ON A NEW GENERALIZATION OF FIBONACCI HYBRID NUMBERS

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ABSTRACT. The hybrid numbers were introduced by Ozdemir [9] as a new generalization of complex, dual, and hyperbolic numbers. A hybrid number is defined by $k = a + bi + c\epsilon + dh$, where a, b, c, d are real numbers and i, ϵ, h are operators such that $i^2 = -1, \epsilon^2 = 0, h^2 = 1$ and $ih = -hi = \epsilon + i$. This work is intended as an attempt to introduce the bi-periodic Horadam hybrid numbers which generalize the classical Horadam hybrid numbers. We give the generating function, the Binet formula, and some basic properties of these new hybrid numbers. Also, we investigate some relationships between generalized bi-periodic Fibonacci hybrid numbers and generalized bi-periodic Lucas hybrid numbers.

1. INTRODUCTION

Non-commutative algebras play important role and have broad applications in many areas, such as mathematics and physics. Hence it is worth to study and investigate the properties of some special types of non-commutative algebras. The real quaternion algebra is the first non-commutative division algebra to be discovered and defined by

$$\mathbb{H} = \{a + bi + cj + dk \mid i^2 = j^2 = k^2 = -1, ij = -ji = k\}$$

where $a, b, c, d \in \mathbb{R}$. For a survey on the properties of quaternions, we refer to Hamilton's book [5], and for some special type of quaternions see [4, 6].

A new non-commutative number system, the hybrid numbers, were introduced by Ozdemir [9] as a generalization of complex numbers, dual numbers, and hyperbolic numbers which are having the form $a + be$ with $e^2 = -1, e^2 = 0$, and $e^2 = 1$, respectively. The set of hybrid numbers are defined as

$$\mathbb{K} = \left\{ a + bi + c\epsilon + dh \mid i^2 = -1, \epsilon^2 = 0, h^2 = 1, ih = -hi = \epsilon + i \right\} \quad (1.1)$$

where $a, b, c, d \in \mathbb{R}$.

The addition, substraction and multiplication of two hybrid numbers $k_1 = a_1 + b_1 i + c_1 \epsilon + d_1 h$ and $k_2 = a_2 + b_2 i + c_2 \epsilon + d_2 h$ are defined as

$$k_1 \pm k_2 = (a_1 \pm a_2) + (b_1 \pm b_2) i + (c_1 \pm c_2) \epsilon + (d_1 \pm d_2) h,$$

$$\begin{aligned} k_1 k_2 = {}& a_1 a_2 - b_1 b_2 + d_1 d_2 + b_1 c_2 + c_1 b_2 \\ & + (a_1 b_2 + b_1 a_2 + b_1 d_2 - d_1 b_2) i \\ & + (a_1 c_2 + c_1 a_2 + b_1 d_2 - d_1 b_2 + d_1 c_2 - c_1 d_2) \epsilon \\ & + (a_1 d_2 + d_1 a_2 + c_1 b_2 - b_1 c_2) h. \end{aligned}$$

The multiplication of a hybrid number $k = a + bi + c\epsilon + dh$ by the real scalar s is defined as

$$sk = sa + sbi + sc\epsilon + sdh,$$

and the norm of a hybrid number k is defined by

$$\|k\| := \sqrt{|C(k)|},$$

where $C(k) := k\overline{k}$ is the character of the hybrid number k and $\overline{k} := a - bi - c\epsilon - dh$ is the conjugate of k. Ozdemir's paper [9] serves as an excellent reference to the algebraic and geometric properties of the hybrid numbers.

Recently, many studies have been devoted to hybrid numbers whose components are taken from special integer sequences such as Fibonacci, Lucas, Pell, Jacobsthal sequences, etc. In particular, Szynal-Liana [13] introduced the Horadam hybrid numbers as

$$\mathbb{K}_{W,n} = W_n + W_{n+1} i + W_{n+2} \epsilon + W_{n+3} h \qquad (1.2)$$

where $\{W_n\}$ is the Horadam sequence defined by $W_n = pW_{n-1} - qW_{n-2}$ with arbitrary initial values W_0, W_1. In [12–15], the authors studied several properties of special type of hybrid numbers. The basic properties of k-Pell hybrid numbers were investigated by Catarino [2]. Also, Morales [8] considered the (p,q)-Fibonacci and (p,q)-Lucas hybrid numbers and gave several relations between them. Recently, motivated by the Szynal-Liana's paper, Senturk et al. [11] derived summation formulas, matrix representations, general bilinear formula, Honsberger formula, etc. regarding to the Horadam hybrid numbers.

This work has been intended as an attempt to introduce a new generalization of Horadam hybrid numbers, called as, bi-periodic Horadam hybrid numbers. The bi-periodic Horadam hybrid numbers generalize the best known hybrid numbers in the literature, such as Horadam hybrid numbers, Fibonacci&Lucas hybrid numbers, k-Pell hybrid numbers, Pell&Pell-Lucas hybrid numbers, Jacobsthal&Jacobsthal-Lucas hybrid numbers, etc. The components of the bi-periodic Horadam hybrid numbers belong to the bi-periodic Horadam sequence $\{w_n\}$ which is defined by the recurrence relation

$$w_n = \chi(n) w_{n-1} + cw_{n-2}, \ n \geq 2 \qquad (1.3)$$

where $\chi(n) = a$ if n is even, $\chi(n) = b$ if n is odd with arbitrary initial conditions w_0, w_1 and nonzero real numbers a, b and c. It is clear that if we take $a = b = p$ and $c = -q$, then it reduces to the classical Horadam sequence in [7]. For the details of the bi-periodic Horadam sequences see [1, 3, 10, 18].

The outline of this paper is as follows: In the rest of this section, we give some necessary definitions and mathematical preliminaries, which is required. In Section 2, we introduce the bi-periodic Horadam hybrid numbers and give the generating function, the Binet formula, matrix representation and several basic properties of these hybrid numbers such as Vajda's identity, Catalan's identity, Cassini's identity, summation and binomial sum formulas. In Section 3, we give some relationships between the generalized bi-periodic Fibonacci hybrid numbers and the generalized bi-periodic Lucas hybrid numbers. The final section is devoted to the conclusions.

