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Research Article

On a New Integral-Type Operator from the Weighted Bergman Space to the Bloch-Type Space on the Unit Ball

Stevo Stević

Mathematical Institute of the Serbian Academy of Science, Knez Mihailova 36/III, 11000 Beograd, Serbia

Correspondence should be addressed to Stevo Stević, sstevic@ptt.rs

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We introduce an integral-type operator, denoted by P_φ^g, on the space of holomorphic functions on the unit ball $\mathbb{B} \subset \mathbb{C}^n$, which is an extension of the product of composition and integral operators on the unit disk. The operator norm of P_φ^g from the weighted Bergman space $A_\alpha^p(\mathbb{B})$ to the Bloch-type space $\mathcal{B}_\mu(\mathbb{B})$ or the little Bloch-type space $\mathcal{B}_{\mu,0}(\mathbb{B})$ is calculated. The compactness of the operator is characterized in terms of inducing functions g and φ. Upper and lower bounds for the essential norm of the operator $P_\varphi^g : A_\alpha^p(\mathbb{B}) \to \mathcal{B}_\mu(\mathbb{B})$, when $p > 1$, are also given.

1. Introduction

Let $\mathbb{B}$ be the open unit ball in the complex vector space $\mathbb{C}^n$, $S = \partial\mathbb{B}$ its boundary, $\mathbb{D}$ the open unit disk in the complex plane $\mathbb{C}$, $dV(z)$ the Lebesgue measure on $\mathbb{B}$, $dV_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dV(z)$, where $\alpha > -1$ and where the constant c_α is chosen such that $V_\alpha(\mathbb{B}) = 1$, $d\sigma$ the normalized rotation invariant measure on S (that is, $\sigma(S) = 1$), $H(\mathbb{B})$ the class of all holomorphic functions on the unit ball and $H^\infty = H^\infty(\mathbb{B})$ the space of all bounded holomorphic functions on $\mathbb{B}$ with the norm

$$\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)|. \tag{1.1}$$

Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$,

$$\langle z, w\rangle = \sum_{k=1}^n z_k \overline{w}_k \tag{1.2}$$

and $|z| = \sqrt{\langle z, z\rangle}$.

For $f \in H(\mathbb{B})$ with the Taylor expansion $f(z) = \sum_{|\beta|\geq 0} a_\beta z^\beta$, let

$$\Re f(z) = \sum_{|\beta|\geq 0} |\beta| a_\beta z^\beta \tag{1.3}$$

be the radial derivative of f, where $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a multi-index, $|\beta| = \beta_1 + \cdots + \beta_n$ and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$. It is well known (see, e.g., [1]) that

$$\Re f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z). \tag{1.4}$$

For $p > 0$ the Hardy space $H^p = H^p(\mathbb{B})$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_p^p = \sup_{0<r<1} \int_S |f(r\zeta)|^p d\sigma(\zeta) < \infty. \tag{1.5}$$

It is well known that for every $f \in H^p$ the radial limit

$$f^*(\zeta) := \lim_{r\to 1} f(r\zeta) \tag{1.6}$$

exists almost everywhere on $\zeta \in S$.

The weighted Bergman space $A_\alpha^p = A_\alpha^p(\mathbb{B})$, $p > 0$, $\alpha > -1$, consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{A_\alpha^p}^p = \int_{\mathbb{B}} |f(z)|^p dV_\alpha(z) < \infty. \tag{1.7}$$

When $p \geq 1$, the weighted Bergman space with the norm $\|\cdot\|_{A_\alpha^p}$ becomes a Banach space. If $p \in (0,1)$, it is a Frechet space with the translation invariant metric

$$d(f,g) = \|f - g\|_{A_\alpha^p}^p. \tag{1.8}$$

Since for every $f \in H^p$

$$\lim_{\alpha\to -1+0} \int_{\mathbb{B}} |f(z)|^p dV_\alpha(z) = \int_S |f^*(\zeta)|^p d\sigma(\zeta), \tag{1.9}$$

we will also use the notation A_{-1}^p for the Hardy space H^p.

A positive continuous function ϕ on $[0,1)$ is called normal (see [2]) if there is $\delta \in [0,1)$ and a and b, $0 < a < b$ such that

$$\begin{aligned} &\frac{\phi(r)}{(1-r)^a} \text{ is decreasing on } [\delta,1), \qquad \lim_{r\to 1} \frac{\phi(r)}{(1-r)^a} = 0; \\ &\frac{\phi(r)}{(1-r)^b} \text{ is increasing on } [\delta,1), \qquad \lim_{r\to 1} \frac{\phi(r)}{(1-r)^b} = \infty. \end{aligned} \tag{1.10}$$

From now on if we say that a function $\mu : \mathbb{B} \to [0,\infty)$ is normal, we will also assume that it is radial, that is, $\mu(z) = \mu(|z|)$, $z \in \mathbb{B}$.

