\alpha f(a).\end{aligned} \tag{43}$$

Repeating the same procedure $n-3$ times, we get

$$\begin{aligned}f(x)&=\frac{1}{(n-1)!}\int_a^x \left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{n-1}\psi(t)\frac{dt}{(t-a)^{1-\alpha}}\\&\quad+\sum_{k=0}^{n-1}\frac{{}_a^kT^\alpha f(a)}{\alpha^k k!}(x-a)^{\alpha k}.\end{aligned} \tag{44}$$

It is clear from (39) that $\psi(t)={}_a^nT^\alpha f(t)$.

Sufficiency is proved by applying the operator ${}_a^nT^\alpha$ to both sides of (38). □

For the right-fractional conformable derivatives, we can state a similar lemma.

Lemma 3.2 *$f\in C^n_{\alpha,b}([a,b])$ if and only if f is presented in the form*

$$\begin{aligned}f(x)&=\frac{1}{(n-1)!}\int_x^b \left(\frac{(b-x)^\alpha-(b-t)^\alpha}{\alpha}\right)^{n-1}\frac{({}^nT_b^\alpha f)(t)}{(b-t)^{1-\alpha}}\,dt\\&\quad+\sum_{k=0}^{n-1}\frac{(-1)^k\,{}^kT_b^\alpha f(b)}{k!}\frac{(b-x)^{\alpha k}}{\alpha^k}.\end{aligned} \tag{45}$$

Proof The proof is similar to the proof of Lemma 3.1. □

In the following theorem we state the fractional derivatives of functions in $C^n_{\alpha,a}$ and $C^n_{\alpha,b}$

Theorem 3.3 *Let $\beta\in\mathbb{C}$, $\mathrm{Re}(\beta)>0$, $n=[\beta]+1$. Then*

(1) *if $f \in C^n_{\alpha,a}([a,b])$, the left-fractional derivative of f of order β exists everywhere and can be represented in the form*

$$ {}^{\beta}_{a}\mathfrak{D}^{\alpha}f(x) = \frac{1}{\Gamma(n-\beta)}\int_a^x \left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{n-\beta-1}\frac{{}^{n}_{a}T^{\alpha}f(t)}{(t-a)^{1-\alpha}}\,dt + \sum_{k=0}^{n-1}\frac{({}^{k}_{a}T^{\alpha}f(a))(x-a)^{\alpha(k-\beta)}}{\alpha^{k-\beta}\Gamma(k-\beta+1)}, \tag{46}$$

(2) *if $f \in C^n_{\alpha,b}([a,b])$, the right-fractional derivative of f of order β exists everywhere and*

$$ {}^{\beta}\mathfrak{D}^{\alpha}_{b}f(x) = \frac{1}{\Gamma(n-\beta)}\int_x^b \left(\frac{(b-x)^\alpha-(b-t)^\alpha}{\alpha}\right)^{n-\beta-1}\frac{{}^{n}T^{\alpha}_{b}f(t)}{(b-t)^{1-\alpha}}\,dt + \sum_{k=0}^{n-1}\frac{((-1)^{k}\,{}^{k}T^{\alpha}_{b}f(b))(b-x)^{\alpha(k-\beta)}}{\alpha^{k-\beta}\Gamma(k-\beta+1)}. \tag{47}$$

Proof We prove (46). The proof of (47) can be done analogously.

Since $f \in C^n_{\alpha,a}[a,b]$, from Lemma 3.1, f should be written as

$$ f(x) = \frac{1}{(n-1)!}\int_a^x \left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{n-1}\frac{({}^{n}_{a}T^{\alpha}f)(t)}{(t-a)^{1-\alpha}}\,dt + \sum_{k=0}^{n-1}\frac{{}^{n}_{a}T^{\alpha}f(a)}{k!}\frac{(x-a)^{\alpha k}}{\alpha^k}. \tag{48}$$