The Binet formula for the bi-periodic Horadam sequence $\{w_n\}$ is

$$w_n = \frac{a^{\xi(n+1)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(A\alpha^n - B\beta^n\right), \tag{1.4}$$

where

$$A := \frac{w_1 - \frac{\beta}{a}w_0}{\alpha - \beta} \text{ and } B := \frac{w_1 - \frac{\alpha}{a}w_0}{\alpha - \beta}. \tag{1.5}$$

Here α and β are the roots of the polynomial $x^2 - abx - abc$, that is, $\alpha = \frac{ab+\sqrt{a^2b^2+4abc}}{2}$ and $\beta = \frac{ab-\sqrt{a^2b^2+4abc}}{2}$, and $\xi(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd. Let assume $a^2b^2 + 4abc > 0$. Also we have $\alpha + \beta = ab$, $\Delta := \alpha - \beta = \sqrt{a^2b^2 + 4abc}$ and $\alpha\beta = -abc$. If we take the initial conditions 0 and 1, we get the Binet formula of the generalized bi-periodic Fibonacci sequence $\{u_n\}$ as

$$u_n = \frac{a^{\xi(n+1)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) \tag{1.6}$$

and by taking the initial conditions 2 and b, we get the Binet formula of the generalized bi-periodic Lucas sequence $\{v_n\}$ as

$$v_n = \frac{a^{-\xi(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(\alpha^n + \beta^n\right). \tag{1.7}$$

The bi-periodic Horadam numbers for negative subscripts is defined as

$$(-c)^n w_{-n} = \left(\frac{b}{a}\right)^{\xi(n)} w_0 u_{n+1} - w_1 u_n. \tag{1.8}$$

Also we have

$$\alpha^m = a^{-1}a^{\frac{m+\xi(m)}{2}}b^{\frac{m-\xi(m)}{2}}\alpha u_m + ca^{\frac{m-\xi(m)}{2}}b^{\frac{m+\xi(m)}{2}}u_{m-1} \tag{1.9}$$

and
$$\beta^m = a^{-1} a^{\frac{m+\xi(m)}{2}} b^{\frac{m-\xi(m)}{2}} \beta u_m + c a^{\frac{m-\xi(m)}{2}} b^{\frac{m+\xi(m)}{2}} u_{m-1}. \qquad (1.10)$$

For details, see [16, 17].

2. The bi-periodic Horadam hybrid numbers

Definition 1. For $n \geq 0$, the bi-periodic Horadam hybrid number $\mathbb{K}_{w,n}$ is defined by the recurrence relation

$$\mathbb{K}_{w,n} = w_n + w_{n+1}i + w_{n+2}\epsilon + w_{n+3}h$$

where w_n is the n-th bi-periodic Horadam number.

From the definition of bi-periodic Horadam hybrid numbers, we have

$$\begin{aligned}
\mathbb{K}_{w,0} &= w_0 + w_1 i + (aw_1 + cw_0)\epsilon + ((ab+c)w_1 + bcw_0)h, \\
\mathbb{K}_{w,1} &= w_1 + (aw_1 + cw_0)i + ((ab+c)w_1 + bcw_0)\epsilon \\
&\quad + (a(ab+2c)w_1 + c(ab+c)w_0)h.
\end{aligned}$$

In the following table we state several number of hybrid numbers in terms of the bi-periodic Horadam hybrid numbers $\mathbb{K}_{w,n}$ according to the initial conditions w_0, w_1 and the related coefficients a, b, c.

$\mathbf{K}_{w,n}$	$(\mathbf{w_0, w_1; a, b, c})$	**bi-periodic Horadam hybrid numbers**
$\mathbb{K}_{u,n}$	$(0,1;a,b,c)$	gen. bi-periodic Fibonacci hybrid numbers
$\mathbb{K}_{v,n}$	$(2,b;a,b,c)$	gen. bi-periodic Lucas hybrid numbers
$\mathbb{K}_{W,n}$	$(W_0,W_1;p,p,-q)$	Horadam hybrid numbers [11, 13]
$\mathbb{K}_{U,n}$	$(0,1;p,p,q)$	(p,q)-Fibonacci hybrid numbers [8]
$\mathbb{K}_{V,n}$	$(2,p;p,p,q)$	(p,q)-Lucas hybrid numbers [8]
$\mathbb{K}_{F,n}$	$(0,1;1,1,1)$	Fibonacci hybrid numbers [12]
$\mathbb{K}_{L,n}$	$(2,1;1,1,1)$	Lucas hybrid numbers [13]
$\mathbb{K}_{P,n}$	$(0,1;2,2,1)$	Pell hybrid numbers [15]
$\mathbb{K}_{Q,n}$	$(2,2;2,2,1)$	Pell-Lucas hybrid numbers [15]
$\mathbb{K}_{kP,n}$	$(0,1;2,2,k)$	k-Pell hybrid numbers [2]
$\mathbb{K}_{J,n}$	$(0,1;1,1,2)$	Jacobsthal hybrid numbers [14]
$\mathbb{K}_{j,n}$	$(2,1;1,1,2)$	Jacobsthal-Lucas hybrid numbers [14]

$Table$ 1 : Special cases of the sequence $\{\mathbb{K}_{w,n}\}$

The norm of the n-th bi-periodic Horadam hybrid number $\mathbb{K}_{w,n}$ is $\|\mathbb{K}_{w,n}\| := \sqrt{|C(\mathbb{K}_{w,n})|}$. Here $C(\mathbb{K}_{w,n})$ is the character of the n-th bi-periodic Horadam

hybrid number $\mathbb{K}_{w,n}$ and defined by

$$C\left(\mathbb{K}_{w,n}\right) = \mathbb{K}_{w,n}\overline{\mathbb{K}_{w,n}} = w_n^2 + (w_{n+1} - w_{n+2})^2 - w_{n+2}^2 - w_{n+3}^2, \qquad (2.1)$$

where $\overline{\mathbb{K}_{w,n}} := w_n - w_{n+1}i - w_{n+2}\epsilon - w_{n+3}h$ is the conjugate of the bi-periodic Horadam hybrid number.

