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On a nonlinear Hadamard type fractional differential equation with p-Laplacian operator and strip condition

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Abstract

Under certain nonlinear growth conditions of the nonlinearity, we investigate the existence of solutions for a nonlinear Hadamard type fractional differential equation with strip condition and p-Laplacian operator. At the end, two examples are given to illustrate our main results.

Keywords: Hadamard fractional differential equations, strip condition, p-Laplacian operator, fixed point.
2010 MSC: 34A08, 34B10.

1. Introduction

In recent years, fractional differential equations have been acquired much attention due to its applications in a number of fields such as physics, mechanics, chemistry, biology, economics, biophysics, capacitor theory, signal and image processing, etc.. For theoretical and practical development of the subject, we refer to the books [7, 19, 20, 25, 35, 46]. Some recent works on fractional differential equations can be found in [2, 8, 10, 12, 14, 26, 28–32, 36–38, 40–43] and the references therein.

It has been noticed that the most of the above mentioned works on the topic are based on Riemann-Liouville or Caputo fractional derivatives. In 1892, Hadamard [16] introduced another fractional derivative, which differs from the above mentioned ones because its definition involves logarithmic function of arbitrary exponent and named as Hadamard derivative. Although many researchers are paying more and more attention to Hadamard type fractional differential equation, the study of the topic is still in its primary

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stage. About the details and recent developments on Hadamard fractional differential equations, we refer the reader to [1, 4, 5, 24, 34, 39].

On the another hand, p-Laplacian operator is extensively applied in the mathematical modelling of several real world phenomena in physics, mechanics, dynamical systems, etc.. While studying the fundamental problem of turbulent flow in a porous medium, Leibenson[21] introduced the p-Laplacian operator $\phi_p(x(t))$ in 1945. Also there has been much interests shown in obtaining the existence and multiplicity of solutions of this class of problems by employing different techniques such as critical point theorems, variational methods and energy functionals, etc (see [3, 6, 9, 22, 33] and the references cited therein). For some works on nonlinear fractional differential equations involving p-Laplacian operator, we refer the reader to a series of papers [11, 13, 15, 17, 18, 23, 44, 45].

Motivated by the work mentioned above, we consider the following Hadamard fractional differential equation with the nonlocal Hadamard fractional integral boundary condition

$$\begin{cases} {}^H D^\beta \phi_p({}^H D^\alpha u(t)) = f(t, u(t)), \quad t \in (1, T), \quad T > 1, \\ u(T) = \lambda {}^H I^\sigma u(\eta), \quad {}^H D^\alpha u(1) = 0, \quad u(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha \le 2, 0 < \beta \le 1, \sigma > 0$, $\lambda \in \mathbb{R}, f \in C([1,T] \times \mathbb{R}, \mathbb{R})$, ${}^H D^\alpha$ and ${}^H I^\sigma$ denote Hadamard fractional derivative of order α and the Hadamard fractional integral of order σ, respectively. $\phi_p(s)$ is a p-Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, $(\phi_p)^{-1}(s) = \phi_q(s)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

To the best of our knowledge, no paper has considered Hadamard type nonlinear fractional differential equation with p-Laplacian operator and strip condition (1.1). Now, this paper attempts to fill this gap in the literature.

The rest of the paper is arranged as follows. Section 2 gives some necessary notations, definitions, auxiliary lemmas and a theorem. In Section 3, we discuss first the existence and uniqueness of solutions for the corresponding linear fractional differential equation. Based on the analysis, we prove the existence of solution for the Hadamard type nonlinear fractional differential equation with p-Laplacian operator and strip condition (1.1). Finally, we propose two applicable examples for illustrating the obtained results.

2. Preliminaries

For the reader's convenience, we present some necessary definitions of fractional calculus theory and lemmas.

Definition 2.1 ([19])**.** The Hadamard fractional integral of order q for a function g is defined as

$$ {}^H I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t (\log \frac{t}{s})^{q-1} \frac{g(s)}{s} ds, \qquad q > 0,$$

provided the integral exists.

