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Research Article

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On a nonlinear system of Riemann-Liouville fractional differential equations with semi-coupled integro-multipoint boundary conditions

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Abstract: We study a nonlinear system of Riemann-Liouville fractional differential equations equipped with nonseparated semi-coupled integro-multipoint boundary conditions. We make use of the tools of the fixed-point theory to obtain the desired results, which are well-supported with numerical examples.

Keywords: Riemann-Liouville fractional derivative, integro-differential inclusions, nonlocal multi-point boundary conditions, existence, fixed point theorems

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1 Introduction

Nonlinear boundary value problems involving Riemann-Liouville fractional derivatives have been studied by many researchers, for example, see [1–6]. In a recent article [7], the authors discussed the existence and Ulam-type stability for nonlinear Riemann-Liouville fractional differential equations with constant delay.

The nonlocal nature of fractional derivative operators significantly contributed to the popularity of fractional calculus. Nowadays, one can find extensive application of fractional-order operators in the mathematical models of several real-world phenomena occurring in physical and applied sciences, such as continuum mechanics [8], bioengineering [9], financial economics [10], fractals [11], etc.

The topic of fractional differential systems also received considerable attention in view of their applications in diverse fields such as anomalous diffusion [12], disease models [13] biological models [14], hybrid systems [15], rheological models [16], diffusion systems [17], ecological models [18], etc. For theoretical details of such systems, see [19–29].

In this paper, we investigate the existence of solutions for a system of nonlinear Riemann-Liouville fractional differential equations

$$\begin{cases} D^{\alpha}u(t) = F(t, u(t), v(t)), & 1 < \alpha \le 2, \ t \in [0, T], \\ D^{\beta}v(t) = G(t, u(t), v(t)), & 1 < \beta \le 2, \ t \in [0, T], \end{cases} \tag{1}$$

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equipped with nonlocal semi-coupled fractional integro-multipoint boundary conditions of the form:

$$\begin{cases} D^{\alpha-2}u(0^+) + a_0 D^{\alpha-2}u(T^-) = \lambda_1, \\ D^{\alpha-1}u(0^+) + a_1 D^{\alpha-1}u(T^-) = \nu I^{\alpha-1}v(\eta_1) + \sum_{i=1}^{m}\mu_i v(\xi_i), \\ D^{\beta-2}v(0^+) + b_0 D^{\beta-2}v(T^-) = \lambda_2, \\ D^{\beta-1}v(0^+) + b_1 D^{\beta-1}v(T^-) = \mu I^{\beta-1}u(\eta_2) + \sum_{j=1}^{n}\sigma_j u(\zeta_j), \end{cases} \tag{2}$$

where D^{χ} is the Riemann-Liouville fractional derivative of order $\chi \in \{\alpha, \beta\}$, $0 < \eta_1 < \eta_2 < \xi_1 < \xi_2 < \ldots < \xi_m < \zeta_1 < \zeta_2 < \ldots < \zeta_m < T$, $a_0, a_1, b_0, b_1, \nu, \mu, \lambda_1, \lambda_2, \mu_i$ $(i = 1, 2, \ldots, m)$, σ_j $(j = 1, 2, \ldots, n)$ are real constants and $a_0 \neq -1$, $a_1 \neq -1$, $b_0 \neq -1$, and $F, G : [0, T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.

Here we emphasize that our results are new with respect to the semi-coupled boundary conditions (2) and enrich the related literature on the topic.

In Section 2, we prove an auxiliary lemma dealing with the linear variant of systems (1)–(2), which plays a fundamental role in establishing the existence and uniqueness results for the given nonlinear problem, presented in Section 3. Illustrative examples demonstrating the application of the obtained results are given in Section 4.

2 Preliminaries

Let us begin this section with some related definitions [30,31].

Definition 2.1. The (left) Riemann-Liouville fractional integral of order $\chi > 0$ for $\varrho \in L_1[a, b]$, $-\infty < a \leq t \leq b < +\infty$, existing almost everywhere on $[a, b]$, is defined as

$$(I^{\chi}_{a+}\varrho)(t) = \frac{1}{\Gamma(\chi)}\int_a^t (t-s)^{\chi-1}\varrho(s)\mathrm{d}s, \quad \chi > 0,$$

where $\Gamma(\cdot)$ is the gamma function and $(I^0_{a+}\varrho)(x) = \varrho(x)$.

Definition 2.2. For $\varrho, \varrho^{(m)} \in L^1[a, b]$, the (left) Riemann-Liouville fractional derivative $D^{\chi}_{a^+}\varrho$ of order $\chi \in (m-1, m]$, $m \in \mathbb{N}$ is defined as

$$D^{\chi}_{a^+}\varrho(t) = \frac{1}{\Gamma(m-\chi)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\int_a^t (t-s)^{m-1-\chi}\varrho(s)\mathrm{d}s, \quad -\infty \leq a < t < b \leq +\infty.$$

In the sequel, we will write Riemann-Liouville fractional integral and derivative operators as I^{χ} and D^{χ} instead of I^{χ}_{a+} and D^{χ}_{a+}, respectively.

Lemma 2.1. *Let $\psi_1, \psi_2 \in C[0, T] \cap L[0, T]$ and $\Lambda \neq 0$. Then the unique solution of the following linear system*

$$\begin{cases} D^{\alpha}u(t) = \psi_1(t), & 1 < \alpha \leq 2, \ t \in [0, T], \\ D^{\beta}v(t) = \psi_2(t), & 1 < \beta \leq 2, \ t \in [0, T], \end{cases} \tag{3}$$

subject to the boundary conditions (2) *is given by*

$$\begin{aligned} u(t) = {} & \lambda_1\rho_1(t) + \lambda_2\omega_1\rho_2(t) - a_1\rho_2(t)I^1\psi_1(T) - b_1\rho_3(t)I^1\psi_2(T) - a_0\rho_1(t)I^2\psi_1(T) - b_0\omega_1\rho_2(t)I^2\psi_2(T) \\ & + \mu\rho_3(t)I^{\alpha+\beta-1}\psi_1(\eta_2) + \nu\rho_2(t)I^{\alpha+\beta-1}\psi_2(\eta_1) + \rho_3(t)\sum_{j=1}^{n}\sigma_j I^{\alpha}\psi_1(\zeta_j) + \rho_2(t)\sum_{i=1}^{m}\mu_i I^{\beta}\psi_2(\xi_i) + I^{\alpha}\psi_1(t), \end{aligned} \tag{4}$$