Theorem 1. *The generating function for the bi-periodic Horadam hybrid sequence $\{\mathbb{K}_{w,n}\}$ is*

$$G\left(x\right) = \frac{\left(1 - (ab+c)\,x^2 + bcx^3\right)\mathbb{K}_{w,0} + x\left(1 + ax - cx^2\right)\mathbb{K}_{w,1}}{1 - (ab+2c)\,x^2 + c^2x^4}.$$

Proof. Let

$$G\left(x\right) = \sum_{n=0}^{\infty}\mathbb{K}_{w,n}x^n = \mathbb{K}_{w,0} + \mathbb{K}_{w,1}x + \mathbb{K}_{w,2}x^2 + \cdots + \mathbb{K}_{w,n}x^n + \cdots.$$

Since the bi-periodic Horadam hybrid numbers satisfy the recurrence relation

$$\mathbb{K}_{w,n} = (ab+2c)\,\mathbb{K}_{w,n-2} - c^2\mathbb{K}_{w,n-4},\ \ n \geq 4,$$

we get

$$\left(1 - (ab+2c)\,x^2 + c^2x^4\right)G\left(x\right)$$

$$\begin{aligned}
= \ \ & \mathbb{K}_{w,0} + \mathbb{K}_{w,1}x + \left(\mathbb{K}_{w,2} - (ab+2c)\,\mathbb{K}_{w,0}\right)x^2 \\
& + \left(\mathbb{K}_{w,3} - (ab+2c)\,\mathbb{K}_{w,1}\right)x^3 \\
& + \sum_{n=4}^{\infty}\left(\mathbb{K}_{w,n} - (ab+2c)\,\mathbb{K}_{w,n-2} + c^2\mathbb{K}_{w,n-4}\right)x^n
\end{aligned}$$

$$\begin{aligned}
= \ \ & \mathbb{K}_{w,0} + \mathbb{K}_{w,1}x + \left((a\mathbb{K}_{w,1} + c\mathbb{K}_{w,0}) - (ab+2c)\,\mathbb{K}_{w,0}\right)x^2 \\
& + \left(((ab+c)\,\mathbb{K}_{w,1} + bc\mathbb{K}_{w,0}) - (ab+2c)\,\mathbb{K}_{w,1}\right)x^3.
\end{aligned}$$

$$\square$$

Next, we state the Binet formula for the bi-periodic Horadam hybrid numbers and so derive some well-known mathematical properties.

Theorem 2. *The Binet formula for the bi-periodic Horadam hybrid numbers is*

$$\mathbb{K}_{w,n} = \frac{a^{\xi(n+1)}}{(ab)^{\left\lfloor\frac{n}{2}\right\rfloor}}\left(A\alpha_{\xi(n)}\alpha^n - B\beta_{\xi(n)}\beta^n\right)$$

where $\alpha_{\xi(n)}$ and $\beta_{\xi(n)}$ are defined as

$$\alpha_{\xi(n)} \quad : \quad = 1 + \frac{1}{a}\left(\frac{a}{b}\right)^{\xi(n)}\alpha i + \frac{1}{ab}\alpha^2\epsilon + \frac{1}{a^2 b}\left(\frac{a}{b}\right)^{\xi(n)}\alpha^3 h,$$

$$\beta_{\xi(n)} \quad : \quad = 1 + \frac{1}{a}\left(\frac{a}{b}\right)^{\xi(n)}\beta i + \frac{1}{ab}\beta^2\epsilon + \frac{1}{a^2 b}\left(\frac{a}{b}\right)^{\xi(n)}\beta^3 h.$$

Proof. By using the definition of the sequence $\{\mathbb{K}_{w,n}\}$ and the Binet formula of $\{w_n\}$, we obtain the desired result. $\qquad\square$

Remark 1. If we take $a = b = p$ and $c = -q$, we obtain the Binet formula of the classical Horadam hybrid numbers in [15].

Lemma 1.

$$\alpha_{\xi(n)}\beta_{\xi(n)} = \begin{cases} \mathbb{K}_{v,0} - \theta + \frac{\Delta}{a}c\left(\mathbb{K}_{u,0} - \eta\right), & \text{if } n \text{ is even} \\ \mathbb{K}_{\widehat{v},0} - \widehat{\theta} + \frac{\Delta}{b}c\left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta}\right), & \text{if } n \text{ is odd} \end{cases} \qquad (2.2)$$

$$\beta_{\xi(n)}\alpha_{\xi(n)} = \begin{cases} \mathbb{K}_{v,0} - \theta - \frac{\Delta}{a}c\left(\mathbb{K}_{u,0} - \eta\right), & \text{if } n \text{ is even} \\ \mathbb{K}_{\widehat{v},0} - \widehat{\theta} - \frac{\Delta}{b}c\left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta}\right), & \text{if } n \text{ is odd} \end{cases} \qquad (2.3)$$

where

$$\eta \quad : \quad = (1-b)\,i + (a-b-c)\,\epsilon + (1+ab+c)\,h,$$

$$\widehat{\eta} \quad : \quad = (1-a)\,i + (b-a-c)\,\epsilon + (1+ab+c)\,h,$$

$$\theta \quad : \quad = 1 - \frac{bc}{a} + bc + \frac{bc^3}{a},$$

$$\widehat{\theta} \quad : \quad = 1 - \frac{ac}{b} + ac + \frac{ac^3}{b}.$$

and the sequences $\left\{\mathbb{K}_{\widehat{u},0}\right\}$ and $\left\{\mathbb{K}_{\widehat{v},0}\right\}$ are the auxiliary sequences that are obtained from $\{\mathbb{K}_{u,0}\}$ and $\{\mathbb{K}_{v,0}\}$ just only switching $a \leftrightarrow b$. That is, $\widehat{u_n} = \left(\frac{b}{a}\right)^{\xi(n+1)} u_n$ and $\widehat{v_n} = \left(\frac{a}{b}\right)^{\xi(n)} v_n$.