Definition 2.2 ([19])**.** The Hadamard derivation of fractional order q for a function $g : [1, \infty) \to \mathbb{R}$ is defined as

$$ {}^H D^q g(t) = \frac{1}{\Gamma(n-q)} (t \frac{d}{dt})^n \int_1^t (\log \frac{t}{s})^{n-q-1} \frac{g(s)}{s} ds, \quad n-1 < q < n, \quad n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Lemma 2.3 ([19])**.** *If $a, \alpha, \beta > 0$, then*

$$({}^H D_a^\alpha (\log \frac{t}{a})^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\log \frac{x}{a})^{\beta-\alpha-1},$$

$$({}^H I_a^\alpha (\log \frac{t}{a})^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\log \frac{x}{a})^{\beta+\alpha-1}.$$

Lemma 2.4 ([19]). *Let $q > 0$ and $x \in C[1,\infty)\bigcap L^1[1,\infty)$. Then the Hadamard fractional differential equation $D^q x(t) = 0$ has the solution*

$$x(t) = \sum_{i=1}^{n} c_i (\log t)^{q-i},$$

and the following formula holds:

$${}^H I^q\ {}^H D^q x(t) = x(t) + \sum_{i=1}^{n} c_i (\log t)^{q-i},$$

where $c_i \in \mathbb{R}, i = 1, 2, ..., n$, and $n - 1 < q < n$.

Theorem 2.5 ([27]). *Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $V = \{u \in X | u = \mu T u, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in X.*

3. Main results

In this section, we investigate the existence of solutions for (1.1). By $C[1,T]$ we denote the Banach space of all continuous functions from $[1,T] \to \mathbb{R}$ endowed with the norm $\|u\|_\infty = \max_{t\in[1,T]} |u(t)|$.

For the sake of simplicity and convenience, we give the following notations:

$$\begin{aligned}\Omega =&(\log T)^{\alpha-1} - \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\sigma)}(\log \eta)^{\alpha+\sigma-1} \neq 0,\\ \Lambda =&\frac{\Gamma(\beta(q-1)+1)(\log T)^{\alpha+\beta(q-1)}}{(\Gamma(\beta+1))^{q-1}}\Big[\frac{1}{\Gamma(\alpha+\beta(q-1)+1)} + \frac{(\log T)^{\alpha-1}}{|\Omega|}\Big(\frac{|\lambda|(\log T)^{\sigma}}{\Gamma(\alpha+\beta(q-1)+\sigma+1)}\\ &+ \frac{1}{\Gamma(\alpha+\beta(q-1)+1)}\Big)\Big].\end{aligned} \tag{3.1}$$

Lemma 3.1. *For any $y \in C[1,T]$, the unique solution of the linear Hadamard fractional integral boundary value problem*

$$\begin{cases} {}^H D^\beta \phi_p({}^H D^\alpha u(t)) = y(t), \quad t \in (1,T), \quad T > 1,\\ u(T) = \lambda {}^H I^\sigma u(\eta), \quad {}^H D^\alpha u(1) = 0, \quad u(1) = 0,\end{cases}$$

is given by

$$\begin{aligned} u(t) =&\frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^s(\log\frac{s}{\tau})^{\beta-1}\frac{y(\tau)}{\tau}d\tau\Big)\frac{ds}{s}\\ &+\frac{\lambda(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\sigma)}\int_1^\eta (\log\frac{\eta}{s})^{\alpha+\sigma-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^s(\log\frac{s}{\tau})^{\beta-1}\frac{y(\tau)}{\tau}d\tau\Big)\frac{ds}{s}\\ &-\frac{(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha)}\int_1^T (\log\frac{T}{s})^{\alpha-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^s(\log\frac{s}{\tau})^{\beta-1}\frac{y(\tau)}{\tau}d\tau\Big)\frac{ds}{s}.\end{aligned}$$