$$v(t) = \lambda_2\rho_1^*(t) + \lambda_1\nu_1\rho_2^*(t) - a_1\rho_3^*(t)I^1\psi_1(T) - b_1\rho_2^*(t)I^1\psi_2(T) - a_0\nu_1\rho_2^*(t)I^2\psi_1(T) - b_0\rho_1^*(t)I^2\psi_2(T) + \mu\rho_2^*(t)I^{\alpha+\beta-1}\psi_1(\eta_2) + \nu\rho_3^*(t)I^{\alpha+\beta-1}\psi_2(\eta_1) + \rho_2^*(t)\sum_{j=1}^{n}\sigma_j I^\alpha\psi_1(\zeta_j) + \rho_3^*(t)\sum_{i=1}^{m}\mu_i I^\beta\psi_2(\xi_i) + I^\beta\psi_2(t), \tag{5}$$

where

$$\begin{aligned}
\rho_1(t) &= t^{\alpha-2}\left[\frac{1}{(1+a_0)\Gamma(\alpha-1)} + \frac{\nu_1}{\Lambda}\frac{a_0T\Gamma(\alpha)}{(1+a_0)\Gamma(\alpha-1)}\left(\omega_2 - \frac{b_0T\Gamma(\beta)}{(1+a_1)\Gamma(\alpha)}\omega_1\right)\right] - t^{\alpha-1}\frac{\nu_1}{\Lambda}\left[\omega_2 - \frac{b_0T\Gamma(\beta)}{(1+a_1)\Gamma(\alpha)}\omega_1\right],\\
\rho_2(t) &= \frac{1}{\Lambda}\left[t^{\alpha-2}\frac{a_0T(b_1+1)\Gamma(\beta)}{(1+a_1)(1+a_0)\Gamma(\alpha-1)} - t^{\alpha-1}\frac{(1+b_1)\Gamma(\beta)}{(1+a_1)\Gamma(\alpha)}\right],\\
\rho_3(t) &= \frac{1}{\Lambda}\left[\omega_2 - \frac{b_0T\Gamma(\beta)}{(1+a_1)\Gamma(\alpha)}\omega_1\right]\left[t^{\alpha-2}\frac{a_0T\Gamma(\alpha)}{(1+a_0)\Gamma(\alpha-1)} - t^{\alpha-1}\right],\\
\rho_1^*(t) &= t^{\beta-2}\left[\frac{1}{(1+b_0)\Gamma(\beta-1)} + \frac{\omega_1}{\Lambda}\frac{b_0T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)}\left(\nu_2 - \frac{a_0T}{(1+a_1)}\nu_1\right)\right] - t^{\beta-1}\frac{\omega_1}{\Lambda}\left[\nu_2 - \frac{a_0T}{(1+a_1)}\nu_1\right],\\
\rho_2^*(t) &= \frac{1}{\Lambda}\left[t^{\beta-2}\frac{b_0T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)} - t^{\beta-1}\right],\\
\rho_3^*(t) &= \frac{1}{\Lambda}\left[\nu_2 - \frac{a_0T}{(1+a_1)}\nu_1\right]\left[t^{\beta-2}\frac{b_0T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)} - t^{\beta-1}\right],\\
\Lambda &= -\Gamma(\beta)(1+b_1) + \left[\nu_2 - \frac{a_0T}{(1+a_1)}\nu_1\right][\Gamma(\alpha)(1+a_1)\omega_2 - b_0T\Gamma(\beta)\omega_1],\\
\nu_1 &= \frac{1}{(1+a_0)\Gamma(\alpha-1)}\left[\mu\eta_2^{\alpha+\beta-3}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+\beta-2)} + \sum_{j=1}^{n}\sigma_j\zeta_j^{\alpha-2}\right],\\
\nu_2 &= \frac{1}{(1+a_1)\Gamma(\alpha)}\left[\mu\eta_2^{\alpha+\beta-2}\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta-1)} + \sum_{j=1}^{n}\sigma_j\zeta_j^{\alpha-1}\right],\\
\omega_1 &= \frac{1}{(1+b_0)\Gamma(\beta-1)}\left[\nu\eta_1^{\alpha+\beta-3}\frac{\Gamma(\beta-1)}{\Gamma(\alpha+\beta-2)} + \sum_{i=1}^{m}\mu_i\xi_i^{\beta-2}\right],\\
\omega_2 &= \frac{1}{(1+a_1)\Gamma(\alpha)}\left[\nu\eta_1^{\alpha+\beta-2}\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta-1)} + \sum_{i=1}^{m}\mu_i\xi_i^{\beta-1}\right].
\end{aligned} \tag{6}$$

Proof. It is well known that the solutions of fractional differential equations in (3) can be written as

$$u(t) = I^\alpha\psi_1(t) + c_0t^{\alpha-2} + c_1t^{\alpha-1}, \tag{7}$$

$$v(t) = I^\beta\psi_2(t) + c_2t^{\beta-2} + c_3t^{\beta-1}, \tag{8}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, 3$ are unknown arbitrary constants. From (7) and (8), we have

$$D^{\alpha-2}u(t) = c_0\Gamma(\alpha-1) + c_1\Gamma(\alpha)t + I^2\psi_1(t), \tag{9}$$

$$D^{\beta-2}v(t) = c_2\Gamma(\beta-1) + c_3\Gamma(\beta)t + I^2\psi_2(t), \tag{10}$$

$$D^{\alpha-1}u(t) = c_1\Gamma(\alpha) + I^1\psi_1(t), \tag{11}$$

$$D^{\beta-1}v(t) = c_3\Gamma(\beta) + I^1\psi_2(t). \tag{12}$$

Using (9) and (10) in (2), we get

$$(1+a_0)\Gamma(\alpha-1)c_0 + a_0T\Gamma(\alpha)c_1 = \lambda_1 - a_0I^2\psi_1(T), \tag{13}$$

$$(1+b_0)\Gamma(\beta-1)c_2 + b_0T\Gamma(\beta)c_3 = \lambda_2 - b_0I^2\psi_2(T). \tag{14}$$