Proof. By using the definition of multiplication of two hybrid numbers, we have

$$\alpha_{\xi(n)}\beta_{\xi(n)} \quad = \quad 1 + \frac{bc}{a}\left(\frac{a}{b}\right)^{2\xi(n)} - \frac{bc^3}{a}\left(\frac{a}{b}\right)^{2\xi(n)} - \left(\frac{a}{b}\right)^{\xi(n)}bc$$

$$+ \left(b\left(\frac{a}{b}\right)^{\xi(n)} + \left(\frac{a}{b}\right)^{2\xi(n)}\frac{bc}{a}\Delta\right)i$$

$$+ \left((ab+2c) + \frac{bc}{a}\left(\frac{a}{b}\right)^{2\xi(n)}\Delta + \frac{c^2}{a}\left(\frac{a}{b}\right)^{\xi(n)}\Delta\right)\epsilon$$

$$+ \left(\left(\frac{a}{b}\right)^{\xi(n)}(ab^2 + 3bc) - \frac{c}{a}\left(\frac{a}{b}\right)^{\xi(n)}\Delta\right)h$$

$$
\begin{aligned}
= \quad & 1 + \left(\frac{a}{b}\right)^{\xi(n)} bc \left(\frac{1}{a}\left(\frac{a}{b}\right)^{\xi(n)} - \frac{c^2}{a}\left(\frac{a}{b}\right)^{\xi(n)} - 1\right) \\
& + \left(\frac{a}{b}\right)^{\xi(n)} \left(\left(\frac{b}{a}\right)^{\xi(n)} 2 + bi + \left(\frac{b}{a}\right)^{\xi(n)} (ab + 2c)\,\epsilon + (b\,(ab + 3c))\,h\right) - 2 \\
& + \Delta \left(\frac{a}{b}\right)^{\xi(n)} \frac{c}{a} \left(\left(\frac{a}{b}\right)^{\xi(n)} bi + \left(b\left(\frac{a}{b}\right)^{\xi(n)} + c\right)\epsilon - h\right)
\end{aligned}
$$

After some necessary simplifications, we get the result (2.2).

Similarly, we can obtain $\beta_{\xi(n)}\alpha_{\xi(n)}$. $\qquad\square$

By using the Lemma 1, we have

$$
\alpha_{\xi(n)}\beta_{\xi(n)} + \beta_{\xi(n)}\alpha_{\xi(n)} = \begin{cases} 2\left(\mathbb{K}_{v,0} - \theta\right), & \text{if } n \text{ is even} \\ 2\left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta}\right), & \text{if } n \text{ is odd.} \end{cases} \tag{2.4}
$$

$$
\alpha_{\xi(n)}\beta_{\xi(n)} - \beta_{\xi(n)}\alpha_{\xi(n)} = \begin{cases} 2\Delta\frac{c}{a}\left(\mathbb{K}_{u,0} - \eta\right), & \text{if } n \text{ is even} \\ 2\Delta\frac{c}{b}\left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta}\right), & \text{if } n \text{ is odd.} \end{cases} \tag{2.5}
$$

Lemma 2.

$$
\alpha_{\xi(n)}\alpha_{\xi(n)} = \begin{cases} \mathbb{K}_{v,0} + \mu_e + \frac{\Delta}{a}\left(\mathbb{K}_{u,0} + \gamma_e\right), & \text{if } n \text{ is even} \\ \mathbb{K}_{\widehat{v},0} + \mu_o + \frac{\Delta}{b}\left(\mathbb{K}_{\widehat{u},0} + \gamma_o\right), & \text{if } n \text{ is odd} \end{cases} \tag{2.6}
$$

$$
\beta_{\xi(n)}\beta_{\xi(n)} = \begin{cases} \mathbb{K}_{v,0} + \mu_e - \frac{\Delta}{a}\left(\mathbb{K}_{u,0} + \gamma_e\right), & \text{if } n \text{ is even} \\ \mathbb{K}_{\widehat{v},0} + \mu_o - \frac{\Delta}{b}\left(\mathbb{K}_{\widehat{u},0} + \gamma_o\right), & \text{if } n \text{ is odd} \end{cases} \tag{2.7}
$$

where

$$
\mu_e \quad : \quad = -1 + \frac{b}{a}c\,(u_5 + 2u_2 - u_1) + b\gamma_e
$$

$$
\mu_o \quad : \quad = -1 + \frac{a}{b}c\left(u_5 + 2\frac{b}{a}u_2 - u_1\right) + a\gamma_o
$$

and

$$
\gamma_e \quad : \quad = \frac{1}{2}\left(\frac{b}{a}u_6 + 2u_3 - \frac{b}{a}u_2\right)
$$

$$
\gamma_o \quad : \quad = \frac{1}{2}\left(u_6 + 2u_3 - u_2\right).
$$

Proof. By considering the relations

$$
\alpha_{\xi(n)}\alpha_{\xi(n)} = 2\alpha_{\xi(n)} - C\left(\alpha_{\xi(n)}\right)
$$

and

$$
\beta_{\xi(n)}\beta_{\xi(n)} = 2\beta_{\xi(n)} - C\left(\beta_{\xi(n)}\right),
$$

where $C\left(\alpha_{\xi(n)}\right)$ is the character of the hybrid number $\alpha_{\xi(n)}$ and using the relations (1.9) and (1.10), we get the desired result. $\qquad\square$

Remark 2. If we take $a = b = p$ and $c = q$, we obtain the analogous relations for (p, q)-Fibonacci hybrid numbers in [8, Lemma 2.9].

Theorem 3. *(Vajda's like identity) For nonnegative integers n, r, and s, we have*

$$\mathbb{K}_{w,n+2r}\mathbb{K}_{w,n+2s} - \mathbb{K}_{w,n}\mathbb{K}_{w,n+2(r+s)}$$
$$= \begin{cases} (-c)^n AB\Delta^2 u_{2r} \left((\mathbb{K}_{v,0} - \theta) u_{2s} - c(\mathbb{K}_{u,0} - \eta) v_{2s}\right), & \text{if } n \text{ is even} \\ (-c)^n AB\Delta^2 u_{2r} \left(\left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta}\right) \frac{b}{a} u_{2s} - c(\mathbb{K}_{\widehat{u},0} - \widehat{\eta}) v_{2s}\right), & \text{if } n \text{ is odd.} \end{cases}$$

