ON A NEW CLASS OF POLYNOMIALS†

by REKHA PANDA

(Received 7 October, 1975)

1. Introduction. The present paper incorporates a preliminary study of a new generalization of several known polynomial systems belonging to (or providing extensions of) the families of the classical Jacobi, Hermite and Laguerre polynomials. It is shown how suitable specializations will yield a number of known or new results in the theory of the special functions considered.

In the usual notation, put

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if} \quad n=0, \\ \lambda(\lambda+1)\ldots(\lambda+n-1), & \text{if} \quad n=1,2,3,\ldots, \end{cases} \tag{1}$$

and let $\Delta(m; \lambda)$ denote the set of m parameters

$$\lambda/m, (\lambda+1)/m, \ldots, (\lambda+m-1)/m \qquad (m \geqq 1),$$

it being understood that the set $\Delta(0; \lambda)$ is empty.

Also let

$$G[z] = \sum_{n=0}^{\infty} \gamma_n z^n \qquad (\gamma_0 \neq 0), \tag{2}$$

and in terms of this power series, define a class of polynomials $\{g_n^c(x, r, s) \,|\, n = 0, 1, 2, \ldots\}$ generated by

$$(1-t)^{-c} G[xt^s/(1-t)^r] = \sum_{n=0}^{\infty} g_n^c(x, r, s) t^n, \tag{3}$$

where c is an arbitrary parameter, r is any integer, positive or negative, and $s = 1, 2, 3, \ldots$.

From (2) and (3), we observe that

$$g_n^c(x, r, s) = \sum_{k=0}^{[n/s]} \frac{(c+rk)_{n-sk} \gamma_k x^k}{(n-sk)!} \qquad (n = 0, 1, 2, \ldots), \tag{4}$$

which would lead fairly readily to the following generating function for $g_n^{c-n}(x, r, s)$:

$$\sum_{n=0}^{\infty} g_n^{c-n}(x, r, s) t^n = (1+t)^{c-1} G[xt^s(1+t)^{r-s}]. \tag{5}$$

Evidently, this last generating function (5) is not contained in the defining relation (3).

The definitions (2) and (3) are motivated by the earlier work of E. D. Rainville [**6**, p. 137, Theorem 48], who considers a special case of (3) when $r = 2$ and $s = 1$, and also by the recent papers by R. C. Singh Chandel ([**2**], [**3**]), who discusses the special cases of (2) and (3) when

† This work was carried out at the University of Victoria while the author was on study leave from Ravenshaw College, Cuttack-3, India.

For an abstract of this paper see *Notices Amer. Math. Soc.* **22** (1975), p. A–308–9, No. 75T–B33.

$\gamma_k = (-r^r)^k/k!$ $(k = 0, 1, 2, \ldots)$ and $s = 1$. Thus it would seem worthwhile to carry out a systematic study of the polynomials $g_n^c(x, r, s)$, which indeed unify several hitherto considered polynomial systems belonging to (or providing extensions of) the families of the classical Jacobi, Hermite and Laguerre polynomials.

2. Hypergeometric forms. For convenience, we shall abbreviate the set of p parameter pairs

$$(a_1, \alpha_1), \ldots, (a_p, \alpha_p)$$

by $((a_p, \alpha_p))$, with similar interpretations for $((b_q, \beta_q))$, etc. We shall also let (a_p) denote the set of p parameters $a_1, \ldots, a_p$, and so on. Thus, if we put

$$\gamma_k = \left\{ \prod_{j=1}^{p} (a_j)_{k\alpha_j} \right\} \left\{ k! \prod_{j=1}^{q} (b_j)_{k\beta_j} \right\}^{-1} \qquad (k = 0, 1, 2, \ldots), \tag{6}$$

the polynomials $g_n^c(x, r, s)$ will assume a hypergeometric form given by

$$h_{n,p,q}^{c,r,s}[((a_p, \alpha_p)); ((b_q, \beta_q)); x]$$
$$= \frac{(c)_n}{n!} {}_{p+2}\Psi_{q+1} \left[\begin{matrix} (-n, s), (c+n, r-s), ((a_p, \alpha_p)); \\ (c, r), ((b_q, \beta_q)); \end{matrix} (-1)^s x \right], \tag{7}$$

where $r > s \geqq 1$; $\alpha_j > 0$ $(j = 1, \ldots, p)$; $\beta_j > 0$ $(j = 1, \ldots, q)$, and ${}_p\Psi_q$ denotes Wright's generalized hypergeometric function.