Therefore we have

$$ \begin{aligned} {}^{\beta}_{a}\mathfrak{D}^{\alpha}f(x) &= \frac{{}^{n}_{a}T^{\alpha}}{(n-1)!\Gamma(n-\beta)}\int_a^x\int_a^t \left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{n-\beta-1}\left(\frac{(t-a)^\alpha-(u-a)^\alpha}{\alpha}\right)^{n} \\ &\quad \times \left({}^{n}_{a}T^{\alpha}f(u)\right)\frac{du}{(u-a)^{1-\alpha}}\frac{dt}{(t-a)^{1-\alpha}} \\ &\quad + \sum_{k=0}^{n-1}\frac{{}^{n}_{a}T^{\alpha}f(a)}{k!\alpha^k}\alpha^{\beta}\frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}(x-a)^{\alpha(k-\beta)}. \end{aligned} \tag{49}$$

Using Lemma 2.3, changing the order of integration, letting $y = \frac{(t-a)^\alpha-(u-a)^\alpha}{(x-a)^\alpha-(u-a)^\alpha}$ and using the properties of the gamma and beta functions, we get

$$ \begin{aligned} {}^{\beta}_{a}\mathfrak{D}^{\alpha}f(x) &= \left(\frac{\Gamma(n-\beta)\Gamma(n){}^{n}_{a}T^{\alpha}}{(n-1)!\Gamma(n-\beta)\Gamma(2n-\beta)}\int_a^x \left(\frac{(x-a)^\alpha-(u-a)^\alpha}{\alpha}\right)^{2n-\beta-1}\frac{({}^{n}_{a}T^{\alpha}f)(u)\,du}{(u-a)^{1-\alpha}}\right) \\ &\quad + \sum_{k=0}^{n-1}\frac{{}^{n}_{a}T^{\alpha}f(a)}{\alpha^{k-\beta}\Gamma(k+1-\beta)}(x-a)^{\alpha(k-\beta)}. \end{aligned} \tag{50}$$

The result is then obtained if the operator ${}^{n}_{a}T^{\alpha}$ is applied to the integral in equation (50). □

Theorem 3.4 *Let* $\mathrm{Re}(\beta) > m > 0$, *where* $m \in \mathbb{N}$. *Then*

$$ {}^{m}_{a}T^{\alpha}\left({}^{\beta}_{a}\mathfrak{I}^{\alpha}f(x)\right) = {}^{\beta-m}_{a}\mathfrak{I}^{\alpha}f(x); \qquad {}^{m}T^{\alpha}_{b}\left({}^{\beta}\mathfrak{I}^{\alpha}_{b}f(x)\right) = {}^{\beta-m}\mathfrak{I}^{\alpha}_{b}f(x). \tag{51}$$

Proof We have

$$\begin{aligned}
{}_a^m T^\alpha \left({}^\beta \mathfrak{J}_b^\alpha f(x)\right) &= {}_a^m T^\alpha \left[\frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{\beta-1} f(t) \frac{dt}{(t-a)^{1-\alpha}}\right] \\
&= {}_a^{m-1} T^\alpha \left[\frac{1}{\Gamma(\beta-1)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{\beta-2} f(t) \frac{dt}{(t-a)^{1-\alpha}}\right] \\
&= {}_a^{m-2} T^\alpha \left[\frac{1}{\Gamma(\beta-2)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{\beta-3} f(t) \frac{dt}{(t-a)^{1-\alpha}}\right] \\
&\vdots \\
&= \frac{1}{\Gamma(\beta-m)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{\beta-m-1} f(t) \frac{dt}{(t-a)^{1-\alpha}} \\
&= {}_a^{\beta-m} I^\alpha f(x).
\end{aligned}$$

