



## Research Article

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# On a new generalization of some Hilbert-type inequalities

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**Abstract:** In this work, by introducing several parameters, a new kernel function including both the homogeneous and non-homogeneous cases is constructed, and a Hilbert-type inequality related to the newly constructed kernel function is established. By convention, the equivalent Hardy-type inequality is also considered. Furthermore, by introducing the partial fraction expansions of trigonometric functions, some special and interesting Hilbert-type inequalities with the constant factors represented by the higher derivatives of trigonometric functions, the Euler number and the Bernoulli number are presented at the end of the paper.

**Keywords:** Hilbert-type inequality, partial fraction expansion, Euler number, Bernoulli number

**MSC 2020:** 26D15, 41A17

## 1 Introduction

Let  $\|f\|_{p,\mu}$  denote the norm of a measurable function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  with respect to a measurable weighted function  $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ , that is,

$$\|f\|_{p,\mu} := \left( \int_{\mathbb{R}^+} \mu(x) f^p(x) dx \right)^{\frac{1}{p}},$$

where  $p \geq 1$ . Under the definition of  $\|f\|_{p,\mu}$ , we define a weighted measured function space  $L_{p,\mu}(\mathbb{R}^+)$  as follows:

$$L_{p,\mu}(\mathbb{R}^+) := \{f : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \|f\|_{p,\mu} < \infty\}.$$

In particular, for  $\mu(x) = 1$ , we have the abbreviated form:  $\|f\|_{p,\mu} = \|f\|_p$  and  $L_{p,\mu}(\mathbb{R}^+) = L_p(\mathbb{R}^+)$ .

Consider two nonnegative real-valued functions  $f \in L_p(\mathbb{R}^+)$  and  $g \in L_q(\mathbb{R}^+)$ , where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Then we have the following classical inequality [1]:

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin \frac{\pi}{p}} \|f\|_p \|g\|_q, \quad (1.1)$$

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where the constant factor  $\frac{\pi}{\sin \frac{\pi}{p}}$  is the best possible. Inequality (1.1) is the so-called Hilbert inequality, which is one of the most important inequalities in analysis and its relevant applications. Additionally, we also have some inequalities similar to inequality (1.1), such as [1]

$$\int_0^{\infty} \int_0^{\infty} \frac{\log \frac{x}{y}}{x-y} f(x)g(y) dx dy < \left( \frac{\pi}{\sin \frac{\pi}{p}} \right)^2 \|f\|_p \|g\|_q. \quad (1.2)$$

In general, such inequalities as inequality (1.2) are called Hilbert-type inequalities. Although (1.1) and (1.2) were put forward more than 100 years ago, mathematicians have always been interested in their extensions, refinements, analogies and high-dimensional generalizations. The following inequality is a classical extension of (1.1), which was established by Yang [2] in 2004, that is,

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x^\beta + y^\beta} dx dy < \frac{\pi}{\beta \sin \frac{\pi}{r}} \|f\|_{p,\mu} \|g\|_{q,\nu}, \quad (1.3)$$

where  $\beta > 0$ ,  $\mu(x) = x^{p(1-\frac{\beta}{r})-1}$ ,  $\nu(y) = y^{q(1-\frac{\beta}{s})-1}$  and  $\frac{1}{r} + \frac{1}{s} = 1$ , and the constant factor on the right-hand side of (1.3) is the best possible. For some other extensions of inequality (1.1), and the relevant research on the discrete and half-discrete cases corresponding to (1.1), we can refer to [3–17]. Furthermore, some analogical forms of inequality (1.1) can be found in [16], such as

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x^{2\beta} + x^\beta y^\beta + y^{2\beta}} dx dy < \frac{2\sqrt{3}\pi}{9\beta} \|f\|_{2,\mu} \|g\|_{2,\nu} \quad (1.4)$$

and

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x^{2\beta} - x^\beta y^\beta + y^{2\beta}} dx dy < \frac{4\sqrt{3}\pi}{9\beta} \|f\|_{2,\mu} \|g\|_{2,\nu}, \quad (1.5)$$

where  $\beta > 0$ ,  $\mu(x) = x^{1-2\beta}$  and  $\nu(y) = y^{1-2\beta}$ .

Regarding inequality (1.2), Yang [18] gave an analogy as follows in 2008:

$$\int_0^{\infty} \int_0^{\infty} \frac{\left| \log \frac{x}{y} \right|}{x+y} f(x)g(y) dx dy < 8c_0 \|f\|_2 \|g\|_2, \quad (1.6)$$

where  $c_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.91596^+$  is the Catalan constant.

Some other inequalities related to (1.2) and (1.6), including the discrete and half-discrete cases, can be found in [7,16,19–21]. In addition, by the introduction of different new kernel functions, multiple parameters and special functions, a large number of new Hilbert-type inequalities were established in the past several decades (see [22–33]).

In this work, we will establish the following Hilbert-type inequalities with the best constant factors:

$$\int_0^{\infty} \int_0^{\infty} \frac{\left( \log \frac{x}{y} \right)^2}{x^2 + xy + y^2} f(x)g(y) dx dy < \frac{16\sqrt{3}\pi}{243} \|f\|_{2,\mu} \|g\|_{2,\nu_1}, \quad (1.7)$$

$$\int_0^{\infty} \int_0^{\infty} \frac{\log \frac{x}{y}}{(x-y)(x^2 + y^2)} f(x)g(y) dx dy < \frac{3\pi^2}{16} \|f\|_{2,\mu} \|g\|_{2,\nu_2}, \quad (1.8)$$

$$\int_0^{\infty} \int_0^{\infty} \frac{\left( \log \frac{x}{y} \right)^2}{x^2 - xy + y^2} f(x)g(y) dx dy < \frac{43\sqrt{3}\pi}{36} \|f\|_{2,\mu} \|g\|_{2,\nu_1}, \quad (1.9)$$

where  $\mu(x) = x^{-1}$ ,  $\nu_1(y) = y^{-1}$  and  $\nu_2(y) = y^{-3}$ .

More generally, we will construct a new kernel including both the homogeneous and non-homogeneous cases, and establish a new Hilbert-type inequality which is a unified extension of inequality (1.1)–(1.9). Furthermore, the equivalent Hardy-type inequality is also considered. The discussions will be closed with some corollaries addressing special Hilbert-type inequalities with the constant factors associated with the higher derivatives of trigonometric functions.

## 2 Definitions and lemmas

**Lemma 2.1.** *Let  $n \in \mathbb{N}^+$ ,  $\delta \in \{1, -1\}$  and  $n\beta > \lambda > 0$ . Let  $\alpha$  be such that  $\alpha > -1$  for  $\delta^n = 1$  and  $\alpha > 0$  for  $\delta^n = -1$ . Define*

$$K(t) := \frac{|\log t|^\alpha}{|1 + \delta t^\beta + \dots + (\delta t^\beta)^n|}. \tag{2.1}$$

Then  $\int_0^\infty K(t)t^{\lambda-1}dt$  converges.

**Proof.** For the case  $\alpha > -1$  and  $\delta^n = 1$ , let  $\theta$  be such that  $0 < \theta < n\beta - \lambda$ . Then

$$\lim_{t \rightarrow +\infty} \frac{K(t)t^{\lambda-1}}{t^{-n\beta+\lambda+\theta-1}} = \lim_{t \rightarrow +\infty} \frac{|\log t|^\alpha t^{\lambda-1}(\delta t^\beta)^{-n}}{t^{-n\beta+\lambda+\theta-1}} = \lim_{t \rightarrow +\infty} \frac{|\log t|^\alpha}{t^\theta} = 0.$$

Since  $-n\beta + \lambda + \theta - 1 < -1$ , it follows that  $\int_2^\infty K(t)t^{\lambda-1}dt$  converges.