Proof. Applying the Hadamard fractional integral of order β to both sides of the equation

$${}^H D^\beta \phi_p({}^H D^\alpha u(t)) = y(t),$$

we have

$$\phi_p({}^H D^\alpha u(t)) = c_0(\log t)^{\beta-1} +{}^H I^\beta y(t).$$

Thus,

$$^{H}D^{\alpha}u(t)=\phi_q\Big[c_0(\log t)^{\beta-1}+\frac{1}{\Gamma(\beta)}\int_1^t(\log\frac{t}{s})^{\beta-1}y(s)\frac{ds}{s}\Big],$$

By the boundary value condition $^{H}D^{\alpha}u(1)=0$, we have $c_0=0$. Then,

$$u(t)={}^{H}I^{\alpha}\phi_q({}^{H}I^{\beta}y(t))+c_1(\log t)^{\alpha-1}+c_2(\log t)^{\alpha-2}. \tag{3.2}$$

The boundary value condition $u(1)=0$ implies $c_2=0$. Therefore, it holds

$$u(t)={}^{H}I^{\alpha}\phi_q({}^{H}I^{\beta}y(t))+c_1(\log t)^{\alpha-1},$$

and

$$^{H}I^{\sigma}u(t)={}^{H}I^{\alpha+\sigma}\phi_q({}^{H}I^{\beta}y(t))+c_1^{H}I^{\sigma}(\log t)^{\alpha-1}.$$

This together with the boundary value condition $u(T)=\lambda^{H}I^{\sigma}u(\eta)$ yields that

$$\begin{aligned}c_1=&\frac{\lambda^{H}I^{\alpha+\sigma}\phi_q({}^{H}I^{\beta}y(\eta))-{}^{H}I^{\alpha}\phi_q({}^{H}I^{\beta}y(T))}{(\log T)^{\alpha-1}-\lambda^{H}I^{\sigma}(\log\eta)^{\alpha-1}}\\
=&\frac{1}{\Omega}\Big[\frac{\lambda}{\Gamma(\alpha+\sigma)}\int_1^{\eta}(\log\frac{\eta}{s})^{\alpha+\sigma-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^{s}(\log\frac{s}{\tau})^{\beta-1}\frac{y(\tau)}{\tau}d\tau\Big)\frac{ds}{s}\\
&-\frac{1}{\Gamma(\alpha)}\int_1^{T}(\log\frac{T}{s})^{\alpha-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^{s}(\log\frac{s}{\tau})^{\beta-1}\frac{y(\tau)}{\tau}d\tau\Big)\frac{ds}{s}\Big].\end{aligned}$$

Substituting the value of c_1 and c_2 into (3.2), we obtain the unique solution of (3.1) as follows:

$$\begin{aligned}u(t)=&\frac{1}{\Gamma(\alpha)}\int_1^{t}(\log\frac{t}{s})^{\alpha-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^{s}(\log\frac{s}{\tau})^{\beta-1}\frac{y(\tau)}{\tau}d\tau\Big)\frac{ds}{s}\\
&+\frac{\lambda(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\sigma)}\int_1^{\eta}(\log\frac{\eta}{s})^{\alpha+\sigma-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^{s}(\log\frac{s}{\tau})^{\beta-1}\frac{y(\tau)}{\tau}d\tau\Big)\frac{ds}{s}\\
&-\frac{(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha)}\int_1^{T}(\log\frac{T}{s})^{\alpha-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^{s}(\log\frac{s}{\tau})^{\beta-1}\frac{y(\tau)}{\tau}d\tau\Big)\frac{ds}{s}.\end{aligned}$$

This completes the proof. □

In relation to (1.1), we define the operator $G: C[1,T]\to C[1,T]$ as

$$\begin{aligned}Gu(t)=&\frac{1}{\Gamma(\alpha)}\int_1^{t}(\log\frac{t}{s})^{\alpha-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^{s}(\log\frac{s}{\tau})^{\beta-1}\frac{f(\tau,u(\tau))}{\tau}d\tau\Big)\frac{ds}{s}\\
&+\frac{\lambda(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha+\sigma)}\int_1^{\eta}(\log\frac{\eta}{s})^{\alpha+\sigma-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^{s}(\log\frac{s}{\tau})^{\beta-1}\frac{f(\tau,u(\tau))}{\tau}d\tau\Big)\frac{ds}{s}\\
&-\frac{(\log t)^{\alpha-1}}{\Omega\Gamma(\alpha)}\int_1^{T}(\log\frac{T}{s})^{\alpha-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^{s}(\log\frac{s}{\tau})^{\beta-1}\frac{f(\tau,u(\tau))}{\tau}d\tau\Big)\frac{ds}{s}.\end{aligned}$$

Obviously, the problem (1.1) has a solution if and only if the operator G has a fixed point.
Now, we are in central position of the paper.