The weighted space $H_\mu^\infty = H_\mu^\infty(\mathbb{B})$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H_\mu^\infty} = \sup_{z\in\mathbb{B}} \mu(z)\left|f(z)\right| < \infty, \tag{1.11}$$

where μ is normal. For $\mu(z) = (1-|z|^2)^\beta$, $\beta > 0$, we obtain the weighted space $H_\beta^\infty = H_\beta^\infty(\mathbb{B})$ (see, e.g., [3–5]).

The little weighted space $H_{\mu,0}^\infty = H_{\mu,0}^\infty(\mathbb{B})$ is a subspace of H_μ^∞ consisting of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1}\mu(z)\left|f(z)\right| = 0. \tag{1.12}$$

The class of all $f \in H(\mathbb{B})$ such that

$$B_\mu(f) = \sup_{z\in\mathbb{B}} \mu(z)\left|\Re f(z)\right| < \infty, \tag{1.13}$$

where μ is normal, is called the Bloch-type space, and is denoted by $\mathcal{B}_\mu = \mathcal{B}_\mu(\mathbb{B})$. With the norm

$$\|f\|_{\mathcal{B}_\mu} = \left|f(0)\right| + B_\mu(f), \tag{1.14}$$

the Bloch-type space becomes a Banach space.

The little Bloch-type space $\mathcal{B}_{\mu,0}$ is a subspace of $\mathcal{B}_\mu$ consisting of those $f \in \mathcal{B}_\mu$ such that

$$\lim_{|z|\to 1}\mu(z)\left|\Re f(z)\right| = 0. \tag{1.15}$$

The α-Bloch space $\mathcal{B}^\alpha$ is obtained for $\mu(z) = (1-|z|^2)^\alpha$, $\alpha \in (0,\infty)$ (see, e.g., [6–11]). For $\alpha = 1$ the space $\mathcal{B}^1 = \mathcal{B}$ becomes the classical Bloch space.

Let φ be a holomorphic self-map of $\mathbb{B}$. For any $f \in H(\mathbb{B})$, the composition operator is defined by

$$C_\varphi f(z) = f\left(\varphi(z)\right), \quad z \in \mathbb{B}. \tag{1.16}$$

It is of interest to provide function theoretic characterizations when φ induces bounded or compact composition operators on spaces of holomorphic functions. For some classical results in the topic (see, e.g., [12]). For some recent results see, for example, [3–5, 7, 13–23] and the references therein.

Let $g \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. For $f \in H(\mathbb{D})$, products of integral-type and composition operator are defined as follows:

$$C_\varphi J_g f(z) = \int_0^{\varphi(z)} f(\zeta)g'(\zeta)d\zeta, \qquad J_g C_\varphi f(z) = \int_0^z f\left(\varphi(\zeta)\right)g'(\zeta)d\zeta. \tag{1.17}$$

When $\varphi(z) = z$, operators in (1.17) are reduced to the integral operator introduced in [24]. For some other results on the operator; see, for example, [25, 26], and related references therein. Some results on related integral-type operators on spaces of holomorphic functions in $\mathbb{C}^n$ can be found, for example, in [27–41] (see also the references therein).

In [42], among other results, we proved the following theorem regarding the boundedness of the operator $J_g C_\varphi : A_\alpha^p(\mathbb{D}) \to \mathcal{B}_\mu(\mathbb{D})$.

Theorem 1.1. *Assume that $p > 0$, $\alpha > -1$, $g \in H(\mathbb{D})$, μ is normal, and φ is a holomorphic self-map of $\mathbb{D}$. Then $J_g C_\varphi : A^p_\alpha(\mathbb{D}) \to \mathcal{B}_\mu(\mathbb{D})$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{\mu(z)\left|g'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(\alpha+2)/p}} < \infty. \tag{1.18}$$

One of the interesting questions is to extend operators in (1.17) in the unit ball settings and to study their function theoretic properties on spaces of holomorphic functions on the unit ball in terms of inducing functions.

Assume that $g \in H(\mathbb{B})$, $g(0) = 0$, and φ is a holomorphic self-map of $\mathbb{B}$. We introduce the following important integral-type operator on the space of holomorphic functions on $\mathbb{B}$:

$$P^g_\varphi(f)(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(\mathbb{B}),\ z \in \mathbb{B}. \tag{1.19}$$

First note that when $n = 1$, the operator is reduced to an operator of the form as the second operator in (1.17). Indeed, since $g \in H(\mathbb{D})$ and $g(0) = 0$, it follows that $g(z) = zg_0(z)$, $z \in \mathbb{D}$ for some $g_0 \in H(\mathbb{D})$. By using this fact and the change of variables $\zeta = tz$, we obtain

$$P^g_\varphi f(z) = \int_0^1 f(\varphi(tz))tzg_0(tz)\frac{dt}{t} = \int_0^z f(\varphi(\zeta))g_0(\zeta)d\zeta. \tag{1.20}$$

Hence operator (1.19) is a natural extension of the second operator in (1.17).