Combining (11) and (12) with (2), we obtain

$$\begin{aligned}
&\Gamma(\alpha)(1+a_1)c_1-\left[\frac{\nu\eta_1^{\alpha+\beta-2}\Gamma(\beta)}{\Gamma(\alpha+\beta-1)}+\sum_{i=1}^{m}\mu_i\xi_i^{\beta-1}-\frac{b_0T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)}\left(\nu\eta_1^{\alpha+\beta-3}\frac{\Gamma(\beta-1)}{\Gamma(\alpha+\beta-2)}+\sum_{i=1}^{m}\mu_i\xi_i^{\beta-2}\right)\right]c_3\\
&\quad=-\frac{b_0}{(1+b_0)\Gamma(\beta-1)}\left[\nu\eta_1^{\alpha+\beta-3}\frac{\Gamma(\beta-1)}{\Gamma(\alpha+\beta-2)}+\sum_{i=1}^{m}\mu_i\xi_i^{\beta-2}\right]I^2\psi_2(t)+\sum_{i=1}^{m}\mu_iI^{\beta}\psi_2(\xi_i)-a_1I^1\psi_1(T)\\
&\qquad+\frac{\lambda_2}{(1+b_0)\Gamma(\beta-1)}\left[\nu\eta_1^{\alpha+\beta-3}\frac{\Gamma(\beta-1)}{\Gamma(\alpha+\beta-2)}+\sum_{i=1}^{m}\mu_i\xi_i^{\beta-2}\right]+\nu I^{\alpha+\beta-1}\psi_2(\eta_1),
\end{aligned}\tag{15}$$

$$\begin{aligned}
&-\left[\mu\eta_2^{\alpha+\beta-2}\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta-1)}+\sum_{j=1}^{n}\sigma_j\zeta_j^{\alpha-1}\right]c_1+\Gamma(\beta)(1+b_1)c_3\\
&\quad=\sum_{j=1}^{n}\sigma_jI^{\alpha}\psi_1(\zeta_j)-b_1I^1\psi_2(T)+\frac{1}{(1+a_0)\Gamma(\alpha-1)}\left[\mu\eta_2^{\alpha+\beta-3}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha+\beta-2)}+\sum_{j=1}^{n}\sigma_j\zeta_j^{\alpha-2}\right]\\
&\qquad\times[\lambda_1-a_0c_1T\Gamma(\alpha)-a_0I^2\psi_1(t)]+\mu I^{\alpha+\beta-1}\psi_1(\eta_2).
\end{aligned}\tag{16}$$

Solving (15) and (16) for c_1 and c_3, we find that

$$\begin{aligned}
c_1=&\left[\frac{1}{\Lambda}\frac{(b_1+1)\Gamma(\beta)}{(1+a_1)\Gamma(\alpha)}\right]\left[-\omega_1\lambda_2+b_0\omega_1I^2\psi_2(T)-\nu I^{\alpha+\beta-1}\psi_2(\eta_1)-\sum_{i=1}^{m}\mu_iI^{\beta}\psi_2(\xi_i)+a_1I^1\psi_1(T)\right]\\
&+\frac{1}{\Lambda}\left[\omega_2-\frac{b_0T\Gamma(\beta)}{(1+a_1)\Gamma(\alpha)}\omega_1\right]\left[-\lambda_1\nu_1-\mu I^{\alpha+\beta-1}\psi_1(\eta_2)-\sum_{j=1}^{n}\sigma_jI^{\alpha}\psi_1(\zeta_j)+b_1I^1\psi_2(T)+a_0\nu_1I^2\psi_1(T)\right],
\end{aligned}\tag{17}$$

$$\begin{aligned}
c_3=\frac{1}{\Lambda}\Bigg[&-\lambda_1\nu_1-\lambda_2\omega_1\left(\nu_2-\frac{a_0T}{(1+a_1)}\nu_1\right)+a_0\nu_1I^2\psi_1(t)+b_0\omega_1\left(\nu_2-\frac{a_0T}{(1+a_1)}\nu_1\right)I^2\psi_2(T)\\
&-\nu\left(\nu_2-\frac{a_0T}{(1+a_1)}\nu_1\right)I^{\alpha+\beta-1}\psi_2(\eta_1)-\mu I^{\alpha+\beta-1}\psi_1(\eta_2)-\left(\nu_2-\frac{a_0T}{(1+a_1)}\nu_1\right)\sum_{i=1}^{m}\mu_iI^{\beta}\psi_2(\xi_i)\\
&-\sum_{j=1}^{n}\sigma_jI^{\alpha}\psi_1(\zeta_j)+a_1\left(\nu_2-\frac{a_0T}{(1+a_1)}\nu_1\right)I^1\psi_1(T)+b_1I^1\psi_2(T)\Bigg].
\end{aligned}\tag{18}$$

Substituting the values of c_1 and c_3 in (13) and (14), respectively, we get

$$\begin{aligned}
c_0=&[\lambda_1-a_0I^2\psi_1(T)]\left[\frac{1}{(1+a_0)\Gamma(\alpha-1)}+\frac{\nu_1}{\Lambda}\frac{a_0T\Gamma(\alpha)}{(1+a_0)\Gamma(\alpha-1)}\left(\omega_2-\frac{b_0T\Gamma(\beta)}{(1+a_1)\Gamma(\alpha)}\omega_1\right)\right]\\
&+\lambda_2\left[\frac{\omega_1}{\Lambda}\frac{a_0(1+b_1)T|\Gamma(\beta)}{(1+a_0)(1+a_1)\Gamma(\alpha-1)}\right]-I^2\psi_2(T)\left[\frac{\omega_1}{\Lambda}\frac{a_0b_0(1+b_1)T\Gamma(\beta)}{(1+a_0)(1+a_1)\Gamma(\alpha-1)}\right]\\
&+I^{\alpha+\beta-1}\psi_2(\eta_1)\left[\frac{\nu}{\Lambda}\frac{a_0(1+b_1)T\Gamma(\beta)}{(1+a_0)(1+a_1)\Gamma(\alpha-1)}\right]+\sum_{i=1}^{m}\mu_iI^{\beta}\psi_2(\xi_i)\left[\frac{1}{\Lambda}\frac{a_0(1+b_1)T\Gamma(\beta)}{(1+a_0)(1+a_1)\Gamma(\alpha-1)}\right]\\
&-I^1\psi_1(T)\left[\frac{1}{\Lambda}\frac{a_0a_1(1+b_1)T\Gamma(\beta)}{(1+a_0)(1+a_1)\Gamma(\alpha-1)}\right]+I^{\alpha+\beta-1}\psi_1(\eta_2)\left[\frac{\mu}{\Lambda}\frac{a_0T\Gamma(\alpha)}{(1+a_0)\Gamma(\alpha-1)}\left(\omega_2-\frac{b_0T\Gamma(\beta)}{(1+a_1)\Gamma(\alpha)}\omega_1\right)\right]\\
&+\sum_{j=1}^{n}\sigma_jI^{\alpha}\psi_1(\zeta_j)\left[\frac{1}{\Lambda}\frac{a_0T\Gamma(\alpha)}{(1+a_0)\Gamma(\alpha-1)}\left(\omega_2-\frac{b_0T\Gamma(\beta)}{(1+a_1)\Gamma(\alpha)}\omega_1\right)\right]\\
&+I^1\psi_2(T)\left[\frac{-b_1}{\Lambda}\frac{a_0T\Gamma(\alpha)}{(1+a_0)\Gamma(\alpha-1)}\left(\omega_2-\frac{b_0T\Gamma(\beta)}{(1+a_1)\Gamma(\alpha)}\omega_1\right)\right],
\end{aligned}$$