If $s > r$, where r is a negative integer, then (7) may be rewritten in the form

$$h_{n,p,q}^{c,r,s}[((a_p, \alpha_p)); ((b_q, \beta_q)); x]$$
$$= \frac{(c)_n}{n!} {}_{p+2}\Psi_{q+1} \left[\begin{matrix} (-n, s), (1-c, -r), ((a_p, \alpha_p)); \\ (1-c-n, s-r), ((b_q, \beta_q)); \end{matrix} x \right], \tag{8}$$

s being a positive integer.

In particular, if $\alpha_j = 1$ $(j = 1, \ldots, p)$ and $\beta_j = 1$ $(j = 1, \ldots, q)$, (7) yields

$$f_{n,p,q}^{c,r,s}[(a_p); (b_q); x]$$
$$= \frac{(c)_n}{n!} {}_{p+r}F_{q+r} \left[\begin{matrix} \Delta(s; -n), \Delta(r-s; c+n), (a_p); \\ \Delta(r; c), (b_q); \end{matrix} \frac{(-s)^s (r-s)^{r-s} x}{r^r} \right], \tag{9}$$

where $r > s \geqq 1$, the case $r = s$ being given by

$$f_{n,p,q}^{c,s,s}[(a_p); (b_q); x]$$
$$= \frac{(c)_n}{n!} {}_{p+s}F_{q+s} \left[\begin{matrix} \Delta(s; -n), (a_p); \\ \Delta(s; c), (b_q); \end{matrix} (-1)^s x \right], \tag{10}$$

and similarly for other possible choices of r and s.

https://doi.org/10.1017/S0017089500003116 Published online by Cambridge University Press

These last polynomials in (10) are essentially the same as the Brafman polynomials defined in [**1**, p. 186] by

$$B_n^s[(a_p);(b_q);x] = {}_{p+s}F_q\left[\begin{matrix}\Delta(s;\,-n),(a_p); \\ (b_q);\end{matrix}\; x\right], \tag{11}$$

it being understood, as before, that the a_j and b_j parameters are independent of n. Indeed, in the notations of (9) and (10), we have

$$B_n^s[(a_p);(b_q);x] = \frac{n!}{(c)_n} f_{n,\,p+s,\,q}^{c,\,s,\,s}[(a_p),\Delta(s;c);(b_q);(-1)^s x]. \tag{12}$$

Next we recall the Gould–Hopper generalization of the classical Hermite polynomials $\{H_n(x)\,|\,n = 0, 1, 2, \ldots\}$ defined in [**4**, p. 58] by

$$g_n^s(x,\lambda) = \sum_{k=0}^{[n/s]} \frac{n!}{k!(n-sk)!}\lambda^k x^{n-sk} = x^n\,{}_sF_0[\Delta(s;\,-n);\,-\,;\lambda(-s/x)^s], \tag{13}$$

which evidently are contained in the Brafman polynomials in (11) with $p = q = 0$. As a matter of fact, it is readily seen that

$$g_n^s(x,\lambda) = x^n B_n^s[-\,;\,-\,;\lambda(-s/x)^s]. \tag{14}$$

Remark 1. Certain obvious special cases or trivial variations of the Gould–Hopper polynomials (and hence also of the Brafman polynomials) have appeared and are still appearing in papers by several subsequent writers too numerous to mention here.