Now we formulate the following big research project related to the operator P^g_φ.

Research project 1. Let X and Y be two Banach spaces of holomorphic functions on the unit ball in $\mathbb{C}^n$ (e.g., the weighted Bergman space A^p_α, the Bloch-type space $\mathcal{B}_\mu$, the Hardy space H^p space, the weighted space H^∞_μ, the Besov space B^p, BMOA etc.) Characterize the boundedness, compactness, essential norms, and other operator theoretic properties of the operator $P^g_\varphi : X \to Y$ in terms of function theoretic properties of inducing functions φ and g.

Another interesting question is to find the exact value of the norm of operators on spaces of holomorphic functions. Majority of papers in the area only find asymptotics of the operator norm of certain linear operators on some spaces of holomorphic functions. There are a few papers which calculate the operator norm of these operators. Recently in [4] we calculated operator norm of the weighted composition operator uC_φ mapping the Bloch space $\mathcal{B}$ to the weighted space H^∞_μ, which motivates us to find the norms of weighted composition and other closely related operators between various spaces of holomorphic functions.

Research project 2. Let X and Y be two Banach spaces of holomorphic functions as in Research project 1. Calculate the operator norm of $P^g_\varphi : X \to Y$ in terms of inducing functions φ and g.

In this paper, among other results, we will calculate the operator norm of $P^g_\varphi : A^p_\alpha(\mathbb{B}) \to \mathcal{B}_\mu(\mathbb{B})$. We will also characterize the boundedness, compactness, and the essential norm of the operator. These results partially solve problems posed in the above research projects.

Throughout the paper, C denotes a positive constant not necessarily the same at each occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $A/C \leq B \leq CA$.

2. Auxiliary results

In this section, we give several auxiliary results, which are used in the proofs of the main results.

Lemma 2.1 (see [43, Corollary 3.5]). *Suppose that* $p \in (0, \infty)$ *and* $\alpha \geq -1$. *Then for all* $f \in A^p_\alpha(\mathbb{B})$ *and* $z \in \mathbb{B}$, *the following inequality holds:*

$$|f(z)| \leq \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{(n+1+\alpha)/p}}. \tag{2.1}$$

The following criterion for the compactness follows by standard arguments (see, e.g., [12, 20, 34–36]). Hence, we omit its proof.

Lemma 2.2. *Suppose that* $0 < p < \infty$, $\alpha \geq -1$, $g \in H(\mathbb{B})$, μ *is normal, and* φ *is a holomorphic self-map of* $\mathbb{B}$. *Then the operator* $P^g_\varphi : A^p_\alpha \to \mathcal{B}_\mu$ *is compact if and only if* $P^g_\varphi : A^p_\alpha \to \mathcal{B}_\mu$ *is bounded and for every bounded sequence* $(f_k)_{k\in\mathbb{N}}$ *in* A^p_α *converging to zero uniformly on compacts of* $\mathbb{B}$, *one has* $\|P^g_\varphi f_k\|_{\mathcal{B}_\mu} \to 0$ *as* $k \to \infty$.

The following result can be found in [44]. For closely related results, see also [11, 45–52] and the references therein.

Lemma 2.3. *Suppose that* $0 < p < \infty$, $\alpha > -1$, *then*

$$\|f\|^p_{A^p_\alpha} \asymp |f(0)|^p + \int_{\mathbb{B}} |\nabla f(z)|^p (1-|z|^2)^{p+\alpha} dV(z), \tag{2.2}$$

for every $f \in A^p_\alpha$ *(here* $\nabla f = ((\partial f/\partial_{z_1}), \ldots, (\partial f/\partial_{z_n}))$*).*

The following lemma can be proved similar to [53, Lemma 1].

Lemma 2.4. *Suppose that* μ *is normal. A closed set* K *in* $\mathcal{B}_{\mu,0}$ *is compact if and only if it is bounded and*

$$\lim_{|z|\to 1} \sup_{f\in K} \mu(z)|\Re f(z)| = 0. \tag{2.3}$$

The following lemma is related to [32, Lemma 1] and [34, Lemma 2].

Lemma 2.5. *Assume that* $f, g \in H(\mathbb{B})$ *and* $g(0) = 0$. *Then*

$$\Re P^g_\varphi(f)(z) = f(\varphi(z))g(z). \tag{2.4}$$

Proof. Since the function $f(\varphi(z))g(z)$ is holomorphic and $g(0) = 0$, it has the Taylor expansion in the following form $\sum_{\alpha\neq 0} a_\alpha z^\alpha$. Then

$$\Re[P^g_\varphi(f)](z) = \Re\int_0^1 \sum_{\alpha\neq 0} a_\alpha (tz)^\alpha \frac{dt}{t} = \Re\left(\sum_{\alpha\neq 0} \frac{a_\alpha}{|\alpha|} z^\alpha\right) = \sum_{\alpha\neq 0} a_\alpha z^\alpha, \tag{2.5}$$

as claimed. □

3. The norm of the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$

In this section, we calculate the norm $\|P_\varphi^g\|_{A_\alpha^p \to \mathcal{B}_\mu}$.