In addition, since  $n \in \mathbb{N}^+$ ,  $\delta \in \{1, -1\}$  and  $\delta^n = 1$ , we can easily obtain

$$\lim_{t \rightarrow 1} |1 + \delta t^\beta + \dots + (\delta t^\beta)^n| = 1 + \delta + \delta^2 + \dots + \delta^n \neq 0.$$

Therefore,

$$\lim_{t \rightarrow 1} \frac{K(t)t^{\lambda-1}}{|1 - t|^\alpha} = \frac{1}{1 + \delta + \delta^2 + \dots + \delta^n} \lim_{t \rightarrow 1} \frac{|\log t|^\alpha}{|1 - t|^\alpha} = \frac{1}{1 + \delta + \delta^2 + \dots + \delta^n}. \tag{2.2}$$

Hence, it follows from (2.2) that  $\int_{\frac{1}{2}}^2 K(t)t^{\lambda-1}dt$  converges.

Finally, it is needed to prove that  $\int_0^{\frac{1}{2}} K(t)t^{\lambda-1}dt$  converges, and this is obvious according to the following equality, that is,

$$\lim_{t \rightarrow 0^+} \frac{K(t)t^{\lambda-1}}{t^{\lambda+\theta-1}} = \lim_{t \rightarrow 0^+} \frac{|\log t|^\alpha}{t^\theta} = 0,$$

where  $-\lambda < \theta < 0$ . Based on the above discussions, it is proved that  $\int_0^\infty K(t)t^{\lambda-1}dt$  converges under the condition  $\alpha > -1$  and  $\delta^n = 1$ . Similarly, it can also be proved that  $\int_0^\infty K(t)t^{\lambda-1}dt$  converges under the condition  $\alpha > 0$  and  $\delta^n = -1$ . □

**Lemma 2.2.** *Let  $n \in \mathbb{N}^+$ ,  $\delta \in \{1, -1\}$  and  $\beta n > \lambda > 0$ . Let  $\alpha$  be such that  $\alpha > -1$  for  $\delta^n = 1$  and  $\alpha > 0$  for  $\delta^n = -1$ . Define*

$$C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) := \sum_{k=0}^\infty \delta^{k(n+1)} \left\{ \frac{1}{(k\beta(n+1) + \lambda)^{\alpha+1}} - \frac{\delta}{(k\beta(n+1) + \beta(n+1) - \lambda)^{\alpha+1}} \right\} + \sum_{k=0}^\infty \delta^{k(n+1)} \left\{ \frac{1}{(k\beta(n+1) + \beta n - \lambda)^{\alpha+1}} - \frac{\delta}{(k\beta(n+1) + \lambda + \beta)^{\alpha+1}} \right\}. \tag{2.3}$$

Then

$$\int_0^\infty K(t)t^{\lambda-1}dt = \Gamma(\alpha + 1)C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2), \tag{2.4}$$

where  $\Gamma(u) := \int_0^\infty x^{u-1}e^{-x}dx$ , ( $u > 0$ ) is the  $\Gamma$ -function [34], and  $\Gamma(u) = (u - 1)!$  for  $u \in \mathbb{N}^+$ .

**Proof.**

$$\begin{aligned} \int_0^\infty K(t)t^{\lambda-1}dt &= \int_0^\infty \frac{|t^{\lambda-1} - \delta t^{\lambda+\beta-1}|}{|1 - (\delta t^\beta)^{n+1}|} |\log t|^\alpha dt \\ &= \int_0^1 \frac{|t^{\lambda-1} - \delta t^{\lambda+\beta-1}|}{|1 - (\delta t^\beta)^{n+1}|} |\log t|^\alpha dt + \int_1^\infty \frac{|t^{\lambda-1} - \delta t^{\lambda+\beta-1}|}{|1 - (\delta t^\beta)^{n+1}|} |\log t|^\alpha dt \\ &:= I_1 + I_2. \end{aligned} \tag{2.5}$$

Observing that  $\beta > 0$ ,  $\delta \in \{1, -1\}$  and  $0 < t < 1$ , we have

$$\frac{1}{1 - (\delta t^\beta)^{n+1}} = \sum_{k=0}^\infty (\delta t^\beta)^{k(n+1)}.$$

Therefore,

$$I_1 = \sum_{k=0}^\infty \delta^{k(n+1)} \left\{ \int_0^1 t^{k\beta(n+1)+\lambda-1} |\log t|^\alpha dt - \delta \int_0^1 t^{k\beta(n+1)+\lambda+\beta-1} |\log t|^\alpha dt \right\} := \sum_{k=0}^\infty \delta^{k(n+1)} (I_{11} - \delta I_{12}). \tag{2.6}$$

Setting  $|\log t| = \frac{u}{k\beta(n+1) + \lambda}$ , we can obtain

$$I_{11} = \frac{1}{(k\beta(n+1) + \lambda)^{\alpha+1}} \int_0^\infty e^{-u} u^\alpha du = \frac{\Gamma(\alpha + 1)}{(k\beta(n+1) + \lambda)^{\alpha+1}}. \tag{2.7}$$

Similarly, we have

$$I_{12} = \frac{1}{(k\beta(n+1) + \lambda + \beta)^{\alpha+1}} \int_0^\infty e^{-u} u^\alpha du = \frac{\Gamma(\alpha + 1)}{(k\beta(n+1) + \lambda + \beta)^{\alpha+1}}. \tag{2.8}$$

It follows from plugging (2.7) and (2.8) into (2.6) that

$$I_1 = \sum_{k=0}^\infty \delta^{k(n+1)} \left\{ \frac{\Gamma(\alpha + 1)}{(k\beta(n+1) + \lambda)^{\alpha+1}} - \frac{\delta \Gamma(\alpha + 1)}{(k\beta(n+1) + \lambda + \beta)^{\alpha+1}} \right\}. \tag{2.9}$$

Similarly, we can deduce that

$$\begin{aligned} I_2 &= \int_0^1 \frac{|t^{\beta n - \lambda - 1} - \delta t^{\beta(n+1) - \lambda - 1}|}{|1 - (\delta t^\beta)^{n+1}|} |\log t|^\alpha dt \\ &= \sum_{k=0}^\infty \delta^{k(n+1)} \left\{ \frac{\Gamma(\alpha + 1)}{(k\beta(n+1) + \beta n - \lambda)^{\alpha+1}} - \frac{\delta \Gamma(\alpha + 1)}{(k\beta(n+1) + \beta(n+1) - \lambda)^{\alpha+1}} \right\}. \end{aligned} \tag{2.10}$$

Plugging (2.9) and (2.10) back into (2.5), and using (2.3), we obtain (2.4). □

**Lemma 2.3.** *Let  $a, b > 0$ ,  $a + b = s$  and  $m \in \mathbb{N}$ . Let  $\varphi_1(x) = \cot x$ . Then*

$$\varphi_1^{(2m)}\left(\frac{a\pi}{s}\right) = \frac{(2m)!s^{2m+1}}{\pi^{2m+1}} \sum_{k=0}^\infty \left( \frac{1}{(sk + a)^{2m+1}} - \frac{1}{(sk + b)^{2m+1}} \right), \tag{2.11}$$

$$\varphi_1^{(2m+1)}\left(\frac{a\pi}{s}\right) = -\frac{(2m+1)!s^{2m+2}}{\pi^{2m+2}} \sum_{k=0}^\infty \left( \frac{1}{(sk + a)^{2m+2}} + \frac{1}{(sk + b)^{2m+2}} \right). \tag{2.12}$$