Proof. From the Binet formula of the bi-periodic Horadam hybrid numbers, we get

$$\mathbb{K}_{w,n+2r}\mathbb{K}_{w,n+2s} - \mathbb{K}_{w,n}\mathbb{K}_{w,n+2(r+s)}$$

$$= \frac{a^{\xi(n+2r+1)}}{(ab)^{\left\lfloor \frac{n+2r}{2} \right\rfloor}} \left(A\alpha_{\xi(n)}\alpha^{n+2r} - B\beta_{\xi(n)}\beta^{n+2r}\right) \frac{a^{\xi(n+2s+1)}}{(ab)^{\left\lfloor \frac{n+2s}{2} \right\rfloor}} \left(A\alpha_{\xi(n)}\alpha^{n+2s} - B\beta_{\xi(n)}\beta^{n+2s}\right)$$

$$- \frac{a^{\xi(n+1)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(A\alpha_{\xi(n)}\alpha^{n} - B\beta_{\xi(n)}\beta^{n}\right) \frac{a^{\xi(n+2(r+s)+1)}}{(ab)^{\left\lfloor \frac{n+2(r+s)}{2} \right\rfloor}} \left(A\alpha_{\xi(n)}\alpha^{n+2(r+s)} - B\beta_{\xi(n)}\beta^{n+2(r+s)}\right)$$

$$= \frac{a^{2\xi(n+1)}}{(ab)^{2\left\lfloor \frac{n}{2} \right\rfloor + r+s}} \left(-AB\alpha_{\xi(n)}\beta_{\xi(n)}\alpha^{n+2r}\beta^{n+2s} - AB\beta_{\xi(n)}\alpha_{\xi(n)}\alpha^{n+2s}\beta^{n+2r}\right.$$

$$\left. + AB\alpha_{\xi(n)}\beta_{\xi(n)}\alpha^{n}\beta^{n+2(r+s)} + AB\beta_{\xi(n)}\alpha_{\xi(n)}\alpha^{n+2(r+s)}\beta^{n}\right)$$

$$= \frac{a^{2\xi(n+1)}}{(ab)^{2\left\lfloor \frac{n}{2} \right\rfloor + r+s}} AB(\alpha\beta)^n \left(\alpha_{\xi(n)}\beta_{\xi(n)}\beta^{2s}\left(\beta^{2r} - \alpha^{2r}\right) + \beta_{\xi(n)}\alpha_{\xi(n)}\alpha^{2s}\left(\alpha^{2r} - \beta^{2r}\right)\right)$$

$$= \frac{a^{2\xi(n+1)}}{(ab)^{2\left\lfloor \frac{n}{2} \right\rfloor + r+s}} AB(\alpha\beta)^n \left(\alpha^{2r} - \beta^{2r}\right) \left(\beta_{\xi(n)}\alpha_{\xi(n)}\alpha^{2s} - \alpha_{\xi(n)}\beta_{\xi(n)}\beta^{2s}\right).$$

If n is even, by considering the relations (2.2) and (2.3), we obtain

$$\mathbb{K}_{w,n+2r}\mathbb{K}_{w,n+2s} - \mathbb{K}_{w,n}\mathbb{K}_{w,n+2(r+s)}$$

$$= \frac{a^{2\xi(n+1)}}{(ab)^{2\left\lfloor \frac{n}{2} \right\rfloor + r+s}} AB(\alpha\beta)^n \left(\alpha^{2r} - \beta^{2r}\right) \left(\beta_{\xi(n)}\alpha_{\xi(n)}\alpha^{2s} - \alpha_{\xi(n)}\beta_{\xi(n)}\beta^{2s}\right)$$

$$= \frac{a^2(-c)^n}{(ab)^{r+s}} AB\left(\alpha^{2r} - \beta^{2r}\right) \left((\mathbb{K}_{v,0} - \theta)\left(\alpha^{2s} - \beta^{2s}\right) - \frac{\Delta}{a}c(\mathbb{K}_{u,0} - \eta)\left(\alpha^{2s} + \beta^{2s}\right)\right)$$

$$= \frac{a^2(-c)^n}{(ab)^{r+s}} AB\frac{(ab)^r}{a}\Delta u_{2r}\left((\mathbb{K}_{v,0} - \theta)\left(\frac{(ab)^s}{a}u_{2s}\Delta\right) - \frac{\Delta}{a}c(\mathbb{K}_{u,0} - \eta)\left((ab)^s v_{2s}\right)\right)$$

$$= (-c)^n ABu_{2r}\Delta^2 \left((\mathbb{K}_{v,0} - \theta) u_{2s} - c(\mathbb{K}_{u,0} - \eta) v_{2s}\right).$$

Similarly, we obtain the desired result for odd n. $\qquad\qquad\square$

Corollary 1. *If we take $s = -r$, we get the Catalan's like identity:*

$$\mathbb{K}_{w,n+2r}\mathbb{K}_{w,n-2r} - \mathbb{K}_{w,n}^2$$
$$= \begin{cases} (-1)^{n+1}\, c^{n-2r}AB\Delta^2 u_{2r}\left((\mathbb{K}_{v,0} - \theta)\, u_{2r} + c\,(\mathbb{K}_{u,0} - \eta)\, v_{2r}\right), & \text{if } n \text{ is even} \\ (-1)^{n+1}\, c^{n-2r}AB\Delta^2 u_{2r}\left(\left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta}\right)\frac{b}{a}u_{2r} + c\left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta}\right) v_{2r}\right), & \text{if } n \text{ is odd.} \end{cases}$$

Corollary 2. *If we take $s = -r$ and $r = 1$, we get the Cassini's like identity:*

$$\mathbb{K}_{w,n+2}\mathbb{K}_{w,n-2} - \mathbb{K}_{w,n}^2$$
$$= \begin{cases} (-1)^{n+1}\, ac^{n-2}AB\Delta^2\left((\mathbb{K}_{v,0} - \theta)\, a + c\,(ab + 2c)\,(\mathbb{K}_{u,0} - \eta)\right), & \text{if } n \text{ is even} \\ (-1)^{n+1}\, ac^{n-2}AB\Delta^2\left(\left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta}\right) b + c\,(ab + 2c)\left(\mathbb{K}_{\widehat{u},0} - \widehat{\eta}\right)\right), & \text{if } n \text{ is odd.} \end{cases}$$