Finally, we turn to a generalization of the Jacobi, Laguerre, Rice, Bessel, and several other polynomials considered recently by R. N. Jain [**5**], who defines the hypergeometric polynomials

$$f_n^{(c,k)}[(a_p);(b_q);x] = \frac{(c)_n}{n!}\,{}_{p+k}F_{q+k}\left[\begin{matrix}-n,\,\Delta(k-1;\,c+n),(a_p); \\ \Delta(k;\,c),(b_q);\end{matrix}\;(k-1)^{k-1}x\right]\quad (n \geqq 0), \tag{15}$$

where k is a positive integer.

A comparison (9) and (15) yields the relationship

$$f_n^{(c,k)}[(a_p);(b_q);x] = f_{n,\,p,\,q}^{c,\,k,\,1}[(a_p);(b_q);\,-k^k x], \tag{16}$$

which exhibits the fact that results involving any of the known polynomial systems, occurring in Jain's paper [**5**, p. 177–8], can be deduced as special cases of those involving the polynomials $g_n^c(x, r, s)$ defined by (3).

3. Recurrence relations. If we put

$$\Phi = (1-t)^{-c} G[xt^s/(1-t)^r], \tag{17}$$

https://doi.org/10.1017/S0017089500003116 Published online by Cambridge University Press

then it is easily verified that

$$x[s+(r-s)t]\frac{\partial \Phi}{\partial x}-t(1-t)\frac{\partial \Phi}{\partial t} = -ct\Phi. \tag{18}$$

Making use of (18) in conjunction with the fact that

$$\Phi = \sum_{n=0}^{\infty} g_n^c(x, r, s)t^n, \tag{19}$$

we are led to the following differential recurrence relations for $g_n^c(x, r, s)$:

$$sxD_x\{g_n^c(x, r, s)\}-ng_n^c(x, r, s)$$
$$= -(c+n-1)g_{n-1}^c(x, r, s)-(r-s)xD_x\{g_{n-1}^c(x, r, s)\}, \tag{20}$$

$$sxD_x\{g_n^c(x, r, s)\}-ng_n^c(x, r, s) = -c\sum_{k=0}^{n-1} g_k^c(x, r, s)-rx\sum_{k=0}^{n-1} D_x\{g_k^c(x, r, s)\} \tag{21}$$

and

$$sxD_x\{g_n^c(x, r, s)\}-ng_n^c(x, r, s) = -\sum_{k=0}^{n-1}(c+kr/s)(1-r/s)^{n-k-1}g_k^c(x, r, s), \tag{22}$$

where $D_x = d/dx$, and $n = 1, 2, 3, \ldots$.

REMARK 2. The recurrence relations (20) through (22), when $r = 2$ and $s = 1$, are substantially the same as those given by Theorem 48 of [**6**, p. 137]. Several other special cases of these results are scattered throughout the literature.

ACKNOWLEDGMENT. The author wishes to express her sincere thanks to Professor H. M. Srivastava for his suggestions during the course of the present investigation.

REFERENCES

1. F. Brafman, Some generating functions for Laguerre and Hermite polynomials, *Canad. J. Math.* **9** (1957), 180–187.

2. R. C. Singh Chandel, Generalized Laguerre polynomials and the polynomials related to them, *Indian J. Math.* **11** (1969), 57–66.

3. R. C. Singh Chandel, A short note on generalized Laguerre polynomials and the polynomials related to them, *Indian J. Math.* **13** (1971), 25–27.

4. H. W. Gould and A. T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math. J.* **29** (1962), 51–63.

5. R. N. Jain, A generalized hypergeometric polynomial, *Ann. Polon. Math.* **19** (1967), 177–184; corrigendum, *ibid.* **20** (1968), 329.

6. E. D. Rainville, *Special functions* (Macmillan Co., 1960).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF VICTORIA
VICTORIA, BRITISH COLUMBIA
CANADA V8W 2Y2

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD
PRINTERS TO THE UNIVERSITY OF GLASGOW