The second assertion in (51) can be proved similarly. □

Corollary 3.5 *If* $\mathrm{Re}(\gamma) < \mathrm{Re}(\beta)$, *then*

$$ {}_a^\gamma \mathfrak{D}^\alpha \left({}_a^\beta \mathfrak{J}^\alpha f(x)\right) = {}_a^{\beta-\gamma} \mathfrak{J}^\alpha f(x); \qquad {}^\gamma \mathfrak{D}_b^\alpha \left({}^\beta \mathfrak{J}_b^\alpha f(x)\right) = {}^{\beta-\gamma} \mathfrak{J}_b^\alpha f(x). \tag{52}$$

Proof The proof is done by using Theorem 2.1 and Theorem 3.4.

$$\begin{aligned}
{}_a^\gamma \mathfrak{D}^\alpha \left({}_a^\beta \mathfrak{J}^\alpha f(x)\right) &= {}_a^m T^\alpha \left({}_a^{m-\gamma} \mathfrak{J}^\alpha \left({}_a^\beta \mathfrak{J}^\alpha f(x)\right)\right) \\
&= {}_a^m T^\alpha \left({}_a^{\beta+m-\gamma} \mathfrak{J}^\alpha f(x)\right) \\
&= {}_a^{\beta-\gamma} \mathfrak{J}^\alpha f(x).
\end{aligned} \tag{53}$$

This proves the first claim in (53). The second claim can be proved analogously. □

Below we state the inverse properties.

Theorem 3.6 *Let* $\beta > 0$ *and* $f \in C_{\alpha,a}^n[a,b]$ $(f \in C_{\alpha,b}^n[a,b])$. *Then*

$$ {}_a^\beta \mathfrak{D}^\alpha \left({}_a^\beta \mathfrak{J}^\alpha f(x)\right) = f(x); \qquad {}^\beta \mathfrak{D}_b^\alpha \left({}^\beta \mathfrak{J}_b^\alpha f(x)\right) = f(x). \tag{54}$$

Proof

$$\begin{aligned}
&{}_a^\beta \mathfrak{D}^\alpha \left({}_a^\beta \mathfrak{J}^\alpha f(x)\right) \\
&\quad = \frac{{}_a^n T^\alpha}{\Gamma(n-\beta)\Gamma(\beta)} \int_a^x \int_a^t \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{n-\beta} \left(\frac{(t-a)^\alpha - (u-a)^\alpha}{\alpha}\right)^{\beta-1} f(u) \\
&\qquad \times \frac{du}{(u-a)^{1-\alpha}} \frac{dt}{(t-a)^{1-\alpha}} \\
&\quad = \frac{{}_a^n T^\alpha}{\Gamma(n-\beta)\Gamma(\beta)} \int_a^x \int_u^x \frac{((x-a)^\alpha - (t-a)^\alpha)^{n-\beta-1}((t-a)^\alpha - (u-a)^\alpha)^{\beta-1}}{\alpha^n} \frac{dt}{(t-a)^{1-\alpha}} \\
&\qquad \times f(u) \frac{du}{(u-a)^{1-\alpha}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{{}^{n}T^{\alpha}}{\alpha^{n-1}\Gamma(n-\beta)\Gamma(\beta)} \int_a^x \big((x-a)^{\alpha} - (u-a)^{\alpha}\big)^{n-1} f(u) \frac{du}{(u-a)^{\alpha-1}} \\
&\quad \times \int_0^1 (1-z)^{n-\beta-1} z^{\beta-1}\, dz \\
&= \frac{{}^{n}_{a}T^{\alpha}}{\Gamma(n)} \int_a^x \left(\frac{(x-a)^{\alpha} - (u-a)^{\alpha}}{\alpha}\right)^{n-1} f(u) \frac{du}{(u-a)^{1-\alpha}} \\
&= {}^{n}_{a}T^{\alpha}\big({}^{n}_{a}I^{\alpha} f(x)\big).
\end{aligned}$$

Here we have used the change of variable z defined in the proof of Theorem 2.1 and utilized the properties of the gamma and beta functions. The last step in the proof is to use Lemma 2.1 in [16]. The second formula in (54) can be proved in a similar manner. □