Note that for even case, the Cassini's like identity can be stated as by means of the following matrix identity:

$$\begin{bmatrix} \mathbb{K}_{w,2n+2} & \mathbb{K}_{w,2n} \\ \mathbb{K}_{w,2n} & \mathbb{K}_{w,2n-2} \end{bmatrix} = \begin{bmatrix} \mathbb{K}_{w,4} & \mathbb{K}_{w,2} \\ \mathbb{K}_{w,2} & \mathbb{K}_{w,0} \end{bmatrix} \begin{bmatrix} ab + 2c & -c^2 \\ 1 & 0 \end{bmatrix}^{n-1}. \quad (2.8)$$

By taking determinant from above to down below of both sides of the matrix equality (2.8), we get

$$\mathbb{K}_{w,2n+2}\mathbb{K}_{w,2n-2} - \mathbb{K}_{w,2n}^2 = c^{2n-2}\left(\mathbb{K}_{w,4}\mathbb{K}_{w,0} - \mathbb{K}_{w,2}^2\right). \quad (2.9)$$

By taking determinant from down below to above of both sides of the matrix equality (2.8), we get

$$\mathbb{K}_{w,2n-2}\mathbb{K}_{w,2n+2} - \mathbb{K}_{w,2n}^2 = c^{2n-2}\left(\mathbb{K}_{w,0}\mathbb{K}_{w,4} - \mathbb{K}_{w,2}^2\right). \quad (2.10)$$

Theorem 4. *For $n \geq 1$, we have*

$$\sum_{r=1}^{n}\mathbb{K}_{w,r} = \frac{c^2\left(\mathbb{K}_{w,n} + \mathbb{K}_{w,n-1} - \mathbb{K}_{w,0} - \mathbb{K}_{w,-1}\right) - \mathbb{K}_{w,n+2} - \mathbb{K}_{w,n+1} + \mathbb{K}_{w,2} + \mathbb{K}_{w,1}}{c^2 - ab - 2c + 1}.$$

Proof. First note that by considering the formula in (1.8), the bi-periodic Horadam hybrid numbers for negative subscripts can be defined as

$$\mathbb{K}_{w,-n} = w_{-n} + w_{-n+1}i + w_{-n+2}\epsilon + w_{-n+3}h.$$

If n is odd, we have

$$\sum_{r=1}^{n}\mathbb{K}_{w,r} = \sum_{r=1}^{\frac{n-1}{2}}\mathbb{K}_{w,2r} + \sum_{r=1}^{\frac{n+1}{2}}\mathbb{K}_{w,2r-1}$$

$$
= \frac{a}{(ab)^{\frac{n-1}{2}}} \left(\frac{A\alpha_{\xi(n)}\alpha^{n+1} - A\alpha_{\xi(n)}\alpha^2 (ab)^{\frac{n-1}{2}}}{\alpha^2 - ab} + \frac{-B\beta_{\xi(n)}\beta^{n+1} + B\beta_{\xi(n)}\beta^2 (ab)^{\frac{n-1}{2}}}{\beta^2 - ab} \right)
$$

$$
+ \frac{ab}{(ab)^{\frac{n+1}{2}}} \left(\frac{A\alpha_{\xi(n)}\alpha^{n+2} - A\alpha_{\xi(n)}\alpha (ab)^{\frac{n+1}{2}}}{\alpha^2 - ab} + \frac{-B\beta_{\xi(n)}\beta^{n+2} + B\beta_{\xi(n)}\beta (ab)^{\frac{n+1}{2}}}{\beta^2 - ab} \right)
$$

$$
= \sum_{r=1}^{\frac{n-1}{2}} \frac{a}{(ab)^r} \left(A\alpha_{\xi(n)}\alpha^{2r} - B\beta_{\xi(n)}\beta^{2r} \right) + \sum_{r=1}^{\frac{n-1}{2}+1} \frac{ab}{(ab)^r} \left(A\alpha_{\xi(n)}\alpha^{2r-1} - B\beta_{\xi(n)}\beta^{2r-1} \right)
$$

$$
= aA\alpha_{\xi(n)} \sum_{r=1}^{\frac{n-1}{2}} \left(\frac{\alpha^2}{ab} \right)^r - aB\beta_{\xi(n)} \sum_{r=1}^{\frac{n-1}{2}} \left(\frac{\beta^2}{ab} \right)^r
$$

$$
+ \frac{ab}{\alpha} A\alpha_{\xi(n)} \sum_{r=1}^{\frac{n-1}{2}+1} \left(\frac{\alpha^2}{ab} \right)^r - \frac{ab}{\beta} B\beta_{\xi(n)} \sum_{r=1}^{\frac{n-1}{2}+1} \left(\frac{\beta^2}{ab} \right)^r
$$

$$
= aA\alpha_{\xi(n)} \left(\frac{\left(\frac{\alpha^2}{ab} \right)^{\frac{n-1}{2}+1} - \frac{\alpha^2}{ab}}{\frac{\alpha^2}{ab} - 1} \right) - aB\beta_{\xi(n)} \left(\frac{\left(\frac{\beta^2}{ab} \right)^{\frac{n-1}{2}+1} - \frac{\beta^2}{ab}}{\frac{\beta^2}{ab} - 1} \right)
$$

$$
+ \frac{ab}{\alpha} A\alpha_{\xi(n)} \left(\frac{\left(\frac{\alpha^2}{ab} \right)^{\frac{n-1}{2}+2} - \frac{\alpha^2}{ab}}{\frac{\alpha^2}{ab} - 1} \right) - \frac{ab}{\beta} B\beta_{\xi(n)} \left(\frac{\left(\frac{\beta^2}{ab} \right)^{\frac{n-1}{2}+2} - \frac{\beta^2}{ab}}{\frac{\beta^2}{ab} - 1} \right)
$$

$$
= \frac{a}{(ab)^{\frac{n-1}{2}} (\alpha^2 - ab) (\beta^2 - ab)} \times
$$
$$
\left((\alpha\beta)^2 \left(A\alpha_{\xi(n)}\alpha^{n-1} - B\beta_{\xi(n)}\beta^{n-1} \right) - ab \left(A\alpha_{\xi(n)}\alpha^{n+1} - B\beta_{\xi(n)}\beta^{n+1} \right) \right.
$$
$$
\left. + (ab)^{\frac{n-1}{2}} \left(-(\alpha\beta)^2 \left(A\alpha_{\xi(n)} - B\beta_{\xi(n)} \right) + ab \left(A\alpha_{\xi(n)}\alpha^2 - B\beta_{\xi(n)}\beta^2 \right) \right) \right)
$$

$$
= \frac{c^2 \left(\mathbb{K}_{w,n-1} - \mathbb{K}_{w,0} + \mathbb{K}_{w,n} - \mathbb{K}_{w,-1} \right) - \mathbb{K}_{w,n+1} - \mathbb{K}_{w,n+2} + \mathbb{K}_{w,2} + \mathbb{K}_{w,1}}{c^2 - ab - 2c + 1}
$$