**Proof.** We start the proof of Lemma 2.3 from the partial fraction expansion of  $\varphi_1(x) = \cot x$  as follows [34]:

$$\varphi_1(x) = \frac{1}{x} + \sum_{k=1}^{\infty} \left( \frac{1}{x + k\pi} + \frac{1}{x - k\pi} \right).$$

Taking the  $2m$ th derivative of  $\varphi_1(x)$ , we have

$$\varphi_1^{(2m)}(x) = (2m)! \lim_{n \rightarrow \infty} \left( \sum_{k=0}^n \frac{1}{(x + k\pi)^{2m+1}} + \sum_{k=1}^n \frac{1}{(x - k\pi)^{2m+1}} \right). \tag{2.13}$$

Setting  $x = \frac{a\pi}{s}$  in (2.13), in view of  $a + b = s$ , it follows that

$$\begin{aligned} \varphi_1^{(2m)}\left(\frac{a\pi}{s}\right) &= \frac{(2m)!s^{2m+1}}{\pi^{2m+1}} \lim_{n \rightarrow \infty} \left( \sum_{k=0}^n \frac{1}{(sk + a)^{2m+1}} + \sum_{k=1}^n \frac{1}{(a - sk)^{2m+1}} \right) \\ &= \frac{(2m)!s^{2m+1}}{\pi^{2m+1}} \lim_{n \rightarrow \infty} \left( \sum_{k=0}^n \frac{1}{(sk + a)^{2m+1}} - \sum_{k=0}^{n-1} \frac{1}{(sk + b)^{2m+1}} \right) \\ &= \frac{(2m)!s^{2m+1}}{\pi^{2m+1}} \sum_{k=0}^{\infty} \left( \frac{1}{(sk + a)^{2m+1}} - \frac{1}{(sk + b)^{2m+1}} \right). \end{aligned}$$

Therefore, the proof of (2.11) is completed. Similarly, we can obtain (2.12). □

**Lemma 2.4.** Let  $a, b > 0, a + b = s$  and  $m \in \mathbb{N}$ . Let  $\varphi_2(x) = \csc x$ . Then

$$\varphi_2^{(2m)}\left(\frac{a\pi}{s}\right) = \frac{(2m)!s^{2m+1}}{\pi^{2m+1}} \sum_{k=0}^{\infty} (-1)^k \left( \frac{1}{(sk + a)^{2m+1}} + \frac{1}{(sk + b)^{2m+1}} \right). \tag{2.14}$$

**Proof.** Take the  $2m$ th derivative the partial fraction expansion of  $\varphi_2(x) = \csc x$  as follows [34]:

$$\varphi_2(x) = \frac{1}{x} + \sum_{k=1}^{\infty} (-1)^k \left( \frac{1}{x + k\pi} + \frac{1}{x - k\pi} \right). \tag{2.15}$$

Then

$$\begin{aligned} \varphi_2^{(2m)}(x) &= (2m)! \lim_{n \rightarrow \infty} \left( \sum_{k=0}^n \frac{(-1)^k}{(x + k\pi)^{2m+1}} + \sum_{k=1}^n \frac{(-1)^k}{(x - k\pi)^{2m+1}} \right) \\ &= (2m)! \lim_{n \rightarrow \infty} \left( \sum_{k=0}^n \frac{(-1)^k}{(k\pi + x)^{2m+1}} + \sum_{k=0}^{n-1} \frac{(-1)^k}{(k\pi + \pi - x)^{2m+1}} \right). \end{aligned} \tag{2.16}$$

Since  $a + b = s$ , then we can arrive at (2.14) by setting  $x = \frac{a\pi}{s}$  in (2.16). □

**Remark 2.5.** Letting  $a = b = 1$  in (2.12), we have

$$\varphi_1^{(2m+1)}\left(\frac{\pi}{2}\right) = -\frac{(2m + 1)!2^{2m+3}}{\pi^{2m+2}} \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^{2m+2}}. \tag{2.17}$$

Since [34]  $\sum_{k=0}^{\infty} \frac{1}{k^{2m+2}} = \frac{(2\pi)^{2m+2}}{2(2m + 2)!} B_{m+1}$ , where  $B_{m+1}$  is a Bernoulli number,  $B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, \dots$ , then we obtain

$$\sum_{k=0}^{\infty} \frac{1}{(2k + 1)^{2m+2}} = \sum_{k=1}^{\infty} \left( \frac{1}{k^{2m+2}} - \frac{1}{(2k)^{2m+2}} \right) = \frac{2^{2m+2} - 1}{2(2m + 2)!} \pi^{2m+2} B_{m+1}. \tag{2.18}$$

Applying (2.18) to (2.17), we obtain

$$\varphi_1^{(2m+1)}\left(\frac{\pi}{2}\right) = -\frac{1}{m + 1} (2^{2m+2} - 1) 2^{2m+1} B_{m+1}. \tag{2.19}$$

Similarly, letting  $a = 1, b = 3$  in (2.12), we can also obtain

$$\varphi_1^{(2m+1)}\left(\frac{\pi}{4}\right) = -\frac{1}{m+1}(2^{2m+2} - 1)4^{2m+1}B_{m+1}. \tag{2.20}$$

In addition, letting  $a = 1, b = 3$  in (2.11) and  $a = b = 1$  in (2.14), in view of [34]  $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2m+1}} = \frac{\pi^{2m+1}}{2^{2m+2}(2m)!}E_m$ , where  $E_m$  is an Euler number,  $E_0 = 1, E_1 = 1, E_2 = 5, E_3 = 61, \dots$ , we obtain

$$E_m = \frac{1}{4^m}\varphi_1^{(2m)}\left(\frac{\pi}{4}\right) = \varphi_2^{(2m)}\left(\frac{\pi}{2}\right). \tag{2.21}$$

### 3 Main results

**Theorem 3.1.** Let  $n \in \mathbb{N}^+, \delta \in \{1, -1\}, \beta n > \lambda > 0$  and  $\beta_1, \beta_2 \neq 0$ . Let  $\alpha$  be such that  $\alpha > -1$  for  $\delta^n = 1$  and  $\alpha > 0$  for  $\delta^n = -1$ . Define  $\mu(x) = x^{p(1-\lambda\beta_1)-1}$  and  $\nu(y) = y^{q(1-\lambda\beta_2)-1}$ , where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Suppose that  $f(x), g(x) \geq 0$  and  $f(x) \in L_{p,\mu}(\mathbb{R}^+), g(x) \in L_{q,\nu}(\mathbb{R}^+)$ .  $K(t)$  is defined via (2.1) and  $C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2)$  is defined via (2.3). Then

$$\int_0^{\infty} \int_0^{\infty} K(x^{\beta_1}y^{\beta_2}) f(x)g(y) dx dy < |\beta_1|^{-\frac{1}{q}}|\beta_2|^{-\frac{1}{p}}\Gamma(\alpha + 1)C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{3.1}$$

where the constant factor  $|\beta_1|^{-\frac{1}{q}}|\beta_2|^{-\frac{1}{p}}\Gamma(\alpha + 1)C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2)$  is the best possible.