Theorem 3.7 *Let* $\mathrm{Re}(\beta) > 0$, $n = -[-\mathrm{Re}(\beta)]$, $f \in L(a,b)$ *and* ${}^{\beta}_{a}\mathfrak{I}^{\alpha} f \in C^{n}_{\alpha,a}[a,b]$ $({}^{\beta}\mathfrak{I}^{\alpha}_{b} f \in C^{n}_{\alpha,b}[a,b])$. *Then*

$$ {}^{\beta}_{a}\mathfrak{I}^{\alpha}\big({}^{\beta}_{a}\mathfrak{D}^{\alpha} f(x)\big) = f(x) - \sum_{j=1}^{n} \frac{{}^{\beta-j}_{a}\mathfrak{D}^{\alpha} f(a)}{\alpha^{\beta-j}\Gamma(\beta-j+1)} (x-a)^{\alpha\beta-\alpha j} \tag{55}$$

and

$$ {}^{\beta}\mathfrak{I}^{\alpha}_{b}\big({}^{\beta}\mathfrak{D}^{\alpha}_{b} f(x)\big) = f(x) - \sum_{j=1}^{n} \frac{(-1)^{n-j}\, {}^{\beta-j}\mathfrak{D}^{\alpha}_{b} f(b)}{\alpha^{\beta-j}\Gamma(\beta-j+1)} (b-x)^{\alpha\beta-\alpha j}. \tag{56}$$

Proof

$$\begin{aligned}
{}^{\beta}_{a}\mathfrak{I}^{\alpha}\big({}^{\beta}_{a}\mathfrak{D}^{\alpha} f(x)\big) &= \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha}\right)^{\beta-1} \big({}^{n}_{a}T^{\alpha}\big({}^{n-\beta}_{a}\mathfrak{I}^{\alpha} f(t)\big)\big) \frac{dt}{(t-a)^{1-\alpha}} \\
&= \frac{{}^{1}_{a}T^{\alpha}}{\Gamma(\beta+1)} \left[\int_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha}\right)^{\beta} \big({}^{n}_{a}T^{\alpha}\big({}^{n-\beta}_{a}\mathfrak{I}^{\alpha} f(t)\big)\big) \frac{dt}{(t-a)^{1-\alpha}}\right].
\end{aligned}$$

Using the integration by parts formula in Theorem 3.3 in [16] n times, we get

$$\begin{aligned}
{}^{\beta}_{a}\mathfrak{I}^{\alpha}\big({}^{\beta}_{a}\mathfrak{D}^{\alpha} f(x)\big) &= {}^{1}_{a}T^{\alpha}\left[\frac{1}{\Gamma(\beta-n+1)} \int_a^x \left(\frac{(x-a)^{\alpha} - (t-a)^{\alpha}}{\alpha}\right)^{\beta-n} \big({}^{n-\beta}_{a}\mathfrak{I}^{\alpha} f(t)\big) \frac{dt}{(t-a)^{1-\alpha}}\right. \\
&\quad \left. - \sum_{j=1}^{n} \frac{({}^{n-j}_{a}T^{\alpha}({}^{n-\beta}_{a}\mathfrak{I}^{\alpha} f(a)))}{\Gamma(\beta+2-j)\alpha^{\beta-j+1}} (x-a)^{\alpha\beta-\alpha j+\alpha}\right] \\
&= {}^{1}_{a}T^{\alpha}\left[{}^{\beta-n+1}_{a}\mathfrak{I}^{\alpha}\big({}^{n-\beta}_{a}\mathfrak{I}^{\alpha} f(x)\big) - \sum_{j=1}^{n} \frac{({}^{n-j}_{a}T^{\alpha}({}^{n-\beta}_{a}\mathfrak{I}^{\alpha} f(a)))}{\Gamma(\beta+2-j)\alpha^{\beta-j+1}} (x-a)^{\alpha\beta-\alpha j+\alpha}\right].
\end{aligned}$$