Theorem 3.1. *Assume that $p > 0$, $\alpha \geq -1$, $g \in H(\mathbb{B})$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$, and $P_\varphi^g : A_\alpha^p(\mathbb{B}) \to \mathcal{B}_\mu(\mathbb{B})$ is bounded. Then*

$$\|P_\varphi^g\|_{A_\alpha^p \to \mathcal{B}_\mu} = \|P_\varphi^g\|_{A_\alpha^p \to \mathcal{B}_{\mu,0}} = \sup_{z\in\mathbb{B}} \frac{\mu(z)\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} =: M. \tag{3.1}$$

Proof. If $f \in A_\alpha^p$, then by Lemmas 2.5 and 2.1 we obtain

$$\|P_\varphi^g f\|_{\mathcal{B}_\mu} = \sup_{z\in\mathbb{B}} \mu(z)\left|g(z) f(\varphi(z))\right| \leq \|f\|_{A_\alpha^p} \sup_{z\in\mathbb{B}} \frac{\mu(z)\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}}, \tag{3.2}$$

from which it follows that

$$\|P_\varphi^g\|_{A_\alpha^p \to \mathcal{B}_\mu} \leq M. \tag{3.3}$$

Now we prove the reverse inequality. For $w \in \mathbb{B}$ fixed, set

$$f_w(z) = \frac{\left(1-|w|^2\right)^{(n+1+\alpha)/p}}{\left(1-\langle z,w\rangle\right)^{2(n+1+\alpha)/p}}, \quad z \in \mathbb{B}. \tag{3.4}$$

We have that $\|f_w\|_{A_\alpha^p} = 1$, for each $w \in \mathbb{B}$. For $\alpha > -1$ this fact is well known. The proof for the case $\alpha = -1$ could be less known, and we give a proof of it for the lack of a specific reference and for the benefit of the reader. Let $z = r\zeta$, $\zeta \in S$, then we have

$$\begin{aligned}
\|f_w\|_p^p &= \sup_{0<r<1} \int_S \frac{\left(1-|w|^2\right)^n}{\left|1-\langle z,w\rangle\right|^{2n}} d\sigma(\zeta) \\
&= \left(1-|w|^2\right)^n \sup_{0<r<1} \int_S \left|\left(1-\langle z,w\rangle\right)^{-n}\right|^2 d\sigma(\zeta) \\
&= \left(1-|w|^2\right)^n \sup_{0<r<1} \int_S \left|\sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{\Gamma(k+1)\Gamma(n)} r^k \langle \zeta,w\rangle^k\right|^2 d\sigma(\zeta) \\
&= \left(1-|w|^2\right)^n \sup_{0<r<1} \int_S \sum_{k=0}^{\infty} \left(\frac{\Gamma(n+k)}{\Gamma(k+1)\Gamma(n)}\right)^2 r^{2k} \left|\langle \zeta,w\rangle\right|^{2k} d\sigma(\zeta) \\
&= \left(1-|w|^2\right)^n \sup_{0<r<1} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{\Gamma(k+1)\Gamma(n)} r^{2k} |w|^{2k} \\
&= \left(1-|w|^2\right)^n \sup_{0<r<1} \frac{1}{\left(1-r^2|w|^2\right)^n} = 1,
\end{aligned} \tag{3.5}$$

where we have used the following formula (see, e.g., [1])

$$\int_S \left|\langle \zeta,w\rangle\right|^{2k} d\sigma(\zeta) = \frac{\Gamma(k+1)\Gamma(n)}{\Gamma(n+k)} |w|^{2k}. \tag{3.6}$$

From this and the boundedness of $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$, we have

$$\begin{aligned}\|P_\varphi^g\|_{A_\alpha^p\to\mathcal{B}_\mu} &= \|f_{\varphi(w)}\|_{A_\alpha^p}\|P_\varphi^g\|_{A_\alpha^p\to\mathcal{B}_\mu} \geq \|P_\varphi^g(f_{\varphi(w)})\|_{\mathcal{B}_\mu} \\ &= \sup_{z\in\mathbb{B}} \mu(z)|g(z)||f_{\varphi(w)}(\varphi(z))| \\ &\geq \mu(w)|g(w)||f_{\varphi(w)}(\varphi(w))| \\ &= \frac{\mu(w)|g(w)|}{(1-|\varphi(w)|^2)^{(n+1+\alpha)/p}}.\end{aligned} \tag{3.7}$$

Taking the supremum in (3.7) over $w \in \mathbb{B}$, we obtain

$$\|P_\varphi^g\|_{A_\alpha^p\to\mathcal{B}_\mu} \geq M. \tag{3.8}$$

From (3.3) and (3.8), it follows that $\|P_\varphi^g\|_{A_\alpha^p\to\mathcal{B}_\mu} = M$.