**Proof.** By Hölder’s inequality and Fubini’s theorem, we have

$$\begin{aligned} & \int_0^{\infty} \int_0^{\infty} K(x^{\beta_1}y^{\beta_2}) f(x)g(y) dx dy \\ &= \int_0^{\infty} \int_0^{\infty} \left( K(x^{\beta_1}y^{\beta_2}) \right)^{\frac{1}{p}} x^{\frac{1-\lambda\beta_1}{q}} y^{\frac{\lambda\beta_2-1}{p}} f(x) \left( K(x^{\beta_1}y^{\beta_2}) \right)^{\frac{1}{q}} y^{\frac{1-\lambda\beta_2}{p}} x^{\frac{\lambda\beta_1-1}{q}} g(y) dx dy \\ &\leq \left( \int_0^{\infty} \int_0^{\infty} K(x^{\beta_1}y^{\beta_2}) y^{\lambda\beta_2-1} x^{\frac{p(1-\lambda\beta_1)}{q}} f^p(x) dx dy \right)^{\frac{1}{p}} \left( \int_0^{\infty} \int_0^{\infty} K(x^{\beta_1}y^{\beta_2}) x^{\lambda\beta_1-1} y^{\frac{q(1-\lambda\beta_2)}{p}} g^q(y) dx dy \right)^{\frac{1}{q}} \\ &= \left( \int_0^{\infty} \omega(x) x^{\frac{p(1-\lambda\beta_1)}{q}} f^p(x) dx \right)^{\frac{1}{p}} \left( \int_0^{\infty} \varpi(y) y^{\frac{q(1-\lambda\beta_2)}{p}} g^q(y) dy \right)^{\frac{1}{q}}, \end{aligned} \tag{3.2}$$

where  $\omega(x) = \int_0^{\infty} K(x^{\beta_1}y^{\beta_2}) y^{\lambda\beta_2-1} dy$  and  $\varpi(y) = \int_0^{\infty} K(x^{\beta_1}y^{\beta_2}) x^{\lambda\beta_1-1} dx$ .

Setting  $x^{\beta_1}y^{\beta_2} = t$ , and using Lemma 2.2, we can obtain

$$\omega(x) = \frac{x^{-\lambda\beta_1}}{|\beta_2|} \int_0^{\infty} K(t)t^{\lambda-1} dt = \frac{1}{|\beta_2|} \Gamma(\alpha + 1)C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2)x^{-\lambda\beta_1} \tag{3.3}$$

and

$$\varpi(y) = \frac{y^{-\lambda\beta_2}}{|\beta_1|} \int_0^{\infty} K(t)t^{\lambda-1} dt = \frac{1}{|\beta_1|} \Gamma(\alpha + 1)C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2)y^{-\lambda\beta_2}. \tag{3.4}$$

Plugging (3.3) and (3.4) into (3.2), we obtain

$$\int_0^\infty \int_0^\infty K(x^{\beta_1}y^{\beta_2}) f(x)g(y) dx dy \leq |\beta_1|^{-\frac{1}{q}}|\beta_2|^{-\frac{1}{p}}\Gamma(\alpha + 1)C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{3.5}$$

If (3.5) takes the form of an equation, then there must exist two constants  $C_1$  and  $C_2$  that are not both equal to zero, such that

$$C_1K(x^{\beta_1}y^{\beta_2})y^{\lambda\beta_2-1}x^{\frac{p(1-\lambda\beta_1)}{q}}f^p(x) = C_2K(x^{\beta_1}y^{\beta_2})x^{\lambda\beta_1-1}y^{\frac{q(1-\lambda\beta_2)}{p}}g^q(y)$$

holds almost everywhere in the domain  $(0, \infty) \times (0, \infty)$ . That is,

$$C_1x^{p(1-\lambda\beta_1)}f^p(x) = C_2y^{q(1-\lambda\beta_2)}g^q(y).$$

Hence, there must be a constant  $C$  such that  $C_1x^{p(1-\lambda\beta_1)}f^p(x) = C$  and  $C_2y^{q(1-\lambda\beta_2)}g^q(y) = C$  hold almost everywhere in  $(0, \infty)$ . Without loss of generality, assuming  $C_1 \neq 0$ , we can obtain  $x^{p(1-\lambda\beta_1)-1}f^p(x) = \frac{C}{C_1x}$  holds almost everywhere in  $(0, \infty)$ . This obviously contradicts the fact that  $f(x) \in L_{p,\mu}(\mathbb{R}^+)$ . Therefore, (3.5) keeps the form of a strict inequality, and (3.1) is obtained.

Finally, it will be proved that the constant factor in (3.1) is the best possible. If the constant factor  $|\beta_1|^{-\frac{1}{q}}|\beta_2|^{-\frac{1}{p}}\Gamma(\alpha + 1)C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2)$  is not the best possible, then there will be a positive constant  $k < |\beta_1|^{-\frac{1}{q}}|\beta_2|^{-\frac{1}{p}}\Gamma(\alpha + 1)C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2)$ , such that (3.1) still holds if  $|\beta_1|^{-\frac{1}{q}}|\beta_2|^{-\frac{1}{p}}\Gamma(\alpha + 1)C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2)$  is replaced by  $k$ . That is,

$$\int_0^\infty \int_0^\infty K(x^{\beta_1}y^{\beta_2}) f(x)g(y) dx dy < k \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{3.6}$$

Consider a sufficiently small positive number  $\varepsilon$ , and define the functions  $f_\varepsilon(x)$  and  $g_\varepsilon(y)$  as follows:

$$f_\varepsilon(x) := \begin{cases} x^{\frac{p(\lambda\beta_1-1)-\beta_1\varepsilon}{p}}, & x \in \Omega_1, \\ 0 & x \in \mathbb{R}^+ \setminus \Omega_1, \end{cases} \quad g_\varepsilon(y) := \begin{cases} y^{\frac{q(\lambda\beta_2-1)+\beta_2\varepsilon}{q}}, & y \in \Omega_2, \\ 0, & y \in \mathbb{R}^+ \setminus \Omega_2, \end{cases}$$

where  $\Omega_1 = \left\{x : x > 0, x^{\frac{\beta_1}{|\beta_1|}} > 1\right\}$  and  $\Omega_2 = \left\{y : y > 0, y^{\frac{\beta_2}{|\beta_2|}} < 1\right\}$ .

Replacing  $f$  and  $g$  in (3.6) with  $f_\varepsilon$  and  $g_\varepsilon$ , respectively, we have

$$\varepsilon \int_0^\infty \int_0^\infty K(x^{\beta_1}y^{\beta_2}) f_\varepsilon(x)g_\varepsilon(y) dx dy < \varepsilon k \left( \int_{\Omega_1} x^{-\beta_1\varepsilon-1} dx \right)^{\frac{1}{p}} \left( \int_{\Omega_2} y^{\beta_2\varepsilon-1} dy \right)^{\frac{1}{q}} = k |\beta_1|^{-\frac{1}{p}} |\beta_2|^{-\frac{1}{q}}. \tag{3.7}$$