Now by using Theorem 2.1, we get

$$\begin{aligned}
{}^{\beta}_{a}\mathfrak{I}^{\alpha}\big({}^{\beta}_{a}\mathfrak{D}^{\alpha} f(x)\big) &= {}^{1}_{a}T\left[{}^{1}_{a}\mathfrak{I}^{\alpha} f(x) - \sum_{j=1}^{n} \frac{({}^{n-j}_{a}T^{\alpha}({}^{n-\beta}_{a}\mathfrak{I}^{\alpha} f(a)))}{\Gamma(\beta+2-j)\alpha^{\beta-j+1}} (x-a)^{\alpha\beta-\alpha j+\alpha}\right] \\
&= f(x) - \sum_{j=1}^{n} \frac{{}^{\beta-j}_{a}\mathfrak{D}^{\alpha} f(a)}{\alpha^{\beta-j}\Gamma(\beta-j+1)} (x-a)^{\alpha\beta-\alpha j}.
\end{aligned}$$

Assertion (56) can be proved likewise. □

4 Fractional conformable derivatives in the Caputo setting

In this section we define the left- and right-fractional conformable derivatives in the sense of Caputo and present their properties.

Definition 4.1 Let, $\alpha > 0$, $\mathrm{Re}(\beta) \geq 0$ and $n = [\mathrm{Re}(\beta)] + 1$. If $f \in C^n_{\alpha,a}$ ($f \in C^n_{\alpha,b}$), we define the left and right Caputo fractional conformable derivatives of f of order β, respectively, as

$$ {}^{C\beta}_{\ a}\mathfrak{D}^{\alpha} f(x) = {}^{\beta}_{a}\mathfrak{D}^{\alpha}\left[f(t) - \sum_{k=0}^{n-1} \frac{{}^{k}_{a}T^{\alpha} f(a)}{k!\alpha^k}(t-a)^{\alpha k}\right](x) \tag{57}$$

and

$$ {}^{C\beta}\mathfrak{D}^{\alpha}_{b} f(x) = {}^{\beta}\mathfrak{D}^{\alpha}_{b}\left[f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k\, {}^{k}T^{\alpha}_{b} f(b)}{k!\alpha^k}(b-t)^{\alpha k}\right](x). \tag{58}$$

Theorem 4.1 *Let* $\mathrm{Re}(\beta) \geq 0$, $n = [\mathrm{Re}(\beta)] + 1$, $f \in C^n_{\alpha,a}([a,b])$ ($f \in C^n_{\alpha,b}([a,b])$). *Then the left- and right-fractional conformable derivatives in the Caputo settings can be written, respectively, as*

$$\begin{aligned} {}^{C\beta}_{\ a}\mathfrak{D}^{\alpha} f(x) &= \frac{1}{\Gamma(n-\beta)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{n-\beta-1} \frac{{}^{n}_{a}T^{\alpha} f(t)}{(t-a)^{1-\alpha}}\, dt \\ &= {}^{n-\beta}_{\ \ a}\mathfrak{I}^{\alpha}\big({}^{n}_{a}T^{\alpha} f(x)\big), \end{aligned} \tag{59}$$

and

$$\begin{aligned} {}^{C\beta}\mathfrak{D}^{\alpha}_{b} f(x) &= \frac{(-1)^n}{\Gamma(n-\beta)} \int_x^b \left(\frac{(b-x)^\alpha - (b-t)^\alpha}{\alpha}\right)^{n-\beta-1} \frac{{}^{n}T^{\alpha}_{b} f(t)}{(b-t)^{1-\alpha}}\, dt \\ &= {}^{n-\beta}\mathfrak{I}^{\alpha}_{b}\big({}^{n}T^{\alpha}_{b} f(x)\big). \end{aligned} \tag{60}$$