Since

$$\|P_\varphi^g\|_{A_\alpha^p\to\mathcal{B}_{\mu,0}} \leq \|P_\varphi^g\|_{A_\alpha^p\to\mathcal{B}_\mu} \tag{3.9}$$

and the proof of (3.8) does not depend on the space $\mathcal{B}_\mu$ (we may replace it by $\mathcal{B}_{\mu,0}$) the second equality in (3.1) also holds. □

Corollary 3.2. *Assume that* $p > 0$, $\alpha \geq -1$, $g \in H(\mathbb{B})$, μ *is normal, and* φ *is a holomorphic self-map of* $\mathbb{B}$. *Then* $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ *is bounded if and only if*

$$\sup_{z\in\mathbb{B}} \frac{\mu(z)|g(z)|}{(1-|\varphi(z)|^2)^{(n+1+\alpha)/p}} < \infty. \tag{3.10}$$

Proof. If $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ is bounded, then (3.10) follows from Theorem 3.1. If (3.10) holds, then the boundedness of $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ follows from (3.3). □

4. The boundedness of the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_{\mu,0}$

Here we characterize the boundedness of the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_{\mu,0}$.

Theorem 4.1. *Assume that* $p > 0$, $\alpha \geq -1$, $g \in H(\mathbb{B})$, μ *is normal, and* φ *is a holomorphic self-map of* $\mathbb{B}$. *Then* $P_\varphi^g : A_\alpha^p \to \mathcal{B}_{\mu,0}$ *is bounded if and only if* $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ *is bounded and* $g \in H_{\mu,0}^\infty$.

Proof. Assume that $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ is bounded and $g \in H_{\mu,0}^\infty$. Then, for each polynomial p, we have

$$\mu(z)|\Re P_\varphi^g p(z)| = \mu(z)|g(z)p(\varphi(z))| \leq \mu(z)|g(z)|\|p\|_\infty \longrightarrow 0, \quad \text{as } |z| \longrightarrow 1 \tag{4.1}$$

from which it follows that $P_\varphi^g(p) \in \mathcal{B}_{\mu,0}$.

Since the set of all polynomials is dense in A_α^p, we have that for every $f \in A_\alpha^p$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \|f - p_k\|_{A_\alpha^p} = 0. \tag{4.2}$$

From this and since the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ is bounded, it follows that

$$\left\|P_\varphi^g f - P_\varphi^g p_k\right\|_{\mathcal{B}_\mu} \le \left\|P_\varphi^g\right\|_{A_\alpha^p \to \mathcal{B}_\mu} \left\|f - p_k\right\|_{A_\alpha^p} \longrightarrow 0, \tag{4.3}$$

as $k \to \infty$. Hence $P_\varphi^g(A_\alpha^p) \subset \mathcal{B}_{\mu,0}$. Since $\mathcal{B}_{\mu,0}$ is a closed subset of $\mathcal{B}_\mu$, the boundedness of $P_\varphi^g : A_\alpha^p \to \mathcal{B}_{\mu,0}$ follows.

Now assume that $P_\varphi^g : A_\alpha^p \to \mathcal{B}_{\mu,0}$ is bounded. Then clearly $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ is bounded. Taking the test function $f(z) = 1 \in A_\alpha^p$, we obtain $g \in H_{\mu,0}^\infty$. □

5. Compactness of the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$

This section is devoted to studying of the compactness of the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$. We prove the following result.

Theorem 5.1. *Assume that $p > 0$, $\alpha \ge -1$, $g \in H(\mathbb{B})$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$, and the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ is bounded. Then the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ is compact if and only if*

$$\lim_{|\varphi(z)| \to 1} \frac{\mu(z)\left|g(z)\right|}{\left(1 - \left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} = 0. \tag{5.1}$$

Proof. First assume that the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ is compact. If $\|\varphi\|_\infty < 1$, then condition (5.1) is vacuously satisfied. Hence, assume that $\|\varphi\|_\infty = 1$ and assume to the contrary that (5.1) does not hold. Then there is a sequence $(z_k)_{k\in\mathbb{N}}$ satisfying the condition $|\varphi(z_k)| \to 1$ as $k \to \infty$ and $\delta > 0$ such that

$$\frac{\mu(z_k)\left|g(z_k)\right|}{\left(1 - \left|\varphi(z_k)\right|^2\right)^{(n+1+\alpha)/p}} \ge \delta, \quad k \in \mathbb{N}. \tag{5.2}$$

For $w \in \mathbb{B}$ fixed, set

$$F_k(z) = f_{\varphi(z_k)}(z), \quad k \in \mathbb{N}, \tag{5.3}$$

where f_w is defined in (3.4). Recall that $\|f_w\|_{A_\alpha^p} = 1$, for each $w \in \mathbb{B}$. Then $\|F_k\|_{A_\alpha^p} = 1$, $k \in \mathbb{N}$ and it is easy to see that $F_k \to 0$ uniformly on compacts of $\mathbb{B}$ as $k \to \infty$. Hence, by Lemma 2.2, it follows that

$$\lim_{k\to\infty} \left\|P_\varphi^g F_k\right\|_{\mathcal{B}_\mu} = 0. \tag{5.4}$$