On the other hand, setting  $x^{\beta_1}y^{\beta_2} = t$ , we have

$$\begin{aligned} \varepsilon \int_0^\infty \int_0^\infty K(x^{\beta_1}y^{\beta_2}) f_\varepsilon(x)g_\varepsilon(y) dx dy &= \varepsilon \int_{\Omega_2} y^{\frac{q(\lambda\beta_2-1)+\beta_2\varepsilon}{q}} \left( \int_{\Omega_1} K(x^{\beta_1}y^{\beta_2}) x^{\frac{p(\lambda\beta_1-1)-\beta_1\varepsilon}{p}} dx \right) dy \\ &= \frac{\varepsilon}{|\beta_1|} \int_{\Omega_2} y^{\beta_2\varepsilon-1} \left( \int_{y^{\beta_2}}^\infty K(t) t^{\lambda-\frac{\varepsilon}{p}-1} dt \right) dy \\ &= \frac{\varepsilon}{|\beta_1|} \int_{\Omega_2} y^{\beta_2\varepsilon-1} \left( \int_1^\infty K(t) t^{\lambda-\frac{\varepsilon}{p}-1} dt \right) dy + \frac{\varepsilon}{|\beta_1|} \int_{\Omega_2} y^{\beta_2\varepsilon-1} \left( \int_{y^{\beta_2}}^1 K(t) t^{\lambda-\frac{\varepsilon}{p}-1} dt \right) dy \\ &= \frac{1}{|\beta_1\beta_2|} \int_1^\infty K(t) t^{\lambda-\frac{\varepsilon}{p}-1} dt + \frac{\varepsilon}{|\beta_1|} \int_{\Omega_2} y^{\beta_2\varepsilon-1} \left( \int_{y^{\beta_2}}^1 K(t) t^{\lambda-\frac{\varepsilon}{p}-1} dt \right) dy. \end{aligned} \tag{3.8}$$

No matter  $\beta_2 > 0$  or  $\beta_2 < 0$ , it follows from Fubini’s theorem that

$$\int_{\Omega_2} y^{\beta_2 \varepsilon - 1} \left( \int_{y^{\beta_2}}^1 K(t) t^{\lambda - \frac{\varepsilon}{p} - 1} dt \right) dy = \frac{1}{|\beta_2| \varepsilon} \int_0^1 K(t) t^{\lambda + \frac{\varepsilon}{q} - 1} dt. \tag{3.9}$$

Applying (3.9) to (3.8), we can obtain

$$\varepsilon \int_0^\infty \int_0^\infty K(x^{\beta_1} y^{\beta_2}) f_\varepsilon(x) g_\varepsilon(y) dx dy = \frac{1}{|\beta_1 \beta_2|} \left\{ \int_1^\infty K(t) t^{\lambda - \frac{\varepsilon}{p} - 1} dt + \int_0^1 K(t) t^{\lambda + \frac{\varepsilon}{q} - 1} dt \right\}. \tag{3.10}$$

Letting  $\varepsilon \rightarrow 0^+$  and using (2.4), we arrive at

$$\varepsilon \int_0^\infty \int_0^\infty K(x^{\beta_1} y^{\beta_2}) f_\varepsilon(x) g_\varepsilon(y) dx dy = \frac{1}{|\beta_1 \beta_2|} \Gamma(\alpha + 1) C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) + o(1). \tag{3.11}$$

Combining (3.7) and (3.11), and letting  $\varepsilon \rightarrow 0^+$ , then we have  $|\beta_1|^{-\frac{1}{q}} |\beta_2|^{-\frac{1}{p}} \Gamma(\alpha + 1) C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) \leq k$ , which obviously contradicts the hypothesis. Therefore, the constant factor in (3.1) is the best possible.  $\square$

By convention, we will establish the following equivalent form of Theorem 3.1, which is usually called the Hardy-Hilbert-type inequality.

**Theorem 3.2.** *Let  $n \in \mathbb{N}^+$ ,  $\delta \in \{1, -1\}$ ,  $\beta n > \lambda > 0$  and  $\beta_1 \beta_2 \neq 0$ . Let  $\alpha$  be such that  $\alpha > -1$  for  $\delta^n = 1$  and  $\alpha > 0$  for  $\delta^n = -1$ . Define  $\mu(x) = x^{p(1-\lambda\beta_1)-1}$  and  $\nu(y) = y^{q(1-\lambda\beta_2)-1}$ , where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Suppose that  $f(x) \geq 0$  with  $f(x) \in L_{p,\mu}(\mathbb{R}^+)$ .  $K(t)$  is defined via (2.1) and  $C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2)$  is defined via (2.3). Then*

$$\int_0^\infty y^{p\lambda\beta_2-1} \left( \int_0^\infty K(x^{\beta_1} y^{\beta_2}) f(x) dx \right)^p dy < \left( |\beta_1|^{-\frac{1}{q}} |\beta_2|^{-\frac{1}{p}} \Gamma(\alpha + 1) C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) \right)^p (\|f\|_{p,\mu})^p, \tag{3.12}$$

where the constant factor  $\left( |\beta_1|^{-\frac{1}{q}} |\beta_2|^{-\frac{1}{p}} \Gamma(\alpha + 1) C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) \right)^p$  is the best possible, and (3.12) is equivalent to (3.1).

**Proof.** Consider  $g(y) := y^{p\lambda\beta_2-1} \left( \int_0^\infty K(x^{\beta_1} y^{\beta_2}) f(x) dx \right)^{p-1}$ . It follows from Theorem 3.1 that

$$\begin{aligned} 0 < (\|g\|_{q,\nu})^{pq} &= \left( \int_0^\infty y^{q(1-\lambda\beta_2)-1} g^q(y) dy \right)^p \\ &= \left( \int_0^\infty y^{p\lambda\beta_2-1} \left( \int_0^\infty K(x^{\beta_1} y^{\beta_2}) f(x) dx \right)^p dy \right)^p \\ &= \left( \int_0^\infty \int_0^\infty K(x^{\beta_1} y^{\beta_2}) f(x) g(y) dx dy \right)^p \\ &\leq \left( |\beta_1|^{-\frac{1}{q}} |\beta_2|^{-\frac{1}{p}} \Gamma(\alpha + 1) C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) \right)^p (\|f\|_{p,\mu})^p (\|g\|_{q,\nu})^p. \end{aligned} \tag{3.13}$$

Therefore,

$$\begin{aligned} 0 < (\|g\|_{q,\nu})^q &= \int_0^\infty y^{p\lambda\beta_2-1} \left( \int_0^\infty K(x^{\beta_1} y^{\beta_2}) f(x) dx \right)^p dy \\ &\leq \left( |\beta_1|^{-\frac{1}{q}} |\beta_2|^{-\frac{1}{p}} \Gamma(\alpha + 1) C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) \right)^p (\|f\|_{p,\mu})^p. \end{aligned} \tag{3.14}$$



Since  $f(x) \in L_{p,\mu}(\mathbb{R}^+)$ , it follows from (3.14) that  $g(x) \in L_{q,\nu}(\mathbb{R}^+)$ . By using Theorem 3.1 again, we can obtain that both (3.13) and (3.14) take the form of strict inequality, and therefore (3.12) is proved.