$$
\begin{aligned}
c_2 = {} & [\lambda_2 - b_0 I^2\psi_2(T)]\left[\frac{1}{(1+b_0)\Gamma(\beta-1)} + \frac{\omega_1}{\Lambda}\frac{b_0 T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)}\left(\nu_2 - \frac{a_0 T}{(1+b_1)}\nu_1\right)\right] \\
& + \lambda_1\left[\frac{\nu_1}{\Lambda}\frac{b_0 T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)}\right] - I^2\psi_1(T)\left[\frac{\nu_1}{\Lambda}\frac{a_0 b_0 T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)}\right] \\
& + I^{\alpha+\beta-1}\psi_2(\eta_1)\left[\frac{\nu}{\Lambda}\frac{b_0 T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)}\left(\nu_2 - \frac{a_0 T}{(1+a_1)}\nu_1\right)\right] \\
& + I^{\alpha+\beta-1}\psi_1(\eta_2)\left[\frac{\mu}{\Lambda}\frac{b_0 T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)}\right] + \sum_{i=1}^{m}\mu_i I^{\beta}\psi_2(\xi_i)\left[\frac{1}{\Lambda}\frac{b_0 T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)}\left(\nu_2 - \frac{a_0 T}{(1+a_1)}\nu_1\right)\right] \\
& - I^1\psi_1(T)\left[\frac{1}{\Lambda}\frac{b_0 a_1 T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)}\left(\nu_2 - \frac{a_0 T}{(1+a_1)}\nu_1\right)\right] \\
& + \sum_{j=1}^{n}\sigma_j I^{\alpha}\psi_1(\zeta_j)\left[\frac{1}{\Lambda}\frac{b_0 T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)}\right] - I^1\psi_2(T)\left[\frac{1}{\Lambda}\frac{b_0 b_1 T\Gamma(\beta)}{(1+b_0)\Gamma(\beta-1)}\right].
\end{aligned}
$$

Inserting the value of c_0, c_1, c_2, and c_3 in (7) and (8) together with the notations (6) leads to the solutions (4) and (5). The converse of this lemma follows by direct computation. The proof is completed. □

3 Main results

Let $C([0,T],\mathbb{R})$ denote the Banach space of all continuous real-valued functions defined on $[0,T]$ with norm $\|u\| = \sup_{t\in[0,T]}|u(t)|$. For $t \in [0,T]$, let $C_r([0,T],\mathbb{R})$ denote the space of all functions u_r such that $u_r \in C([0,T],\mathbb{R})$, which is a Banach space endowed with norm $\|u\|_r = \sup_{t\in[0,T]}\{t^r|u(t)|\}$. Let $X = \{u : u \in C_{2-\alpha}([0,T],\mathbb{R})\}$ and $Y = \{v : v \in C_{2-\beta}([0,T],\mathbb{R})\}$ be equipped with the norm $\|u\|_X = \sup_{t\in[0,T]}\{t^{2-\alpha}|u(t)|\}$ and $\|v\|_Y = \sup_{t\in[0,T]}\{t^{2-\beta}|v(t)|\}$, respectively. Then the product space $(X\times Y, \|\cdot\|_{X\times Y})$ is a Banach space with the norm

$$\|(u,v)\|_{X\times Y} = \|u\|_X + \|v\|_Y.$$

Next we introduce an operator $\mathcal{P} : X\times Y \to X\times Y$ by

$$\mathcal{P}(u,v)(t) = (\mathcal{P}_1(u,v)(t), \mathcal{P}_2(u,v)(t)),$$

where

$$
\begin{aligned}
\mathcal{P}_1(u,v)(t) = {} & \lambda_1\rho_1(t) + \lambda_2\omega_1\rho_2(t) - a_1\rho_2(t)I^1F(T,u(T),v(T)) - b_1\rho_3(t)I^1G(T,u(T)v(T)) \\
& - a_0\rho_1(t)I^2F(T,u(T),v(T)) - b_0\omega_1\rho_2(t)I^2G(T,u(T),v(T)) \\
& + \mu\rho_3(t)I^{\alpha+\beta-1}F(\eta_2,u(\eta_2),v(\eta_2)) + \nu\rho_2(t)I^{\alpha+\beta-1}G(\eta_1,u(\eta_1),v(\eta_1)) \\
& + \rho_3(t)\sum_{j=1}^{n}\sigma_j I^{\alpha}F(\zeta_j,u(\zeta_j),v(\zeta_j)) + \rho_2(t)\sum_{i=1}^{m}\mu_i I^{\beta}G(\xi_i,u(\xi_i),v(\xi_i)) \\
& + I^{\alpha}F(t,u(t),v(t)), \quad t\in[0,T],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}_2(u,v)(t) = {} & \lambda_2\rho_1^*(t) + \lambda_1\nu_1\rho_2^*(t) - a_1\rho_3^*(t)I^1F(T,u(T),v(T)) - b_1\rho_2^*(t)I^1G(T,u(T),v(T)) \\
& - a_0\nu_1\rho_2^*(t)I^2F(T,u(T),v(T)) - b_0\rho_1^*(t)I^2G(T,u(T),v(T)) \\
& + \mu\rho_2^*(t)I^{\alpha+\beta-1}F(\eta_2,u(\eta_2),v(\eta_2)) + \nu\rho_3^*(t)I^{\alpha+\beta-1}G(\eta_1,u(\eta_1),v(\eta_1)) \\
& + \rho_2^*(t)\sum_{j=1}^{n}\sigma_j I^{\alpha}F(\zeta_j,u(\zeta_j),v(\zeta_j)) + \rho_3^*(t)\sum_{i=1}^{m}\mu_i I^{\beta}G(\xi_i,u(\xi_i),v(\xi_i)) \\
& + I^{\beta}G(t,u(t),v(t)), \quad t\in[0,T].
\end{aligned}
\tag{19}
$$