On the other hand, by Lemma 2.5 and (5.2), we obtain

$$\left\|P_\varphi^g F_k\right\|_{\mathcal{B}_\mu} = \sup_{z\in\mathbb{B}} \mu(z)\left|g(z)\right|\left|F_k(\varphi(z))\right| \ge \frac{\mu(z_k)\left|g(z_k)\right|}{\left(1 - \left|\varphi(z_k)\right|^2\right)^{(n+1+\alpha)/p}} \ge \delta > 0, \tag{5.5}$$

for every $k \in \mathbb{N}$, which contradicts with (5.4).

Now assume that (5.1) holds. Then for every $\varepsilon > 0$ there is an $r \in (0,1)$ such that when $r < |\varphi(z)| < 1$,

$$\frac{\mu(z)\left|g(z)\right|}{\left(1 - \left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} < \varepsilon. \tag{5.6}$$

On the other hand, since the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ is bounded, for $f \equiv 1 \in A_\alpha^p$, we obtain $\|g\|_{H_\mu^\infty} < \infty$.

Assume that $(h_k)_{k\in\mathbb{N}}$ is a bounded sequence in A_α^p converging to zero uniformly on compacts of $\mathbb{B}$ as $k \to \infty$. Let $\sup_{k\in\mathbb{N}}\|h_k\|_{A_\alpha^p} = M_1$. Then by Lemma 2.1 and (5.6), for $r < |\varphi(z)| < 1$, we obtain

$$\mu(z)\left|g(z)\right|\left|h_k(\varphi(z))\right| \le \sup_{k\in\mathbb{N}} \|h_k\|_{A_\alpha^p} \frac{\mu(z)\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} < M_1\varepsilon. \tag{5.7}$$

If $|\varphi(z)| \le r$, we have

$$\mu(z)\left|g(z)\right|\left|h_k(\varphi(z))\right| \le \|g\|_{H_\mu^\infty} \sup_{|w|\le r} \left|h_k(w)\right| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{5.8}$$

From (5.7) and (5.8), it follows that $\|P_\varphi^g h_k\|_{\mathcal{B}_\mu} \to 0$ as $k \to \infty$, from which the compactness of the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_\mu$ follows. □

6. Compactness of the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_{\mu,0}$

This section characterizes the compactness of the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_{\mu,0}$.

Theorem 6.1. *Assume $p > 0$, $\alpha \ge -1$, $g \in H(\mathbb{B})$, μ is normal, φ is a holomorphic self-map of $\mathbb{B}$, and the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_{\mu,0}$ is bounded. Then the operator $P_\varphi^g : A_\alpha^p \to \mathcal{B}_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{\mu(z)\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} = 0. \tag{6.1}$$

Proof. Assume $P_\varphi^g : A_\alpha^p \to \mathcal{B}_{\mu,0}$ is compact. Then clearly $P_\varphi^g : A_\alpha^p \to \mathcal{B}_{\mu,0}$ is bounded and as in Theorem 4.1 we have that $g \in H_{\mu,0}^\infty$.

Hence if $\|\varphi\|_\infty < 1$, then

$$\lim_{|z|\to 1} \frac{\mu(z)\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} \le \lim_{|z|\to 1} \frac{\mu(z)\left|g(z)\right|}{\left(1-\|\varphi\|_\infty^2\right)^{(n+1+\alpha)/p}} = 0, \tag{6.2}$$

from which the result follows in this case.

Now assume $\|\varphi\|_\infty = 1$. By using the test functions $F_k(z) = f_{\varphi(z_k)}(z)$, $k \in \mathbb{N}$, defined in (5.3) we obtain that condition (5.1) holds, which implies that for every $\varepsilon > 0$, there is an $r \in (0,1)$ such that for $r < |\varphi(z)| < 1$, condition (5.6) holds.

Since $g \in H_{\mu,0}^\infty$, there is $\sigma \in (0,1)$ such that for $\sigma < |z| < 1$

$$\mu(z)\left|g(z)\right| < \varepsilon\left(1-r^2\right)^{(n+1+\alpha)/p}. \tag{6.3}$$

Hence, if $|\varphi(z)| \le r$ and $\sigma < |z| < 1$, we have

$$\frac{\mu(z)\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{(n+1+\alpha)/p}} \le \frac{\mu(z)\left|g(z)\right|}{\left(1-r^2\right)^{(n+1+\alpha)/p}} < \varepsilon. \tag{6.4}$$

From (5.6) and (6.4), condition (6.1) follows.