Theorem 3.2. *Assume that:*

(A_1) *There exist nonnegative functions* $a(t), b(t) \in C[1,T]$ *such that*

$$|f(t,u)| \leq a(t) + b(t)|u(t)|^{p-1}, \quad \forall t \in [1,T], \quad u \in \mathbb{R}.$$

(A_2) *The following inequality holds*

$$\Lambda \|b\|_\infty^{q-1} < 1.$$

Then, the problem (1.1) *has at least one solution.*

Proof. We show, as a first step, that the operator G is completely continuous. Clearly, continuity of the operator G follows from the continuity of f. Let $\Omega \subset C[1,T]$ be an open bounded subset, we can get that $G(\overline{\Omega})$ is bounded. Moreover, there exists a constant $M > 0$ such that $|{}^H I^\beta f(t,u(t))| \leq M$, for all $u \in \overline{\Omega}, t \in [1,T]$. Next, we show that the operator G is equicontinuous. For $1 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
&|Gu(t_2) - Gu(t_1)| \\
&\quad = \Big| \frac{1}{\Gamma(\alpha)} \int_1^{t_2} (\log \frac{t_2}{s})^{\alpha-1} \phi_q\Big(\frac{1}{\Gamma(\beta)} \int_1^s (\log \frac{s}{\tau})^{\beta-1} \frac{f(\tau,u(\tau))}{\tau} d\tau \Big) \frac{ds}{s} \\
&\qquad - \frac{1}{\Gamma(\alpha)} \int_1^{t_1} (\log \frac{t_1}{s})^{\alpha-1} \phi_q\Big(\frac{1}{\Gamma(\beta)} \int_1^s (\log \frac{s}{\tau})^{\beta-1} \frac{f(\tau,u(\tau))}{\tau} d\tau \Big) \frac{ds}{s} \\
&\qquad + \Big[\frac{1}{\Omega} \Big[\frac{\lambda}{\Gamma(\alpha+\sigma)} \int_1^{\eta} (\log \frac{\eta}{s})^{\alpha+\sigma-1} \phi_q\Big(\frac{1}{\Gamma(\beta)} \int_1^s (\log \frac{s}{\tau})^{\beta-1} \frac{f(\tau,u(\tau))}{\tau} d\tau \Big) \frac{ds}{s} \\
&\qquad - \frac{1}{\Gamma(\alpha)} \int_1^{T} (\log \frac{T}{s})^{\alpha-1} \phi_q\Big(\frac{1}{\Gamma(\beta)} \int_1^s (\log \frac{s}{t})^{s-1} \frac{f(\tau,u(\tau))}{\tau} d\tau \Big) \frac{ds}{s} \Big] [(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}] \Big| \\
&\quad \leq \frac{M^{q-1}}{\Gamma(\alpha)} \Big| \int_1^{t_1} \Big[(\log \frac{t_2}{s})^{\alpha-1} - (\log \frac{t_1}{s})^{\alpha-1} \Big] \frac{ds}{s} + \int_{t_1}^{t_2} (\log \frac{t_2}{s})^{\alpha-1} \frac{ds}{s} \Big| \\
&\qquad + \Big| \frac{1}{\Omega} \Big[\frac{\lambda}{\Gamma(\alpha+\sigma)} \int_1^{\eta} (\log \frac{\eta}{s})^{\alpha+\sigma-1} \phi_q\Big(\frac{1}{\Gamma(\beta)} \int_1^s (\log \frac{s}{\tau})^{\beta-1} \frac{f(\tau,u(\tau))}{\tau} d\tau \Big) \frac{ds}{s} \\
&\qquad - \frac{1}{\Gamma(\alpha)} \int_1^{T} (\log \frac{T}{s})^{\alpha-1} \phi_q\Big(\frac{1}{\Gamma(\beta)} \int_1^s (\log \frac{s}{t})^{s-1} \frac{f(\tau,u(\tau))}{\tau} d\tau \Big) \frac{ds}{s} \Big] \Big| [(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}] \\
&\quad \leq \frac{M^{q-1}}{\Gamma(\alpha+1)} [\log t_2)^{\alpha} - (\log t_1)^{\alpha}] \\
&\qquad + \frac{1}{|\Omega|} \Big[\Big| \frac{\lambda}{\Gamma(\alpha+\sigma)} \int_1^{\eta} (\log \frac{\eta}{s})^{\alpha+\sigma-1} \phi_q\Big(\frac{1}{\Gamma(\beta)} \int_1^s (\log \frac{s}{\tau})^{\beta-1} \frac{f(\tau,u(\tau))}{\tau} d\tau \Big) \frac{ds}{s} \Big| \\
&\qquad + \Big| \frac{1}{\Gamma(\alpha)} \int_1^{T} (\log \frac{T}{s})^{\alpha-1} \phi_q\Big(\frac{1}{\Gamma(\beta)} \int_1^s (\log \frac{s}{t})^{s-1} \frac{f(\tau,u(\tau))}{\tau} d\tau \Big) \frac{ds}{s} \Big| \Big] [(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}]
\end{aligned}$$