For the sake of brevity, we set

$$
\begin{aligned}
N_1 = |\mu|\delta_3\left[\frac{\eta_2^{2\alpha+\beta-3}\Gamma(\alpha-1)}{\Gamma(2\alpha+\beta-2)} + \frac{\eta_2^{\alpha+2\beta-3}\Gamma(\beta-1)}{\Gamma(\alpha+2\beta-2)}\right] + |a_0|\delta_1\left[\frac{T^\alpha\Gamma(\alpha-1)}{\Gamma(\alpha+1)} + \frac{T^\beta\Gamma(\beta-1)}{\Gamma(\beta+1)}\right] \\
+ \delta_3\sum_{j=1}^{n}|\sigma_j|\left[\frac{\zeta_j^{2\alpha-2}\Gamma(\alpha-1)}{\Gamma(2\alpha-1)} + \frac{\zeta_j^{\alpha+\beta-2}\Gamma(\beta-1)}{\Gamma(\alpha+\beta-1)}\right] + |a_1|\delta_2\left[\frac{T^{\alpha-1}}{(\alpha-1)} + \frac{T^{\beta-1}}{(\beta-1)}\right] \\
+ \frac{T^\alpha\Gamma(\alpha-1)}{\Gamma(2\alpha-1)} + \frac{T^\beta\Gamma(\beta-1)}{\Gamma(\alpha+\beta-1)},
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
N_2 = |\nu|\delta_2\left[\frac{\eta_1^{2\alpha+\beta-3}\Gamma(\alpha-1)}{\Gamma(2\alpha+\beta-2)} + \frac{\eta_1^{\alpha+2\beta-3}\Gamma(\beta-1)}{\Gamma(\alpha+2\beta-2)}\right] + |b_0||\omega_1|\delta_2\left[\frac{T^\alpha\Gamma(\alpha-1)}{\Gamma(\alpha+1)} + \frac{T^\beta\Gamma(\beta-1)}{\Gamma(\beta+1)}\right] \\
+ \delta_2\sum_{i=1}^{m}|\mu_i|\left[\frac{\xi_i^{\alpha+\beta-2}\Gamma(\alpha-1)}{\Gamma(\alpha+\beta-1)} + \frac{\xi_i^{2\beta-2}\Gamma(\beta-1)}{\Gamma(2\beta-1)}\right] + |b_1|\delta_3\left[\frac{T^{\alpha-1}}{\alpha-1} + \frac{T^{\beta-1}}{\beta-1}\right],
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
N_1^* = |\mu|\delta_2^*\left[\frac{\eta_2^{2\alpha+\beta-3}\Gamma(\alpha-1)}{\Gamma(2\alpha+\beta-2)} + \frac{\eta_2^{\alpha+2\beta-3}\Gamma(\beta-1)}{\Gamma(\alpha+2\beta-2)}\right] + |a_0\nu_1|\delta_2^*\left[\frac{T^\alpha\Gamma(\alpha-1)}{\Gamma(\alpha+1)} + \frac{T^\beta\Gamma(\beta-1)}{\Gamma(\beta+1)}\right] \\
+ \delta_2^*\sum_{j=1}^{n}|\sigma_j|\left[\frac{\zeta_j^{2\alpha-2}\Gamma(\alpha-1)}{\Gamma(2\alpha-1)} + \frac{\zeta_j^{\alpha+\beta-2}\Gamma(\beta-1)}{\Gamma(\alpha+\beta-1)}\right] + |a_1|\delta_3^*\left[\frac{T^{\alpha-1}}{(\alpha-1)} + \frac{T^{\beta-1}}{(\beta-1)}\right],
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
N_2^* = |\nu|\delta_3^*\left[\frac{\eta_1^{2\alpha+\beta-3}\Gamma(\alpha-1)}{\Gamma(2\alpha+\beta-2)} + \frac{\eta_1^{\alpha+2\beta-3}\Gamma(\beta-1)}{\Gamma(\alpha+2\beta-2)}\right] + |b_0|\delta_1^*\left[\frac{T^\alpha\Gamma(\alpha-1)}{\Gamma(\alpha+1)} + \frac{T^\beta\Gamma(\beta-1)}{\Gamma(\beta+1)}\right] \\
+ \delta_3^*\sum_{i=1}^{m}|\mu_i|\left[\frac{\xi_i^{\alpha+\beta-2}\Gamma(\alpha-1)}{\Gamma(\alpha+\beta-1)} + \frac{\xi_i^{2\beta-2}\Gamma(\beta-1)}{\Gamma(2\beta-1)}\right] + |b_1|\delta_2^*\left[\frac{T^{\alpha-1}}{\alpha-1} + \frac{T^{\beta-1}}{\beta-1}\right] \\
+ \frac{T^\alpha\Gamma(\alpha-1)}{\Gamma(\alpha+\beta-1)} + \frac{T^\beta\Gamma(\beta-1)}{\Gamma(2\beta-1)}.
\end{aligned} \tag{23}
$$

Theorem 3.1. *Assume that*:

(*A1*) $F, G : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist positive constants* L_1 *and* L_2 *such that, for all* $t \in [0, T]$ *and* $u_i, v_i \in \mathbb{R}$, $i = 1, 2$,

$$
|F(t, u_1, v_1) - F(t, u_2, v_2)| \le L_1(|u_1 - u_2| + |v_1 - v_2|),
$$
$$
|G(t, u_1, v_1) - G(t, u_2, v_2)| \le L_2(|u_1 - u_2| + |v_1 - v_2|).
$$

Then the system (1)–(2) *has a unique solution on* $[0, T]$, *provided that*

$$
L_1(N_1 + N_1^*) + L_2(N_2 + N_2^*) < 1, \tag{24}
$$

where, N_1, N_2, N_1^*, *and* N_2^* *are, respectively, given by* (20), (21), (22), *and* (23).

Proof. Define $\sup_{t\in[0,T]}F(t, 0, 0) = M_1$, $\sup_{t\in[0,T]}G(t, 0, 0) = M_2$, and choose $r > 0$ such that

$$
r \ge \frac{|\lambda_1|(\delta_1 + \delta_2^*|\nu_1|) + |\lambda_2|(|\omega_1|\delta_2 + \delta_1^*) + M_1(e_1 + e_1^*) + M_2(e_2 + e_2^*)}{1 - [L_1(N_1 + N_1^*) + L_2(N_2 + N_2^*)]}, \tag{25}
$$

where

$$
\begin{aligned}
e_1 = |a_0|\delta_1\frac{T^2}{2} + |a_1|\delta_2 T + |\mu|\delta_3\frac{\eta_2^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \delta_3\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^\alpha}{\Gamma(\alpha+1)} + \frac{T^2}{\Gamma(\alpha+1)}, \\
e_2 = |b_1|\delta_3 T + |\nu|\delta_2\frac{\eta_1^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \delta_2\sum_{i=1}^{m}|\mu_i|\frac{\xi_i^\beta}{\Gamma(\beta+1)} + |b_0\omega_1|\delta_2\frac{T^2}{2},
\end{aligned} \tag{26}
$$

$$e_1^* = |a_0\nu_1|\delta_2^*\frac{T^2}{2} + |a_1|\delta_3^*T + |\mu|\delta_2^*\frac{\eta_2^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \delta_2^*\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^{\alpha}}{\Gamma(\alpha+1)},$$
$$e_2^* = |b_1|\delta_2^*T + |\nu|\delta_3^*\frac{\eta_1^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \delta_3^*\sum_{i=1}^{m}|\mu_i|\frac{\xi_i^{\beta}}{\Gamma(\beta+1)} + |b_0|\delta_1^*\frac{T^2}{2} + \frac{T^2}{\Gamma(\beta+1)},$$
$$\delta_m = \sup_{t\in[0,T]}\{t^{2-\alpha}|\rho_m(t)|\}, \quad \delta_m^* = \sup_{t\in[0,T]}\{t^{2-\beta}|\rho_m^*(t)|\}, \quad m = 1, 2, 3. \tag{27}$$