Proof Using (57), Lemma 2.3 and Theorem 3.3, we have

$$\begin{aligned} {}^{C\beta}\mathfrak{D}^{\alpha}_{b} f(x) &= {}^{\beta}_{a}\mathfrak{D}^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{{}^{k}T^{\alpha}_{a} f(a)}{\alpha^{k-\beta} k!} \frac{\Gamma(k+1)}{\Gamma(k-\beta+1)}(x-a)^{k\alpha-\beta\alpha} \\ &= {}^{\beta}_{a}\mathfrak{D}^{\alpha} f(x) - \sum_{k=0}^{n-1} \frac{{}^{k}T^{\alpha}_{a} f(a)}{\alpha^{k-\beta}\Gamma(k-\beta+1)}(x-a)^{k\alpha-\beta\alpha} \\ &= \frac{1}{\Gamma(n-\beta)} \int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\rho}\right)^{n-\beta-1} \frac{{}^{n}_{a}T^{\alpha} f(t)}{(t-a)^{1-\alpha}}\, dt \\ &= {}^{n-\beta}_{\ \ a}\mathfrak{I}^{\alpha}\big({}^{n}_{a}T^{\alpha} f(x)\big). \end{aligned}$$

The identity (60) is proved by using (58), Lemma 2.3 and Theorem 3.3 as well. □

The fractional derivative in (59) coincides with the Caputo derivative (8) when $a = 0$ and $\alpha = 1$, the Caputo Hadamard fractional derivative (15) if $a = 0$ and $\alpha \to 0$ and with the generalized fractional integral (22) when $a = 0$. Meanwhile the fractional derivative in (60) coincides with the Caputo derivative (9) when $b = 0$ and $\alpha = 1$, it coincides with the Hadamard fractional integral (17) once $b = 0$ and $\alpha \to 0$ and with the generalized fractional integral (23) when $b = 0$.

Before we state the inverse properties and the composition of two Caputo fractional conformable derivatives, we shall consider the following lemmas.

Lemma 4.2 *Let* $\operatorname{Re}(\beta) > 0$, $n = [\operatorname{Re}(\beta)] + 1$, $\operatorname{Re}(\beta) \notin \mathbb{N}$ *and* $f \in C[a,b]$. *Then* ${}^{\beta-k}_{a}\mathfrak{I}^{\alpha} f(a) = 0$ *and* ${}^{\beta-k}\mathfrak{I}^{\alpha}_{b} f(b) = 0$ *for* $k = 0,1,\dots,n-1$.

Proof It can be easily obtained that

$$\left|{}^{\beta-k}_{a}\mathfrak{I}^{\alpha} f(x)\right| \leq \frac{\|f\|_C}{|\Gamma(\beta-k)|(\operatorname{Re}(\beta)-k)} \frac{(x-a)^{\alpha(\operatorname{Re}(\beta)-k)}}{\alpha^{\operatorname{Re}(\beta)-k}}.$$

The result is obtained by replacing x by a. The second identity in can be proved similarly. □

Lemma 4.3 *Let* $R(\beta) \geq 0$, $n = [\operatorname{Re}(\beta)] + 1$ *and* ${}^{n}_{a}T^{\alpha} \in C[a,b]$ (${}^{n}T^{\alpha}_{b} \in C[a,b]$). *Then* ${}^{C\beta}_{a}D^{\alpha} f(a) = 0$ *and* ${}^{C\beta}D^{\alpha}_{b} f(b) = 0$.

Proof The identities in hold since

$$\left|{}^{C\beta}_{a}\mathfrak{D}^{\alpha} f(x)\right| \leq \frac{\|{}^{n}_{a}T^{\alpha} f\|_C}{|\Gamma(n-\beta)|(n-\operatorname{Re}(\beta))} \frac{(x-a)^{\alpha(n-\operatorname{Re}(\beta))}}{\alpha^{n-\operatorname{Re}(\beta)}}$$

and

$$\left|{}^{C\beta}\mathfrak{D}^{\alpha}_{b} f(x)\right| \leq \frac{\|{}^{n}T^{\alpha}_{b} f\|_C}{|\Gamma(n-\beta)|(n-\operatorname{Re}(\beta))} \frac{(b-x)^{\alpha(n-\operatorname{Re}(\beta))}}{\alpha^{n-\operatorname{Re}(\beta)}}.$$ □

Theorem 4.4 *Let* $\operatorname{Re}(\beta) > 0$, $n = [\operatorname{Re}(\beta)] + 1$, $f \in C[a,b]$.