$$\leq \frac{M^{q-1}}{\Gamma(\alpha+1)}[(\log t_2)^{\alpha} - (\log t_1)^{\alpha}] + \frac{M^{q-1}}{|\Omega|}\Big[\frac{|\lambda|(\log\eta)^{\alpha+\sigma}}{\Gamma(\alpha+\sigma+1)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}\Big][(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}].$$

Thus, by the Arzela-Ascoli theorem, the operator $G : C[1,T] \to C[1,T]$ is completely continuous.

Next we consider the set $V = \{u \in C[1,T] | u = \mu Gu, \mu \in (0,1)\}$ and show that the set V is bounded. By (A_1), we have

$$\begin{aligned}
|u(t)| =& |\mu(Gu)(t)| \\
\leq& \frac{1}{\Gamma(\alpha)}\int_1^t (\log\frac{t}{s})^{\alpha-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^s(\log\frac{s}{\tau})^{\beta-1}\frac{a(\tau)+b(\tau)|u(\tau)|^{p-1}}{\tau}d\tau\Big)\frac{ds}{s} \\
&+ \frac{|\lambda|(\log t)^{\alpha-1}}{|\Omega|\Gamma(\alpha+\sigma)}\int_1^\eta(\log\frac{\eta}{s})^{\alpha+\sigma-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^s(\log\frac{s}{\tau})^{\beta-1}\frac{a(\tau)+b(\tau)|u(\tau)|^{p-1}}{\tau}d\tau\Big)\frac{ds}{s} \\
&+ \frac{(\log t)^{\alpha-1}}{|\Omega|\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi_q\Big(\frac{1}{\Gamma(\beta)}\int_1^s(\log\frac{s}{\tau})^{\beta-1}\frac{a(\tau)+b(\tau)|u(\tau)|^{p-1}}{\tau}d\tau\Big)\frac{ds}{s} \\
\leq& \frac{1}{\Gamma(\alpha)}\int_1^t(\log\frac{t}{s})^{\alpha-1}\phi_q\Big(\frac{\|a\|_\infty+\|b\|_\infty\|u\|_\infty^{p-1}}{\Gamma(\beta+1)}(\log s)^{\beta}\Big)\frac{ds}{s} \\
&+ \frac{|\lambda|(\log t)^{\alpha-1}}{|\Omega|\Gamma(\alpha+\sigma)}\int_1^\eta(\log\frac{\eta}{s})^{\alpha+\sigma-1}\phi_q\Big(\frac{\|a\|_\infty+\|b\|_\infty\|u\|_\infty^{p-1}}{\Gamma(\beta+1)}(\log s)^{\beta}\Big)\frac{ds}{s} \\
&+ \frac{(\log t)^{\alpha-1}}{|\Omega|\Gamma(\alpha)}\int_1^T(\log\frac{T}{s})^{\alpha-1}\phi_q\Big(\frac{\|a\|_\infty+\|b\|_\infty\|u\|_\infty^{p-1}}{\Gamma(\beta+1)}(\log s)^{\beta}\Big)\frac{ds}{s} \\