Now assume that condition (6.1) holds. Then the quantity M in Theorem 3.1 is finite. Using this fact and the following inequality

$$\mu(z)\left|\Re P_{\varphi}^{g} f(z)\right| \leq \mu(z)\left|g(z) f(\varphi(z))\right| \leq \|f\|_{A_{\alpha}^{p}} \frac{\mu(z)\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^{2}\right)^{(n+1+\alpha)/p}}, \tag{6.5}$$

it follows that the set $P_{\varphi}^{g}(\{f : \|f\|_{A_{\alpha}^{p}} \leq 1\})$ is bounded in $\mathcal{B}_{\mu}$, moreover, in view of (6.1), it is bounded in $\mathcal{B}_{\mu,0}$. Taking the supremum in the last inequality over the unit ball in A_{α}^{p}, then letting $|z| \to 1$, using condition (6.1) and employing Lemma 2.4, we obtain the compactness of the operator $P_{\varphi}^{g} : A_{\alpha}^{p} \to \mathcal{B}_{\mu,0}$, as desired. □

7. Essential norm of $P_{\varphi}^{g} : A_{\alpha}^{p} \to \mathcal{B}_{\mu}$

Let X and Y be Banach spaces, and let $L : X \to Y$ be a bounded linear operator. The essential norm of the operator $L : X \to Y$, denoted by $\|L\|_{e,X\to Y}$, is defined as follows:

$$\|L\|_{e,X\to Y} = \inf\left\{\|L+K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\right\}, \tag{7.1}$$

where $\|\cdot\|_{X\to Y}$ denote the operator norm.

From this definition and since the set of all compact operators is a closed subset of the set of bounded operators, it follows that operator L is compact if and only if $\|L\|_{e,X\to Y} = 0$.

In this section, we study the essential norm of the operator $P_{\varphi}^{g} : A_{\alpha}^{p} \to \mathcal{B}_{\mu}$ for the case $p > 1$.

Theorem 7.1. *Assume that $p \in (1,\infty)$, $\alpha \geq -1$, $g \in H(\mathbb{B})$, $g(0) = 0$, φ is a holomorphic self-map of $\mathbb{B}$, and $P_{\varphi}^{g} : A_{\alpha}^{p} \to \mathcal{B}_{\mu}$ is bounded. Then the following inequalities hold:*

$$\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^{2}\right)^{(n+1+\alpha)/p}} \leq \left\|P_{\varphi}^{g}\right\|_{e,A_{\alpha}^{p}\to\mathcal{B}_{\mu}} \leq 2\limsup_{|\varphi(z)|\to 1} \frac{\mu(z)\left|g(z)\right|}{\left(1-\left|\varphi(z)\right|^{2}\right)^{(n+1+\alpha)/p}}. \tag{7.2}$$

Proof. Assume that $(\varphi(z_k))_{k\in\mathbb{N}}$ is a sequence in $\mathbb{B}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Note that the sequence $(f_{\varphi(z_k)})_{k\in\mathbb{N}}$ (where f_w is defined in (3.4)) is such that $\|f_{\varphi(z_k)}\|_{A_{\alpha}^{p}} = 1$ for each $k \in \mathbb{N}$ and it converges to zero uniformly on compacts of $\mathbb{B}$. From this and by [11, Theorems 2.12 and 4.50], it follows that $f_{\varphi(z_k)} \to 0$ weakly in A_{α}^{p}, as $k \to \infty$ (here we use the condition $p > 1$). Hence, for every compact operator $K : A_{\alpha}^{p} \to \mathcal{B}_{\mu}$, we have that $\|K f_{\varphi(z_k)}\|_{\mathcal{B}_{\mu}} \to 0$ as $k \to \infty$. Thus, for every such sequence and for every compact operator $K : A_{\alpha}^{p} \to \mathcal{B}_{\mu}$, we have that

$$\begin{aligned}
\left\|P_{\varphi}^{g} + K\right\|_{A_{\alpha}^{p}\to\mathcal{B}_{\mu}} &\geq \limsup_{k\to\infty} \frac{\left\|P_{\varphi}^{g} f_{\varphi(z_k)}\right\|_{\mathcal{B}_{\mu}} - \left\|K f_{\varphi(z_k)}\right\|_{\mathcal{B}_{\mu}}}{\left\|f_{\varphi(z_k)}\right\|_{A_{\alpha}^{p}}} \\
&= \limsup_{k\to\infty} \left\|P_{\varphi}^{g} f_{\varphi(z_k)}\right\|_{\mathcal{B}_{\mu}} \\
&\geq \limsup_{k\to\infty} \mu(z_k)\left|g(z_k) f_{\varphi(z_k)}(\varphi(z_k))\right| \\
&= \limsup_{n\to\infty} \frac{\mu(z_k)\left|g(z_k)\right|}{\left(1-\left|\varphi(z_k)\right|^{2}\right)^{(n+1+\alpha)/p}}.
\end{aligned} \tag{7.3}$$

Taking the infimum in (7.3) over the set of all compact operators $K : A_\alpha^p \to \mathcal{B}_\mu$, we obtain

$$\|P_\varphi^g\|_{e,A_\alpha^p\to\mathcal{B}_\mu} \ge \limsup_{n\to\infty} \frac{\mu(z_k)|g(z_k)|}{(1-|\varphi(z_k)|^2)^{(n+1+\alpha)/p}}, \tag{7.4}$$

from which the first inequality follows.