On the other hand, if (3.12) is valid, by Hölder’s inequality, we have

$$\begin{aligned} \int_0^\infty \int_0^\infty K(x^{\beta_1}y^{\beta_2}) f(x)g(y) dx dy &= \int_0^\infty \left( y^{\lambda\beta_2-1+\frac{1}{q}} \int_0^\infty K(x^{\beta_1}y^{\beta_2}) f(x) dx \right) \left( y^{1-\lambda\beta_2-\frac{1}{q}} g(y) \right) dy \\ &\leq \left( \int_0^\infty y^{p\lambda\beta_2-1} \left( \int_0^\infty K(x^{\beta_1}y^{\beta_2}) f(x) dx \right)^p dy \right)^{\frac{1}{p}} \|g\|_{q,v}. \end{aligned} \tag{3.15}$$

Plugging (3.12) into (3.15), we can get (3.1). Therefore, (3.1) is equivalent to (3.12). According to the equivalence of (3.1) and (3.12), it can be shown that the constant factor  $\left( |\beta_1|^{\frac{1}{q}} |\beta_2|^{\frac{1}{p}} \Gamma(\alpha + 1) C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) \right)^p$  in (3.12) is the best possible. The proof of Theorem 3.2 is completed.  $\square$

### 4 Applications

Letting  $\delta = 1, \alpha = 2m$  ( $m \in \mathbb{N}$ ) in (2.3), and using (2.11), we can obtain

$$C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) = \frac{\pi^{2m+1}}{(2m)! [\beta(n+1)]^{2m+1}} \left\{ \varphi_1^{(2m)} \left( \frac{\lambda\pi}{\beta(n+1)} \right) - \varphi_1^{(2m)} \left( \frac{(\beta+\lambda)\pi}{\beta(n+1)} \right) \right\}.$$

Therefore, setting  $\delta = 1, \alpha = 2m$  ( $m \in \mathbb{N}$ ),  $\beta_1 = 1$  and  $\beta_2 = -1$  in Theorem 3.1, and replacing  $g(y)y^{\beta n}$  with  $g(y)$ , we can obtain the following corollary.

**Corollary 4.1.** *Let  $n \in \mathbb{N}^+, m \in \mathbb{N}$  and  $\beta n > \lambda > 0$ . Let  $\varphi_1(x) = \cot x, \mu(x) = x^{p(1-\lambda)-1}$  and  $\nu(y) = y^{q(1+\lambda-\beta n)-1}$ , where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Suppose that  $f(x), g(x) \geq 0$  with  $f(x) \in L_{p,\mu}(\mathbb{R}^+)$  and  $g(x) \in L_{q,\nu}(\mathbb{R}^+)$ . Then*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{\left( \log \frac{x}{y} \right)^{2m} f(x)g(y)}{x^{\beta n} + x^\beta y^{\beta(n-1)} + \dots + y^{\beta n}} dx dy \\ &< \frac{\pi^{2m+1}}{[\beta(n+1)]^{2m+1}} \left\{ \varphi_1^{(2m)} \left( \frac{\lambda\pi}{\beta(n+1)} \right) - \varphi_1^{(2m)} \left( \frac{(\beta+\lambda)\pi}{\beta(n+1)} \right) \right\} \|f\|_{p,\mu} \|g\|_{q,\nu} \end{aligned} \tag{4.1}$$

Let  $n = 1$  in (4.1), in view of

$$\varphi_1^{(2m)} \left( \frac{\lambda\pi}{2\beta} \right) - \varphi_1^{(2m)} \left( \frac{(\beta+\lambda)\pi}{2\beta} \right) = 2^{2m+1} \varphi_2^{(2m)} \left( \frac{\lambda\pi}{\beta} \right),$$

where  $\varphi_2(x) = \csc x$ , then (4.1) is transformed to

$$\int_0^\infty \int_0^\infty \frac{\left( \log \frac{x}{y} \right)^{2m}}{x^\beta + y^\beta} f(x)g(y) dx dy < \left( \frac{\pi}{\beta} \right)^{2m+1} \varphi_2^{(2m)} \left( \frac{\lambda\pi}{\beta} \right) \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{4.2}$$

where  $\mu(x) = x^{p(1-\lambda)-1}$  and  $\nu(y) = y^{q(1+\lambda-\beta)-1}$ . Let  $m = 0, \lambda = \frac{\beta}{r}$  in (4.2), where  $\frac{1}{r} + \frac{1}{s} = 1$ , then (4.2) reduces to (1.3). Additionally, let  $\lambda = \frac{\beta}{2}$  in (4.2), then it follows from (2.21) that

$$\int_0^\infty \int_0^\infty \frac{\left( \log \frac{x}{y} \right)^{2m}}{x^\beta + y^\beta} f(x)g(y) dx dy < \left( \frac{\pi}{\beta} \right)^{2m+1} E_m \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{4.3}$$

where  $\mu(x) = x^{p(1-\frac{\beta}{2})-1}$  and  $\nu(y) = y^{q(1-\frac{\beta}{2})-1}$ .

Let  $n = 2$  in (4.1), then we have

$$\int_0^\infty \int_0^\infty \frac{\left(\log \frac{x}{y}\right)^{2m}}{x^{2\beta} + x^\beta y^\beta + y^{2\beta}} f(x)g(y) dx dy < \left(\frac{\pi}{3\beta}\right)^{2m+1} \left\{ \varphi_1^{(2m)}\left(\frac{\lambda\pi}{3\beta}\right) - \varphi_1^{(2m)}\left(\frac{(\beta + \lambda)\pi}{3\beta}\right) \right\} \|f\|_{p,\mu} \|g\|_{q,v}, \tag{4.4}$$

where  $\mu(x) = x^{p(1-\lambda)-1}$  and  $v(y) = y^{q(1+\lambda-2\beta)-1}$ . Letting  $m = 0$ ,  $\lambda = \beta$  and  $p = q = 2$  in (4.4), we obtain (1.4). Letting  $m = 1$ ,  $\lambda = \beta = 1$  and  $p = q = 2$  in (4.4), we obtain (1.7). Letting  $\lambda = \frac{\beta}{2}$  in (4.4), since  $\varphi_1^{(2m)}\left(\frac{\pi}{2}\right) = 0$ , we can also obtain

$$\int_0^\infty \int_0^\infty \frac{\left(\log \frac{x}{y}\right)^{2m}}{x^{2\beta} + x^\beta y^\beta + y^{2\beta}} f(x)g(y) dx dy < \left(\frac{\pi}{3\beta}\right)^{2m+1} \varphi_1^{(2m)}\left(\frac{\pi}{6}\right) \|f\|_{p,\mu} \|g\|_{q,v}, \tag{4.5}$$

where  $\mu(x) = x^{p\left(1-\frac{\beta}{2}\right)-1}$  and  $v(y) = y^{q\left(1-\frac{3\beta}{2}\right)-1}$ .

Let  $n = 3$  and  $\lambda = \beta$  in (4.1). In view of  $\varphi_1^{(2m)}\left(\frac{\pi}{2}\right) = 0$ , and using (2.21), we have

$$\int_0^\infty \int_0^\infty \frac{(\log(xy))^{2m}}{(x^\beta + y^\beta)(x^{2\beta} + y^{2\beta})} f(x)g(y) dx dy < \left(\frac{\pi}{\beta}\right)^{2m+1} \frac{E_m}{4^{m+1}} \|f\|_{p,\mu} \|g\|_{q,v}, \tag{4.6}$$

where  $\mu(x) = x^{p(1-\beta)-1}$  and  $v(y) = y^{q(1-2\beta)-1}$ .

Setting  $\delta = 1$ ,  $\alpha = 2m$  ( $m \in \mathbb{N}$ ) and  $\beta_1 = \beta_2 = 1$  in Theorem 3.1, we can have the following Hilbert-type inequality with a non-homogeneous kernel.