(1) *If* $\operatorname{Re}(\beta) \notin \mathbb{N}$ *or* $\beta \in \mathbb{N}$, *then*

$${}^{C\beta}_{a}\mathfrak{D}^{\alpha}\left({}^{\beta}_{a}\mathfrak{I}^{\alpha} f(x)\right) = f(x); \qquad {}^{C\beta}\mathfrak{D}^{\alpha}_{b}\left({}^{\beta}\mathfrak{I}^{\alpha}_{b} f(x)\right) = f(x). \tag{61}$$

(2) *If* $\operatorname{Re}(\beta) \neq 0$ *and* $\operatorname{Re}(\alpha) \in \mathbb{N}$, *then*

$${}^{C\beta}_{a}\mathfrak{D}^{\alpha}\left({}^{\beta}_{a}\mathfrak{I}^{\alpha} f(x)\right) = f(x) - \frac{{}^{\beta+1-n}_{a}\mathfrak{I}^{\alpha} f(a)}{\alpha^{n-\beta}\Gamma(n-\beta)}(x-a)^{\alpha n-\alpha\beta}, \tag{62}$$

$${}^{C\beta}\mathfrak{D}^{\alpha}_{b}\left({}^{\beta}\mathfrak{I}^{\alpha}_{b} f(x)\right) = f(x) - \frac{{}^{\beta+1-n}\mathfrak{I}^{\alpha}_{b} f(b)}{\alpha^{n-\beta}\Gamma(n-\beta)}(b-x)^{\alpha n-\alpha\beta}. \tag{63}$$

Proof From the definition (57) we have

$${}^{C\beta}_{a}\mathfrak{D}^{\alpha}\left({}^{\beta}_{a}\mathfrak{I}^{\alpha} f(x)\right) = {}^{\beta}_{a}\mathfrak{D}^{\alpha}\left({}^{\beta}_{a}\mathfrak{I}^{\alpha} f(x)\right) - \sum_{k=0}^{n-1} \frac{{}^{k}_{a}T^{\alpha}({}^{\beta}_{a}\mathfrak{I}^{\alpha} f(a))(x-a)}{\alpha^{k-\beta}\Gamma(k-\beta+1)}.$$

Using Theorem 3.4 and Theorem 3.6, we get

$${}^{C\beta}_{a}\mathfrak{D}^{\alpha}\left({}^{\beta}_{a}\mathfrak{I}^{\alpha} f(x)\right) = f(x) - \sum_{k=0}^{n-1} \frac{{}^{\beta-k}_{a}\mathfrak{I}^{\alpha} f(a)(x-a)^{\alpha-k-\beta\alpha}}{\alpha^{k-\beta}\Gamma(k-\beta+1)}.$$

If $\mathrm{Re}(\beta) \notin \mathbb{N}$, by Lemma 4.2, we have ${}_a^{\beta-k}\mathfrak{I}^\alpha f(a) = 0$. Thus the first identity in (61) is proved. The second identity can be proved by using the same arguments.

The case $\beta \in \mathbb{N}$ is trivial. Now if $\mathrm{Re}(\beta) \in \mathbb{N}$, it can be proved that ${}_a^{\beta-k}\mathfrak{I}^\alpha f(a) = 0$ for $k = 0,1,\ldots,n-2$ using the steps used in proving Lemma 4.2. Thus (62) is proved. Equation (63) can be proved similarly. □