In the sequel we prove the second inequality. Assume that $(r_l)_{l\in\mathbb{N}}$ is a sequence which increasingly converges to 1. Consider the operators defined by

$$(P_{r_l\varphi}^g f)(z) = \int_0^1 g(tz) f(r_l\varphi(tz)) \frac{dt}{t}, \quad l \in \mathbb{N}. \tag{7.5}$$

We prove that these operators are compact. Indeed, since $|r_l\varphi(z)| \le r_l < 1$, it follows that condition (5.1) in Theorem 5.1 is vacuously satisfied, from which the claim follows.

Recall that $g \in H_\mu^\infty$. Let $\rho \in (0,1)$ be fixed for a moment. Employing Lemma 2.1, and using the fact

$$\|f - f_{r_l}\|_{A_\alpha^p} \le 2\|f\|_{A_\alpha^p}, \quad l \in \mathbb{N}, \tag{7.6}$$

which follows by using the triangle inequality for the norm, the monotonicity of the integral means

$$M_p^p(f,r) = \int_S |f(r\zeta)|^p d\sigma(\zeta) \tag{7.7}$$

and the polar coordinates, we have

$$\begin{aligned}
\|P_\varphi^g - P_{r_l\varphi}^g\|_{A_\alpha^p\to\mathcal{B}_\mu} &= \sup_{\|f\|_{A_\alpha^p}\le 1} \sup_{z\in\mathbb{B}} \mu(z)|g(z)||f(\varphi(z)) - f(r_l\varphi(z))| \\
&\le \sup_{\|f\|_{A_\alpha^p}\le 1} \sup_{|\varphi(z)|\le\rho} \mu(z)|g(z)||f(\varphi(z)) - f(r_l\varphi(z))| \\
&\quad + \sup_{\|f\|_{A_\alpha^p}\le 1} \sup_{|\varphi(z)|>\rho} \mu(z)|g(z)||f(\varphi(z)) - f(r_l\varphi(z))| \\
&\le \|g\|_{H_\mu^\infty} \sup_{\|f\|_{A_\alpha^p}\le 1} \sup_{|\varphi(z)|\le\rho} |f(\varphi(z)) - f(r_l\varphi(z))| \\
&\quad + 2 \sup_{|\varphi(z)|>\rho} \frac{\mu(z)|g(z)|}{(1-|\varphi(z)|^2)^{(n+1+\alpha)/p}}.
\end{aligned} \tag{7.8}$$

Let

$$I_l := \sup_{\|f\|_{A_\alpha^p}\le 1} \sup_{|\varphi(z)|\le\rho} |f(\varphi(z)) - f(r_l\varphi(z))|. \tag{7.9}$$

If $\alpha > -1$, then by using the mean value theorem, the subharmonicity of the partial derivatives of f and Lemma 2.3, we have

$$I_l \leq \sup_{\|f\|_{A^p_\alpha}\leq 1} \sup_{|\varphi(z)|\leq\rho} (1-r_l)\big|\varphi(z)\big| \sup_{|w|\leq\rho} \big|\nabla f(w)\big| \tag{7.10}$$

$$\leq C_\rho(1-r_l) \sup_{\|f\|_{A^p_\alpha}\leq 1} \left(\int_{|w|\leq(1+\rho)/2} \big|\nabla f(w)\big|^p \left(1-|w|^2\right)^{p+\alpha} dV(w)\right)^{1/p}$$

$$\leq C_\rho(1-r_l) \sup_{\|f\|_{A^p_\alpha}\leq 1} \left(\int_{\mathbb{B}} \big|f(w)\big|^p \left(1-|w|^2\right)^{\alpha} dV(w)\right)^{1/p}$$

$$\leq C_\rho(1-r_l) \longrightarrow 0 \quad \text{as } l \longrightarrow \infty. \tag{7.11}$$

If $\alpha = -1$, then applying in (7.10) the known fact that for each compact $K \subset \mathbb{B}$,

$$\sup_{w\in K} \big|\nabla f(w)\big| \leq C\|f\|_p, \tag{7.12}$$

for some C independent of f (see [11]), we obtain that (7.11) also holds in this case.

Letting $l \to \infty$ in (7.8), using (7.11), and then letting $\rho \to 1$, the second inequality in (7.2) follows, finishing the proof of the theorem. □

Motivated by Theorem 7.1, we leave the following open problem.

Open problem 1. Find the exact value of the essential norm of the operator $P^g_\varphi : A^p_\alpha \to \mathcal{B}_\mu$.

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