If n is even, we have

$$
\sum_{r=1}^{n} \mathbb{K}_{w,r} = \sum_{r=1}^{\frac{n}{2}} \mathbb{K}_{w,2r} + \sum_{r=1}^{\frac{n}{2}} \mathbb{K}_{w,2r-1}.
$$

In a similar manner, we get the desired result. □

Theorem 5. *For nonnegative integers n and r, we have*

$(i) \sum_{i=0}^{n} \binom{n}{i} (-c)^{n-i} \mathbb{K}_{w,2i+r} = (ab)^{\frac{n}{2}} \left(\frac{a}{b}\right)^{\frac{\xi(n+r)-\xi(r)}{2}} \mathbb{K}_{w,n+r}.$

$(ii) \sum_{i=0}^{n} \binom{n}{i} c^{n-i} (ab)^{\frac{i}{2}} \left(\frac{a}{b}\right)^{\frac{\xi(i+r)-\xi(r)}{2}} \mathbb{K}_{w,i+r} = \mathbb{K}_{w,2n+r}.$

Proof. (i) From the Binet formula of the bi-periodic Horadam hybrid numbers, we get

$$\sum_{i=0}^{n} \binom{n}{i} (-c)^{n-i} \mathbb{K}_{w,2i+r}$$

$$= \sum_{i=0}^{n} \binom{n}{i} (-c)^{n-i} \frac{a^{\xi(2i+r+1)}}{(ab)^{\left\lfloor \frac{2i+r}{2} \right\rfloor}} \left(A\alpha_{\xi(n)}\alpha^{2i+r} - B\beta_{\xi(n)}\beta^{2i+r} \right)$$

$$= \frac{a^{\xi(r+1)}}{(ab)^{\left\lfloor \frac{r}{2} \right\rfloor}} A\alpha_{\xi(n)}\alpha^r \sum_{i=0}^{n} \binom{n}{i} (-c)^{n-i} \left(\frac{\alpha^2}{ab}\right)^i - \frac{a^{\xi(r+1)}}{(ab)^{\left\lfloor \frac{r}{2} \right\rfloor}} B\beta_{\xi(n)}\beta^r \sum_{i=0}^{n} \binom{n}{i} (-c)^{n-i} \left(\frac{\beta^2}{ab}\right)^i$$

$$= \frac{a^{\xi(r+1)}}{(ab)^{\left\lfloor \frac{r}{2} \right\rfloor}} A\alpha_{\xi(n)}\alpha^r \left(\frac{\alpha^2}{ab} - c\right)^n - \frac{a^{\xi(r+1)}}{(ab)^{\left\lfloor \frac{r}{2} \right\rfloor}} B\beta_{\xi(n)}\beta^r \left(\frac{\beta^2}{ab} - c\right)^n$$

$$= \frac{a^{\xi(r+1)}}{(ab)^{\left\lfloor \frac{r}{2} \right\rfloor}} \left(A\alpha_{\xi(n)}\alpha^{n+r} - B\beta_{\xi(n)}\beta^{n+r} \right) = \frac{a^{\xi(r+1)}}{(ab)^{\left\lfloor \frac{r}{2} \right\rfloor}} \frac{(ab)^{\left\lfloor \frac{n+r}{2} \right\rfloor}}{a^{\xi(n+r+1)}} \mathbb{K}_{w,n+r}$$

$$= \frac{a^{-\xi(r)+\xi(n+r)}}{(ab)^{\left\lfloor \frac{r}{2} \right\rfloor - \left\lfloor \frac{n+r}{2} \right\rfloor}} \mathbb{K}_{w,n+r} = (ab)^{\frac{n}{2}} \left(\frac{a}{b}\right)^{\frac{-\xi(r)+\xi(n+r)}{2}} \mathbb{K}_{w,n+r}.$$

(ii) It can be proven similarly. $\square$

3. Some relations between generalized bi-periodic Fibonacci and Lucas hybrid numbers

Now we state some relations between bi-periodic Fibonacci and bi-periodic Lucas hybrid numbers. To do this, we consider the generalized bi-periodic Fibonacci hybrid numbers $\mathbb{K}_{u,n}$ and the generalized bi-periodic Lucas hybrid numbers $\mathbb{K}_{v,n}$ which are stated in Table 1.

The Binet formula of $\mathbb{K}_{u,n}$ is

$$\mathbb{K}_{u,n} = \frac{a^{\xi(n+1)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(\frac{\alpha_{\xi(n)}\alpha^n - \beta_{\xi(n)}\beta^n}{\alpha - \beta} \right), \tag{3.1}$$

and the Binet formula of $\mathbb{K}_{v,n}$ is

$$\mathbb{K}_{v,n} = \frac{a^{-\xi(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left(\alpha_{\xi(n)} \alpha^n + \beta_{\xi(n)} \beta^n \right). \qquad (3.2)$$

Theorem 6. *For any natural number m, n with $n > m$, we have*

(i) $\mathbb{K}_{u,n+1} + c\mathbb{K}_{u,n-1} = \left(\dfrac{a}{b} \right)^{\xi(n)} \mathbb{K}_{v,n}$

(ii) $\mathbb{K}_{v,n+1} + c\mathbb{K}_{v,n-1} = \left(\dfrac{a}{b} \right)^{\xi(n)} \Delta^2 \mathbb{K}_{u,n}$