In the first step, it will be shown that $\mathcal{P}\mathbb{B}_r \subset \mathbb{B}_r$, where $\mathbb{B}_r = \{(u, v) \in X \times Y : \|(u, v)\|_{X\times Y} \le r\}$. By the assumption (A_1), for $(u, v) \in \mathbb{B}_r$, $t \in [0, T]$, we have

$$|F(t, u(t), v(t))| \le |F(t, u(t), v(t)) - F(t, 0, 0)| + |F(t, 0, 0)| \le L_1(|u| + |v|) + M_1$$

and

$$|G(t, u(t), v(t))| \le |G(t, u(t), v(t)) - G(t, 0, 0)| + |G(t, 0, 0)| \le L_2(|u| + |v|) + M_2.$$

In consequence, by using the relation for beta function $B(\cdot,\cdot)$:

$$B(a, b) = \int_0^1 u^{a-1}(1-u)^{b-1}\mathrm{d}u = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

we obtain

$$\begin{aligned}
\|\mathcal{P}_1(u, v)\|_X = {} & \sup_{t\in[0,T]}\{t^{2-\alpha}|\mathcal{P}_1(u, v)(t)|\} \\
\le {} & \sup_{t\in[0,T]}\Bigg\{t^{2-\alpha}\Bigg[|\lambda_1||\rho_1(t)| + |\lambda_2\omega_1||\rho_2(t)| + |a_0||\rho_1(t)|\int_0^T\frac{(T-s)}{\Gamma(2)}(L_1(|u| + |v|) + M_1)\mathrm{d}s \\
& + |a_1||\rho_2(t)|\int_0^T(L_1(|u| + |v|) + M_1)\mathrm{d}s + |b_1||\rho_3(t)|\int_0^T(L_2(|u| + |v|) + M_2)\mathrm{d}s \\
& + |\nu||\rho_2(t)|\int_0^{\eta_1}\frac{(\eta_1 - s)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}(L_2(|u| + |v|) + M_2)\mathrm{d}s + |\mu||\rho_3(t)|\int_0^{\eta_2}\frac{(\eta_2 - s)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}(L_1(|u| + |v|) + M_1)\mathrm{d}s \\
& + |\rho_2(t)|\sum_{i=1}^{m}|\mu_i|\int_0^{\xi_i}\frac{(\xi_i - s)^{\beta-1}}{\Gamma(\beta)}(L_2(|u| + |v|) + M_2)\mathrm{d}s + |\rho_3(t)|\sum_{j=1}^{n}|\sigma_j|\int_0^{\zeta_j}\frac{(\zeta_j - s)^{\alpha-1}}{\Gamma(\alpha)}(L_1(|u| + |v|) + M_1)\mathrm{d}s \\
& + |b_0\omega_1||\rho_2(t)|\int_0^T\frac{(T-s)}{\Gamma(2)}(L_2(|u| + |v|) + M_2)\mathrm{d}s + \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(L_1(|u| + |v|) + M_1)\mathrm{d}s\Bigg]\Bigg\} \\
\le {} & |\lambda_1|\delta_1 + |\lambda_2\omega_1|\delta_2 + |a_0|\delta_1L_1\int_0^T(T-s)[s^{\alpha-2}\|u\|_X + s^{\beta-2}\|v\|_Y]\mathrm{d}s + M_1|a_0|\delta_1\frac{T^2}{2} \\
& + |a_1|\delta_2L_1\int_0^T[s^{\alpha-2}\|u\|_X + s^{\beta-2}\|v\|_Y]\mathrm{d}s + |a_1|\delta_2M_1T + |b_1|\delta_3L_2\int_0^T[s^{\alpha-2}\|u\|_X + s^{\beta-2}\|v\|_Y]\mathrm{d}s \\
& + M_2|b_1|\delta_3T + |\nu|\delta_2L_2\int_0^{\eta_1}\frac{(\eta_1 - s)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}[s^{\alpha-2}\|u\|_X + s^{\beta-2}\|v\|_Y]\mathrm{d}s \\
& + M_2|\nu|\delta_2\frac{(\eta_1)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + |\mu|\delta_3L_1\int_0^{\eta_2}\frac{(\eta_2 - s)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}[s^{\alpha-2}\|u\|_X + s^{\beta-2}\|v\|_Y]\mathrm{d}s \\
& + M_1|\mu|\delta_3\frac{(\eta_2)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \delta_2\sum_{i=1}^{m}|\mu_i|L_2\int_0^{\xi_i}\frac{(\xi_i - s)^{\beta-1}}{\Gamma(\beta)}[s^{\alpha-2}\|u\|_X + s^{\beta-2}\|v\|_Y]\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+ M_2\delta_2\sum_{i=1}^{m}\frac{|\mu_i|\xi_i^{\beta}}{\Gamma(\beta+1)} + \delta_3\sum_{j=1}^{n}|\sigma_j|L_1\int_0^{\zeta_j}\frac{(\zeta_j-s)^{\alpha-1}}{\Gamma(\alpha)}[s^{\alpha-2}\|u\|_X + s^{\beta-2}\|v\|_Y]\mathrm{d}s\\
&+ M_1\delta_3\sum_{j=1}^{n}\frac{|\sigma_j|\zeta_j^{\alpha}}{\Gamma(\alpha+1)} + |b_0\omega_1|\delta_2L_2\int_0^{T}(T-s)[s^{\alpha-2}\|u\|_X + s^{\beta-2}\|v\|_Y]\mathrm{d}s\\
&+ M_1|b_0\omega_1|\delta_2\frac{T^2}{2} + L_1t^{2-\alpha}\int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[s^{\alpha-2}\|u\|_X + s^{\beta-2}\|v\|_Y]\mathrm{d}s + M_1\frac{T^2}{\Gamma(\alpha+1)}\\
&\leq |\lambda_1|\delta_1 + |\lambda_2\omega_1|\delta_2 + [L_1N_1 + L_2N_2]r + M_1e_1 + M_2e_2.
\end{aligned}
\tag{28}
$$

Similarly, one can get

$$
||\mathcal{P}_2(u,v)||_Y \leq |\lambda_2|\delta_1^* + |\lambda_1\nu_1|\delta_2^* + [L_1N_1^* + L_2N_2^*]r + M_1e_1^* + M_2e_2^*. \tag{29}
$$