**Corollary 4.2.** *Let  $n \in \mathbb{N}^+$ ,  $m \in \mathbb{N}$  and  $\beta n > \lambda > 0$ . Let  $\varphi_1(x) = \cot x$ ,  $\mu(x) = x^{p(1-\lambda)-1}$  and  $v(y) = y^{q(1-\lambda)-1}$ , where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Suppose that  $f(x), g(x) \geq 0$  with  $f(x) \in L_{p,\mu}(\mathbb{R}^+)$  and  $g(x) \in L_{q,v}(\mathbb{R}^+)$ . Then*

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{(\log(xy))^{2m} f(x)g(y)}{1 + (xy)^\beta + \dots + (xy)^{\beta n}} dx dy \\ & < \frac{\pi^{2m+1}}{[\beta(n+1)]^{2m+1}} \left\{ \varphi_1^{(2m)}\left(\frac{\lambda\pi}{\beta(n+1)}\right) - \varphi_1^{(2m)}\left(\frac{(\beta + \lambda)\pi}{\beta(n+1)}\right) \right\} \|f\|_{p,\mu} \|g\|_{q,v}. \end{aligned} \tag{4.7}$$

Let  $n = 1$  in (4.7), then we have

$$\int_0^\infty \int_0^\infty \frac{(\log(xy))^{2m}}{1 + (xy)^\beta} f(x)g(y) dx dy < \left(\frac{\pi}{\beta}\right)^{2m+1} \varphi_2^{(2m)}\left(\frac{\lambda\pi}{\beta}\right) \|f\|_{p,\mu} \|g\|_{q,v}, \tag{4.8}$$

where  $\varphi_2(x) = \csc x$ ,  $\mu(x) = x^{p(1-\lambda)-1}$  and  $v(y) = y^{q(1-\lambda)-1}$ .

Similarly, taking special values for the parameters in Corollary 4.2, we can obtain some other special cases of Corollary 4.2. For instance, setting  $n = 2$ ,  $\lambda = \beta$  and  $n = 3$ ,  $\lambda = \beta$  in (4.7), respectively, we obtain

$$\int_0^\infty \int_0^\infty \frac{(\log(xy))^{2m}}{1 + (xy)^\beta + (xy)^{2\beta}} f(x)g(y) dx dy < 2 \left(\frac{\pi}{3\beta}\right)^{2m+1} \varphi_1^{(2m)}\left(\frac{\pi}{3}\right) \|f\|_{p,\mu} \|g\|_{q,v}, \tag{4.9}$$

$$\int_0^\infty \int_0^\infty \frac{(\log(xy))^{2m}}{(1 + (xy)^\beta)(1 + (xy)^{2\beta})} f(x)g(y) dx dy < \left(\frac{\pi}{\beta}\right)^{2m+1} \frac{E_m}{4^{m+1}} \|f\|_{p,\mu} \|g\|_{q,v}, \tag{4.10}$$

where  $\mu(x) = x^{p(1-\beta)-1}$  and  $v(y) = y^{q(1-\beta)-1}$ .

Let  $\delta = n = \alpha = 1$  and  $\lambda = \frac{\beta}{2}$  in (2.3), we can obtain

$$C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) = \frac{8}{\beta^2} \sum_{k=0}^{\infty} \left\{ \frac{1}{(4k+1)^2} - \frac{1}{(4k+3)^2} \right\} = \frac{8}{\beta^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = \frac{8c_0}{\beta^2},$$

where  $c_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.91596\dots$  is the Catalan constant. Therefore, setting  $\delta = n = \alpha = 1$ ,  $\lambda = \frac{\beta}{2}$ ,  $\beta_1 = 1$  and  $\beta_2 = -1$  in Theorem 3.1, and replacing  $g(y)y^\beta$  with  $g(y)$ , we can obtain another corollary.

**Corollary 4.3.** *Let  $\beta > 0$ ,  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Let  $\mu(x) = x^{p(1-\frac{\beta}{2})-1}$  and  $\nu(y) = y^{q(1-\frac{\beta}{2})-1}$ . Suppose that  $f(x), g(x) \geq 0$  with  $f(x) \in L_{p,\mu}(\mathbb{R}^+)$  and  $g(x) \in L_{q,\nu}(\mathbb{R}^+)$ . Then*

$$\int_0^{\infty} \int_0^{\infty} \frac{|\log \frac{x}{y}|}{x^\beta + y^\beta} f(x)g(y) dx dy < \frac{8c_0}{\beta^2} \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.11}$$

Let  $\beta = 1$  and  $p = q = 2$ , then (4.11) reduces to (1.6).

Furthermore, setting  $\delta = \alpha = 1$ ,  $n = 3$ ,  $\lambda = \beta$ ,  $\beta_1 = 1$  and  $\beta_2 = -1$  in Theorem 3.1, and replacing  $g(y)y^{3\beta}$  with  $g(y)$ , we can obtain Corollary 4.4.

**Corollary 4.4.** *Let  $\beta > 0$ ,  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Let  $\mu(x) = x^{p(1-\beta)-1}$  and  $\nu(y) = y^{q(1-2\beta)-1}$ . Suppose that  $f(x), g(x) \geq 0$  with  $f(x) \in L_{p,\mu}(\mathbb{R}^+)$  and  $g(x) \in L_{q,\nu}(\mathbb{R}^+)$ . Then*

$$\int_0^{\infty} \int_0^{\infty} \frac{|\log \frac{x}{y}|}{(x^\beta + y^\beta)(x^{2\beta} + y^{2\beta})} f(x)g(y) dx dy < \frac{c_0}{\beta^2} \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.12}$$

**Remark 4.5.** Corollaries 4.3 and 4.4 can be regarded as supplements to (4.2) and (4.6), respectively. It should be noted that if  $n$  takes natural numbers other than 1 and 3, the conclusion expressed by the Catalan constant like Corollaries 4.3 and 4.4 cannot be obtained.

Let  $\delta = -1$ ,  $n = 2l - 1$  ( $l \in \mathbb{N}^+$ ) and  $\alpha = 2m + 1$  ( $m \in \mathbb{N}$ ) in (2.3). By using (2.13), we can obtain

$$C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) = \frac{\pi^{2m+2}}{(2m+1)!(2\beta l)^{2m+2}} \left\{ \varphi_1^{(2m+1)}\left(\frac{\lambda\pi}{2\beta l}\right) + \varphi_1^{(2m+1)}\left(\frac{(\beta+\lambda)\pi}{2\beta l}\right) \right\}.$$

Therefore, setting  $\delta = -1$ ,  $n = 2l - 1$  ( $l \in \mathbb{N}^+$ ),  $\alpha = 2m + 1$  ( $m \in \mathbb{N}$ ),  $\beta_1 = 1$  and  $\beta_2 = -1$  in Theorem 3.1, and replacing  $g(y)y^{(2l-1)\beta}$  with  $g(y)$ , we can obtain Corollary 4.6.