(iii) $\mathbb{K}_{u,n}\mathbb{K}_{v,m} - \mathbb{K}_{u,m}\mathbb{K}_{v,n} = \begin{cases} 2 \left(-c \right)^m u_{n-m} \left(\mathbb{K}_{v,0} - \theta \right), & \text{if } n \text{ is even} \\ 2 \left(\frac{a}{b} \right)^{-\xi(m)} \left(-c \right)^m u_{n-m} \left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta} \right), & \text{if } n \text{ is odd.} \end{cases}$

(iv) $\mathbb{K}_{v,n}^2 - \mathbb{K}_{u,n}^2 = \begin{cases} \left(\frac{\Delta^2 - a^2}{\Delta^2} \right) \left(\mathbb{K}_{v,0} + \mu_e \right) v_{2n} + \left(\frac{\Delta^2 - a^2}{a^2} \right) \left(\mathbb{K}_{u,0} + \gamma_e \right) u_{2n} \\ \qquad + 2 \left(-c \right)^n \left(\frac{\Delta^2 + a^2}{\Delta^2} \right) \left(\mathbb{K}_{v,0} - \theta \right), & \text{if } n \text{ is even} \\[4pt] \left(\frac{\Delta^2 - a^2}{\Delta^2} \right) \left(\mathbb{K}_{\widehat{v},0} + \mu_o \right) \frac{bv_{2n}}{a} + \left(\frac{\Delta^2 - a^2}{a^2} \right) \left(\mathbb{K}_{\widehat{u},0} + \gamma_o \right) u_{2n} \\ \qquad + \frac{2b}{a} \left(-c \right)^n \left(\frac{\Delta^2 + a^2}{\Delta^2} \right) \left(\mathbb{K}_{\widehat{v},0} - \widehat{\theta} \right), & \text{if } n \text{ is odd.} \end{cases}$

Proof. (i) From the relations (3.1) and (3.2), we have,

$\mathbb{K}_{u,n+1} + c\mathbb{K}_{u,n-1}$

$$= \frac{a^{\xi(n)}}{(ab)^{\left\lfloor \frac{n+1}{2} \right\rfloor}} \left(\frac{\alpha_{\xi(n)} \alpha^{n+1} - \beta_{\xi(n)} \beta^{n+1}}{\alpha - \beta} \right) + c \frac{a^{\xi(n)}}{(ab)^{\left\lfloor \frac{n-1}{2} \right\rfloor}} \left(\frac{\alpha_{\xi(n)} \alpha^{n-1} - \beta_{\xi(n)} \beta^{n-1}}{\alpha - \beta} \right)$$

$$= \frac{a^{\xi(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor + \xi(n)}} \left(\frac{\alpha_{\xi(n)} \alpha^{n+1} - \beta_{\xi(n)} \beta^{n+1} - \alpha_{\xi(n)} \alpha^n \beta + \beta_{\xi(n)} \beta^n \alpha}{\alpha - \beta} \right)$$

$$= \frac{a^{\xi(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor + \xi(n)}} \left(\frac{\alpha_{\xi(n)} \alpha^n \left(\alpha - \beta \right) + \beta_{\xi(n)} \beta^n \left(\alpha - \beta \right)}{\alpha - \beta} \right)$$

$$= \frac{a^{\xi(n)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor + \xi(n)}} \left(\alpha_{\xi(n)} \alpha^n + \beta_{\xi(n)} \beta^n \right) = \left(\frac{a}{b} \right)^{\xi(n)} \mathbb{K}_{v,n}$$

(ii) The proof can be done similarly as (i) by using the relations (3.1) and (3.2).

(iii) By using the Binet formulas for $\mathbb{K}_{u,n}$ and $\mathbb{K}_{v,n}$, and considering the relation (2.4), we get the desired result.

(iv) By using the Binet formulas for $\mathbb{K}_{u,n}$ and $\mathbb{K}_{v,n}$, we have

$\Delta^2 \left(\mathbb{K}_{v,n}\mathbb{K}_{v,n} - \mathbb{K}_{u,n}\mathbb{K}_{u,n} \right)$

$$= \frac{a^{2\xi(n+1)}}{(ab)^{2\lfloor \frac{n}{2} \rfloor}}$$

$$\frac{\Delta^2}{a^2} \left(\alpha_{\xi(n)} \alpha_{\xi(n)} \alpha^{2n} + \beta_{\xi(n)} \beta_{\xi(n)} \beta^{2n} + (\alpha\beta)^n \left(\alpha_{\xi(n)} \beta_{\xi(n)} + \beta_{\xi(n)} \alpha_{\xi(n)} \right) \right)$$

$$- \left(\alpha_{\xi(n)} \alpha_{\xi(n)} \alpha^{2n} + \beta_{\xi(n)} \beta_{\xi(n)} \beta^{2n} - (\alpha\beta)^n \left(\alpha_{\xi(n)} \beta_{\xi(n)} + \beta_{\xi(n)} \alpha_{\xi(n)} \right) \right)$$

$$= \frac{a^{2\xi(n+1)}}{(ab)^{2\lfloor \frac{n}{2} \rfloor}} \left(\left(\frac{\Delta^2}{a^2} - 1 \right) \left(\alpha_{\xi(n)} \alpha_{\xi(n)} \alpha^{2n} + \beta_{\xi(n)} \beta_{\xi(n)} \beta^{2n} \right) \right.$$

$$\left. + \left(\frac{\Delta^2}{a^2} + 1 \right) (\alpha\beta)^n \left(\alpha_{\xi(n)} \beta_{\xi(n)} + \beta_{\xi(n)} \alpha_{\xi(n)} \right) \right).$$

By considering the relations (2.4), (2.6), and (2.7), we get the desired result. □

4. Conclusion

In recent years, many studies have been devoted to investigate the hybrid numbers whose components are from the special number sequences such as Fibonacci, Lucas, Pell, Horadam numbers, etc. This work provides a new generalization for hybrid numbers whose coefficients are from the Horadam numbers. Most of the results of this study generalize the results of those were given in [8,11,13]. It would also be interesting to study the algebraic structure of these new hybrid numbers.

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