=& \Big(\frac{\|a\|_\infty+\|b\|_\infty\|u\|_\infty^{p-1}}{\Gamma(\beta+1)}\Big)^{q-1}[{}^HI^{\alpha}(\log t)^{\beta(q-1)} + \frac{|\lambda|}{|\Omega|}{}^HI^{\alpha+\sigma}(\log\eta)^{\beta(q-1)}(\log t)^{\alpha-1} \\
&+ \frac{1}{|\Omega|}{}^HI^{\alpha}(\log T)^{\beta(q-1)}(\log t)^{\alpha-1}] \\
\leq& \frac{(\|a\|_\infty+\|b\|_\infty\|u\|_\infty^{p-1})^{q-1}}{(\Gamma(\beta+1))^{q-1}}\Big[\frac{\Gamma(\beta(q-1)+1)}{\Gamma(\alpha+\beta(q-1)+1)}(\log t)^{\alpha+\beta(q-1)} \\
&+ \frac{|\lambda|\Gamma(\beta(q-1)+1)}{|\Omega|\Gamma(\alpha+\beta(q-1)+\sigma+1)}(\log\eta)^{\alpha+\beta(q-1)+\sigma}(\log t)^{\alpha-1} \\
&+ \frac{\Gamma(\beta(q-1)+1)}{|\Omega|\Gamma(\alpha+\beta(q-1)+1)}(\log T)^{\alpha+\beta(q-1)}(\log t)^{\alpha-1}\Big] \\
\leq& \frac{(\|a\|_\infty+\|b\|_\infty\|u\|_\infty^{p-1})^{q-1}}{(\Gamma(\beta+1))^{q-1}}\Big[\frac{\Gamma(\beta(q-1)+1)}{\Gamma(\alpha+\beta(q-1)+1)}(\log T)^{\alpha+\beta(q-1)} \\
&+ \frac{|\lambda|\Gamma(\beta(q-1)+1)}{|\Omega|\Gamma(\alpha+\beta(q-1)+\sigma+1)}(\log T)^{2\alpha+\beta(q-1)+\sigma-1} \\
&+ \frac{\Gamma(\beta(q-1)+1)}{|\Omega|\Gamma(\alpha+\beta(q-1)+1)}(\log T)^{2\alpha+\beta(q-1)-1}\Big]
\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{(\|a\|_\infty+\|b\|_\infty\|u\|_\infty^{p-1})^{q-1}\Gamma(\beta(q-1)+1)(\log T)^{\alpha+\beta(q-1)}}{(\Gamma(\beta+1))^{q-1}}\Big[\frac{1}{\Gamma(\alpha+\beta(q-1)+1)} \\
& + \frac{(\log T)^{\alpha-1}}{|\Omega|}\Big(\frac{|\lambda|(\log T)^{\sigma}}{\Gamma(\alpha+\beta(q-1)+\sigma+1)}+\frac{1}{\Gamma(\alpha+\beta(q-1)+1)}\Big)\Big] \\
= & (\|a\|_\infty+\|b\|_\infty\|u\|_\infty^{p-1})^{q-1}\Lambda \\
\leq & \Lambda\|a\|_\infty^{q-1}+\Lambda\|b\|_\infty^{q-1}\|u\|_\infty,
\end{aligned}
$$

which implies that there exists a constant $N > 0$ such that $\|u\|_\infty \leq N$. So the set V is bounded. Thus, by the conclusion of Theorem 2.5, the operator G has at least one fixed point, which implies that nonlinear Hadamard fractional differential equation (1.1) has at least one solution. □