Theorem 4.5 *Let $f \in C^n_{\alpha,a}[a,b]$ ($f \in C^n_{\alpha,b}[a,b]$), $\beta \in \mathbb{C}$. Then*

$$ {}_a^\beta\mathfrak{I}^\alpha \left({}_a^{C\beta}\mathfrak{D}^\alpha f(x)\right) = f(x) - \sum_{k=0}^{n-1} \frac{{}_a^k T^\alpha f(a)(x-a)^{\alpha k}}{\alpha^k k!}, \tag{64} $$

$$ {}^\beta\mathfrak{I}_b^\alpha \left({}^{C\beta}\mathfrak{D}_b^\alpha f(x)\right) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k\, {}^k T_b^\alpha f(b)(b-x)^{\alpha k}}{\alpha^k k!}. \tag{65} $$

Proof

$$ \begin{aligned} {}_a^\beta\mathfrak{I}^\alpha \left({}_a^{C\beta}\mathfrak{D}^\alpha f(x)\right) &= {}_a^\beta\mathfrak{I}^\alpha \left({}_a^{n-\beta}\mathfrak{I}^\alpha \left({}_a^n T^\alpha f(x)\right)\right) \\ &= {}_a^n\mathfrak{I}^\alpha \left({}_a^n T^\alpha f(x)\right) \\ &= f(x) - \sum_{j=1}^{n} \frac{{}_a^{n-j} D^\alpha f(a)}{\alpha^{n-j}\Gamma(n-j+1)}(x-a)^{(n-j)\alpha} \\ &= f(x) - \sum_{k=0}^{n-1} \frac{{}_a^n T^\alpha f(a)}{\alpha^k k!}(x-a)^{k\alpha}. \end{aligned} $$

This proves (64) and (65) can be proved by a similar way. □

Theorem 4.6 *Let $f \in C^{m+n}_{\alpha,a}[a,b]$ ($f \in C^{m+n}_{\alpha,b}[a,b]$), $\mathrm{Re}(\beta) \geq 0$, $\mathrm{Re}(\gamma) \geq 0$, $n-1 < \mathrm{Re}(\beta) \leq n$ and $m-1 < \mathrm{Re}(\gamma) \leq m$. Then*

$$ {}_a^{C\beta}\mathfrak{D}^\alpha \left({}_a^{C\gamma}\mathfrak{D}^\alpha f(x)\right) = {}_a^{C(\beta+\gamma)}\mathfrak{D}^\alpha f(x); \qquad {}^{C\beta}\mathfrak{D}_b^\alpha \left({}^{C\gamma}\mathfrak{D}_b^\alpha f(x)\right) = {}^{C(\beta+\gamma)}\mathfrak{D}_b^\alpha f(x). \tag{66} $$

Proof The proof can be done by using Theorem 2.1, Theorem 3.6, Theorem 4.1 and Lemma 4.3. □

5 Conclusion

This paper was devoted to an investigation of the fractional derivatives and integrals obtained by iterating conformable integrals. We obtained left- and right-fractional conformable integrals. With a standard fractional procedure we found left- and right-fractional conformable derivatives in the sense of Riemann-Liouville and Caputo. We proved that these fractional integrals and derivatives have properties similar to the standard fractional integrals and derivatives. We also define the fractional derivatives of functions belonging to specific spaces in order to find the relation between these new fractional differential operators. The presented left- and right-fractional integrals are different from those defined by Katugampola since their kernels depend on the end points a and b and hence need a different convolution theory when the conformable Laplace is applied.

The classical fractional calculus was applied successfully to extract the hidden information from the dynamics of complex systems. However, each nonlocal system has its own behavior which may not be described properly by the existing fractional integrals and derivatives. This gives rise to the need of new fractional operators that may better describe such a system. Our proposed fractional operators are reduced to well-established fractional operators (Riemann-Liouville, Caputo, Hadamard) and the newly introduced generalized fractional operators under some conditions but they are different and outside of these operators. Therefore, suppose that these newly suggested operators may provide new insights for fractional variational problems, optimal control problems and modeling of complex systems. Another advantage of these operators is that they depend on two parameters naturally. The one which comes from the conformable operator will play an important role in better detection of the memory.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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