In view of (28) and (29) together with (25), we have

$$
\|\mathcal{P}(u,v)\|_{X\times Y} \leq |\lambda_1|[\delta_1 + |\nu_1|\delta_2^*] + |\lambda_2|[\delta_2|\omega_1| + \delta_1^*] + [L_1(N_1+N_1^*) + L_2(N_2+N_2^*)]r + M_1(e_1+e_1^*) + M_2(e_2+e_2^*) \leq r.
$$

Now, for $(u_1, v_1), (u_2, v_2) \in X \times Y$, and for any $t \in [0, T]$, we get

$$
\begin{aligned}
&t^{2-\alpha}|\mathcal{P}_1(u_2,v_2)(t) - \mathcal{P}_1(u_1,v_1)(t)|\\
&\leq \sup_{t\in[0,T]}\Bigg\{t^{2-\alpha}\Bigg[|a_0\rho_1(t)|\int_0^T(T-s)|F(s,u_2,v_2) - F(s,u_1,v_1)|\mathrm{d}s\\
&+ |a_1\rho_2(t)|\int_0^T|F(s,u_2,v_2) - F(s,u_1,v_1)|\mathrm{d}s + |b_1\rho_3(t)|\int_0^T|G(s,u_2,v_2) - G(s,u_1,v_1)|\mathrm{d}s\\
&+ |\nu\rho_2(t)|\int_0^{\eta_1}\frac{(\eta_1-s)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}|G(s,u_2,v_2) - G(s,u_1,v_1)|\mathrm{d}s\\
&+ |\mu\rho_3(t)|\int_0^{\eta_2}\frac{(\eta_2-s)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}|F(s,u_2,v_2) - F(s,u_1,v_1)|\mathrm{d}s\\
&+ |\rho_2(t)|\sum_{i=1}^{m}|\mu_i|\int_0^{\xi_i}\frac{(\xi_i-s)^{\beta-1}}{\Gamma(\beta)}|G(s,u_2,v_2) - G(s,u_1,v_1)|\mathrm{d}s\\
&+ |\rho_3(t)|\sum_{j=1}^{n}|\sigma_j|\int_0^{\zeta_j}\frac{(\zeta_i-s)^{\alpha-1}}{\Gamma(\alpha)}|F(s,u_2,v_2) - F(s,u_1,v_1)|\mathrm{d}s\\
&+ |b_0\omega_1\rho_2(t)|\int_0^T(T-s)|G(s,u_2,v_2) - G(s,u_1,v_1)|\mathrm{d}s + \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|F(s,u_2,v_2) - F(s,u_1,v_1)|\mathrm{d}s\Bigg]\Bigg\}.
\end{aligned}
\tag{30}
$$

Using (A_1) and the relation $|u_2 - u_1| + |v_2 - v_1| \leq t^{\alpha-2}\|u_2 - u_1\|_X + t^{\beta-2}\|v_2 - v_1\|_Y$ in (30) yields

$$
\|\mathcal{P}_1(u_2,v_2) - \mathcal{P}_1(u_1,v_1)\|_X \leq (L_1N_1 + L_2N_2)[\|u_2 - u_1\|_X + \|v_2 - v_1\|_Y]. \tag{31}
$$

In a similar manner, one can get

$$
\|\mathcal{P}_2(u_2,v_2) - \mathcal{P}_2(u_1,v_1)\|_Y \leq (L_1N_1^* + L_2N_2^*)[\|u_2 - u_1\|_X + \|v_2 - v_1\|_Y]. \tag{32}
$$

Thus, it follows from (31) and (32) that

$$
\|\mathcal{P}(u_2,v_2) - \mathcal{P}(u_1,v_1)\|_{X\times Y} \leq [L_1(N_1+N_1^*) + L_2(N_2+N_2^*)][\|u_2 - u_1\|_X + \|v_2 - v_1\|_Y],
$$

which, in view of condition (24), implies that $\mathcal{P}$ is a contraction. Hence, by Banach's fixed point theorem, the operator $\mathcal{P}$ has a unique fixed point, which is indeed a unique solution of the problem (1)–(2) on $[0, T]$. This completes the proof. □

In the following result, we present the sufficient conditions ensuring the existence of solutions for the problem (1)–(2). We apply Leray-Schauder alternative [32] to prove this result.

Theorem 3.2. *Assume that*

(A1) $F, G : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist real constants* $k_i, \gamma_i \geq 0, (i = 1, 2)$ *and* $k_0 > 0$ *and* $\gamma_0 > 0$ *such that, for all* $t \in [0, T]$ *and* $u, v \in \mathbb{R}$,

$$|F(t, u, v)| \leq k_0 + k_1|u| + k_2|v|, \quad |G(t, u, v)| \leq \gamma_0 + \gamma_1|u| + \gamma_2|v|.$$

(A2) $k(N_1 + N_1^*) + \gamma(N_2 + N_2^*) < 1$, *where* $k = \max\{k_1, k_2\}$, $\gamma = \max\{\gamma_1, \gamma_2\}$.

Then the system (1)–(2) *has at least one solution on* $[0, T]$.

Proof. Let us first note that continuity of the operator $\mathcal{P}$ follows from that of the functions F and G. Let $\mathbb{B} \subset X \times Y$ be bounded such that $|F(t, u, v)| \leq K_F$, $|G(t, u, v)| \leq K_G$, $\forall (u, v) \in \mathbb{B}$, for positive constants K_F and K_G. Then, for any $(u, v) \in \mathbb{B}$, we have

$$\begin{aligned}
&t^{2-\alpha}|\mathcal{P}_1(u, v)(t)| \\
&\leq |\lambda_1|\delta_1 + |\lambda_2\omega_1|\delta_2 + K_F\left[|a_0|\delta_1\frac{T^2}{2} + |a_1|T\delta_2 + |\mu|\delta_3\frac{\eta_2^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \delta_3\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^2}{\Gamma(\alpha+1)}\right] \\
&\quad + K_G\left[|\nu|\delta_2\frac{\eta_1^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + |b_1|\delta_3 T + \delta_2\sum_{i=1}^{m}|\mu_i|\frac{\xi_i^{\beta}}{\Gamma(\beta+1)} + |b_0\omega_1|\delta_2\frac{T^2}{2}\right] \\
&= |\lambda_1|\delta_1 + |\lambda_2\omega_1|\delta_2 + K_F e_1 + K_G e_2,
\end{aligned}$$

which implies that

$$\|\mathcal{P}_1(u, v)\|_X \leq |\lambda_1|\delta_1 + |\lambda_2\omega_1|\delta_2 + K_F e_1 + K_G e_2.$$

Similarly, one can show that

$$\|\mathcal{P}_2(u, v)\|_Y \leq |\lambda_2|\delta_1^* + |\lambda_1\nu_1|\delta_2^* + K_F e_1^* + K_G e_2^*.$$

In consequence, we get

$$\|\mathcal{P}(u, v)\|_{X\times Y} \leq |\lambda_1|\delta_1 + |\lambda_2|\delta_1^* + |\lambda_2\omega_1|\delta_2 + |\lambda_1\nu_1|\delta_2^* + K_F(e_1 + e_1^*) + K_G(e_2 + e_2^*) < \infty,$$

which shows that the operator $\mathcal{P}$ is uniformly bounded.