Remark 3.3. Assume that (A_1) holds, for the special case where $T = e$, we get the problem (1.1) has at least one solution, provided that

$$
\begin{aligned}
& \frac{\|b\|_\infty^{q-1}\Gamma(\beta(q-1)+1)}{(\Gamma(\beta+1))^{q-1}}\Big[\frac{1}{\Gamma(\alpha+\beta(q-1)+1)}+\frac{\lambda}{|\Omega|\Gamma(\alpha+\beta(q-1)+\sigma+1)} \\
& \quad + \frac{1}{|\Omega|\Gamma(\alpha+\beta(q-1)+1)}\Big] < 1,
\end{aligned}
$$

where

$$
\Omega = 1-\frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\sigma)}(\log\eta)^{\alpha+\sigma-1}.
$$

4. Examples

Example 4.1. We consider the following Hadamard fractional boundary value problem:

$$
\begin{cases}
{}^HD^{\frac{1}{2}}\phi_5({}^HD^{\frac{3}{2}}u(t)) = \frac{1}{100}e^{-t^2}u^4(t)+\arctan(1+t), & t\in(1,e), \\
u(e)=\frac{1}{2}\,{}^HI^{\frac{1}{2}}u(e), \quad {}^HD^{\frac{3}{2}}u(1)=0, \quad u(1)=0, &
\end{cases}
\tag{4.1}
$$

where $p = 5, q = \frac{5}{4}, \alpha = \frac{3}{2}, \beta = \sigma = \lambda = \frac{1}{2}, \eta = T = e$. Clearly,

$$
f(t,u(t)) = \frac{1}{100}e^{-t^2}u^4(t)+\arctan(1+t) \leq \frac{\pi}{2}+\frac{|u|^4}{100e}.
$$

Further,

$$
\Lambda\|b\|_\infty^{q-1} \leq \frac{\Gamma(\frac{1}{8}+1)}{(100\Gamma(\frac{3}{2}))^{\frac{1}{4}}}\Big[\frac{1}{\Gamma(\frac{21}{8})}+\frac{1}{(2-\frac{\sqrt{\pi}}{2})\Gamma(\frac{25}{8})}+\frac{1}{(1-\frac{\sqrt{\pi}}{4})\Gamma(\frac{21}{8})}\Big] \approx 0.7114 < 1.
$$

Thus, all hypotheses of Theorem 3.2 hold. Therefore, the conclusion of Theorem 3.2 implies that the Hadamard fractional integral boundary value problem (4.1) has at least one solution.

Example 4.2. We consider the following Hadamard fractional boundary value problem:

$$
\begin{cases}
{}^HD^{\frac{3}{4}}\phi_2({}^HD^{\frac{5}{4}}u(t)) = \frac{1}{1+t}\sin|u(t)|+\frac{1}{1+t^2}, & t\in(1,e), \\
u(e)=\frac{1}{4}\,{}^HI^{\frac{1}{4}}u(e), \quad {}^HD^{\frac{5}{4}}u(1)=0, \quad u(1)=0. &
\end{cases}
\tag{4.2}
$$

We see that $p = q = 2, \alpha = \frac{5}{4}, \beta = \frac{3}{4}, \sigma = \lambda = \frac{1}{4}, \eta = T = e$, and

$$f(t, u(t)) = \frac{1}{1+t} \sin |u(t)| + \frac{1}{1+t^2}.$$

Choose $a(t) = \frac{1}{2}, b(t) = \frac{1}{1+t}$, then

$$\Lambda \|b\|_\infty^{q-1} = \frac{1}{2}\Big[\frac{1}{\Gamma(3)} + \frac{1}{(4 - \frac{\Gamma(\frac{1}{4})}{2\sqrt{\pi}})\Gamma(\frac{13}{4})} + \frac{1}{(1 - \frac{\Gamma(\frac{1}{4})}{8\sqrt{\pi}})\Gamma(3)}\Big] \approx 0.6518 < 1.$$

It can easily be verified that all assumptions of Theorem 3.2 hold. Therefore, the conclusion of Theorem 3.2 is applied to nonlocal Hadamard fractional integral boundary value problem (4.2).

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