**Corollary 4.6.** *Let  $l \in \mathbb{N}^+$ ,  $m \in \mathbb{N}$  and  $(2l - 1)\beta > \lambda > 0$ . Let  $\varphi_1(x) = \cot x$ ,  $\mu(x) = x^{p(1-\lambda)-1}$  and  $\nu(y) = y^{q(1+\lambda+\beta-2\beta)-1}$ , where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Suppose that  $f(x), g(x) \geq 0$  with  $f(x) \in L_{p,\mu}(\mathbb{R}^+)$  and  $g(x) \in L_{q,\nu}(\mathbb{R}^+)$ . Then*

$$\int_0^{\infty} \int_0^{\infty} \frac{|\log \frac{x}{y}|^{2m+1}}{|x^{(2l-1)\beta} - x^{\beta(2l-2)}y^\beta + \dots - y^{(2l-1)\beta}|} f(x)g(y) dx dy < \frac{\pi^{2m+2}}{(2\beta)^{2m+2}} \left\{ \varphi_1^{(2m+1)}\left(\frac{\lambda\pi}{2\beta}\right) + \varphi_1^{(2m+1)}\left(\frac{(\beta+\lambda)\pi}{2\beta}\right) \right\} \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.13}$$

Let  $l = 1$  in (4.13), in view of

$$\varphi_1^{(2m+1)}\left(\frac{\lambda\pi}{2\beta}\right) + \varphi_1^{(2m+1)}\left(\frac{(\beta+\lambda)\pi}{2\beta}\right) = 2^{2m+2} \varphi_1^{(2m+1)}\left(\frac{\lambda\pi}{\beta}\right),$$

then it follows from (4.13) that

$$\int_0^\infty \int_0^\infty \frac{(\log \frac{x}{y})^{2m+1}}{x^\beta - y^\beta} f(x)g(y) dx dy < -\left(\frac{\pi}{\beta}\right)^{2m+2} \varphi_1^{(2m+1)}\left(\frac{\lambda\pi}{\beta}\right) \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{4.14}$$

where  $\mu(x) = x^{p(1-\lambda)-1}$  and  $\nu(y) = y^{q(1+\lambda-\beta)-1}$ . Let  $m = 0$ ,  $\beta = 1$  and  $\lambda = \frac{1}{q}$  in (4.14), then we obtain (1.2). Let  $\lambda = \frac{\beta}{2}$  in (4.14), by using (2.19), then we obtain

$$\int_0^\infty \int_0^\infty \frac{(\log \frac{x}{y})^{2m+1}}{x^\beta - y^\beta} f(x)g(y) dx dy < \left(\frac{\pi}{\beta}\right)^{2m+2} \frac{2^{2m+2} - 1}{m + 1} 2^{2m+1} B_{m+1} \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{4.15}$$

where  $\mu(x) = x^{p(1-\frac{\beta}{2})-1}$  and  $\nu(y) = y^{q(1-\frac{\beta}{2})-1}$ . Letting  $\lambda = \frac{\beta}{4}$  in (4.14), and using (2.20), we obtain

$$\int_0^\infty \int_0^\infty \frac{(\log \frac{x}{y})^{2m+1}}{x^\beta - y^\beta} f(x)g(y) dx dy < \left(\frac{\pi}{\beta}\right)^{2m+2} \frac{2^{2m+2} - 1}{m + 1} 4^{2m+1} B_{m+1} \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{4.16}$$

where  $\mu(x) = x^{p(1-\frac{\beta}{4})-1}$  and  $\nu(y) = y^{q(1-\frac{3\beta}{4})-1}$ .

Let  $l = 2$  and  $\beta = \lambda$  in (4.13), then we have

$$\int_0^\infty \int_0^\infty \frac{(\log \frac{x}{y})^{2m+1} f(x)g(y)}{(x^\beta - y^\beta)(x^{2\beta} + y^{2\beta})} dx dy < \left(\frac{\pi}{4\beta}\right)^{2m+2} \frac{2^{2m+2} - 1}{m + 1} (2^{2m+1} + 4^{2m+1}) B_{m+1} \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{4.17}$$

where  $\mu(x) = x^{p(1-\beta)-1}$  and  $\nu(y) = y^{q(1-2\beta)-1}$ . Let  $m = 0$ ,  $\beta = 1$  and  $p = q = 2$ , then we have (1.8).

Let  $\delta = -1$ ,  $n = 2l$  ( $l \in \mathbb{N}^+$ ) and  $\alpha = 2m$  ( $m \in \mathbb{N}$ ) in (2.3). By using (2.13), we can obtain

$$C(\alpha, \beta, \lambda, \delta, n, \beta_1, \beta_2) = \frac{\pi^{2m+1}}{(2m)! [(2l + 1)\beta]^{2m+1}} \left\{ \varphi_2^{(2m)}\left(\frac{\lambda\pi}{(2l + 1)\beta}\right) + \varphi_2^{(2m)}\left(\frac{(\beta + \lambda)\pi}{(2l + 1)\beta}\right) \right\}.$$

Therefore, setting  $\delta = -1$ ,  $n = 2l$  ( $l \in \mathbb{N}^+$ ),  $\alpha = 2m$  ( $m \in \mathbb{N}$ ),  $\beta_1 = 1$  and  $\beta_2 = -1$  in Theorem 3.1, and replacing  $g(y)y^{2\beta}$  with  $g(y)$ , we can obtain the last corollary.

**Corollary 4.7.** *Let  $l \in \mathbb{N}^+$ ,  $m \in \mathbb{N}$  and  $2l\beta > \lambda > 0$ . Let  $\varphi_2(x) = \csc x$ ,  $\mu(x) = x^{p(1-\lambda)-1}$  and  $\nu(y) = y^{q(1+\lambda-2l\beta)-1}$ , where  $p > 1$  and  $\frac{1}{p} + \frac{1}{q} = 1$ . Suppose that  $f(x), g(x) \geq 0$  with  $f(x) \in L_{p,\mu}(\mathbb{R}^+)$  and  $g(x) \in L_{q,\nu}(\mathbb{R}^+)$ . Then*

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{(\log \frac{x}{y})^{2m}}{|x^{2l\beta} - x^{(2l-1)\beta}y^\beta + \dots + y^{2l\beta}|} f(x)g(y) dx dy \\ & < \frac{\pi^{2m+1}}{[(2l + 1)\beta]^{2m+1}} \left\{ \varphi_2^{(2m)}\left(\frac{\lambda\pi}{(2l + 1)\beta}\right) + \varphi_2^{(2m)}\left(\frac{(\beta + \lambda)\pi}{(2l + 1)\beta}\right) \right\} \|f\|_{p,\mu} \|g\|_{q,\nu}. \end{aligned} \tag{4.18}$$

Let  $l = 1$  in (4.18), then we have

$$\int_0^\infty \int_0^\infty \frac{(\log \frac{x}{y})^{2m} f(x)g(y)}{x^{2\beta} - x^\beta y^\beta + y^{2\beta}} dx dy < \left(\frac{\pi}{3\beta}\right)^{2m+1} \left\{ \varphi_2^{(2m)}\left(\frac{\lambda\pi}{3\beta}\right) + \varphi_2^{(2m)}\left(\frac{(\beta + \lambda)\pi}{3\beta}\right) \right\} \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{4.19}$$

where  $\mu(x) = x^{p(1-\lambda)-1}$  and  $\nu(y) = y^{q(1+\lambda-2\beta)-1}$ . Let  $m = 0$ ,  $\lambda = \beta$  and  $p = q = 2$  in (4.19), we can obtain (1.5). In addition, let  $m = 1$ ,  $\lambda = \beta = 1$  and  $p = q = 2$  in (4.19), then we can arrive at (1.9). At last, letting  $\lambda = \frac{\beta}{2}$  in (4.19), and using (2.21), we have

$$\int_0^{\infty} \int_0^{\infty} \frac{\left(\log \frac{x}{y}\right)^{2m}}{x^{2\beta} - x^{\beta}y^{\beta} + y^{2\beta}} f(x)g(y) dx dy < \left(\frac{\pi}{3\beta}\right)^{2m+1} \left(\varphi_2^{(2m)}\left(\frac{\pi}{6}\right) + E_m\right) \|f\|_{p,\mu} \|g\|_{q,v}, \quad (4.20)$$

where  $\mu(x) = x^{p\left(1-\frac{\beta}{2}\right)-1}$  and  $v(y) = y^{q\left(1-\frac{3\beta}{2}\right)-1}$ .

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