Next, we show that $\mathcal{P}$ is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 > t_2$. Then we have

$$\begin{aligned}
&|t_1^{2-\alpha}\mathcal{P}_1(u, v)(t_1) - t_2^{2-\alpha}\mathcal{P}_1(u, v)(t_2)| \\
&\leq |\lambda_1||t_1^{2-\alpha}\rho_1(t_1) - t_2^{2-\alpha}\rho_1(t_2)| + |\lambda_2\omega_1||t_1^{2-\alpha}\rho_2(t_1) - t_2^{2-\alpha}\rho_2(t_2)| \\
&\quad + K_F\Bigg[|a_0||t_1^{2-\alpha}\rho_1(t_1) - t_2^{2-\alpha}\rho_1(t_2)|\frac{T^2}{2} + |a_1|T||t_1^{2-\alpha}\rho_2(t_1) - t_2^{2-\alpha}\rho_2(t_2)| \\
&\quad + |\mu||t_1^{2-\alpha}\rho_3(t_1) - t_2^{2-\alpha}\rho_3(t_2)|\frac{|\eta_2|^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + |t_1^{2-\alpha}\rho_3(t_1) - t_2^{2-\alpha}\rho_3(t_2)|\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^{\alpha}}{\Gamma(\alpha+1)} \\
&\quad + \left|t_1^{2-\alpha}\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds - t_2^{2-\alpha}\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right|\Bigg] \\
&\quad + K_G\Bigg[|\nu|\frac{|\eta_1|^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}|t_1^{2-\alpha}\rho_2(t_1) - t_2^{2-\alpha}\rho_2(t_2)| + |b_1|T|t_1^{2-\alpha}\rho_3(t_1) - t_2^{2-\alpha}\rho_3(t_2)|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}|\mu_i|\frac{\xi_i^{\beta}}{\Gamma(\beta+1)}|t_1^{2-\alpha}\rho_2(t_1)-t_2^{2-\alpha}\rho_2(t_2)|+|b_0\omega_1|\frac{T^2}{2}|t_1^{2-\alpha}\rho_2(t_1)-t_2^{2-\alpha}\rho_2(t_2)|\Bigg]\\
&\le |t_1-t_2|\Bigg\{\Bigg[|\lambda_1|+|a_0|\frac{T^2}{2}K_F\Bigg]\Bigg[\Bigg|\frac{\nu_1}{\Lambda}\Bigg|\Bigg|\omega_2-\frac{b_0T\Gamma(\beta)}{\Gamma(\alpha)(1+a_1)}\omega_1\Bigg|\Bigg]\\
&+\Bigg[|\lambda_2\omega_1|+|a_1|TK_F+|\nu|K_G\frac{\eta_1^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+K_G\sum_{i=1}^{m}|\mu_i|\frac{\xi_i^{\beta}}{\Gamma(\beta+1)}+K_G|b_0\omega_1|\frac{T^2}{2}\Bigg]\Bigg|\frac{\Gamma(\beta)(b_1+1)}{\Lambda\Gamma(\alpha)(a_1+1)}\Bigg|\\
&+\Bigg[\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^{\alpha}}{\Gamma(\alpha+1)}K_G+|\mu|\frac{|\eta_2|^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}K_F+|b_1|TK_G\Bigg]\Bigg[\frac{1}{|\Lambda|}\Bigg|\omega_2-\frac{b_0T\Gamma(\beta)}{\Gamma(\alpha)(1+a_1)}\omega_1\Bigg|\Bigg]\Bigg\}\\
&+\frac{2K_Ft_1^{2-\alpha}}{\Gamma(\alpha+1)}|t_1-t_2|^{\alpha}+K_F\frac{|t_1^2-t_2^2|}{\Gamma(\alpha+1)}\to 0\quad\text{as } t_2\to t_1,
\end{aligned}
$$

independent of $(u, v) \in \mathbb{B}$. Also, we have

$$
\begin{aligned}
&|t_1^{2-\beta}\mathcal{P}_2(u,v)(t_1)-t_2^{2-\beta}\mathcal{P}_2(u,v)(t_2)|\\
&\le |t_1-t_2|\Bigg\{\Bigg[|\lambda_2|+|b_0|K_G\frac{T^2}{2}\Bigg]\Bigg[\Bigg|\frac{\omega_1}{\Lambda}\Bigg|\Bigg|\nu_2-\frac{a_0T}{(1+a_1)}\nu_1\Bigg|\Bigg]\\
&+\Bigg[|\lambda_1\nu_1|+|a_0\nu_1|K_F\frac{T^2}{2}+\mu K_F\frac{\eta_2^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+|b_1|TK_G+K_F\sum_{j=1}^{n}|\sigma_j|\frac{\zeta_j^{\alpha}}{\Gamma(\alpha)}\Bigg]\Bigg[\frac{1}{|\Lambda|}\Bigg]\\
&+\Bigg[|a_1|TK_F+|\nu|K_G\frac{\eta_1^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}+K_G\sum_{i=1}^{m}|\mu_i|\frac{\xi_i^{\beta}}{\Gamma(\beta+1)}\Bigg]\Bigg[\frac{1}{|\Lambda|}\Bigg|\nu_2-\frac{a_0T}{(1+a_1)}\nu_1\Bigg|\Bigg]\Bigg\}\\
&+\frac{2K_Gt_1^{2-\beta}}{\Gamma(\beta+1)}|t_1-t_2|^{\beta}+K_G\frac{|t_1^2-t_2^2|}{\Gamma(\beta+1)}\to 0\quad\text{as } t_2\to t_1,
\end{aligned}
$$

independently of $(u, v) \in \mathbb{B}$. Thus, the operator $\mathcal{P}$ is equicontinuous. Thus, we deduce that the operator $\mathcal{P}$ is completely continuous.

Finally, we consider the set

$$V=\{(u,v)\in X\times Y|(u,v)=m\mathcal{P}(u,v),\ 0\le m\le 1\}$$

and show that it is bounded. Let $(u, v) \in V$ with $(u, v) = m\mathcal{P}(u, v)$, $u = m\mathcal{P}_1(u, v)$, and $v = m\mathcal{P}_2(u, v)$. Then we have

$$
\begin{aligned}
\|u\|_X &\le |\lambda_1|\delta_1+|\lambda_2\omega_1|\delta_2+(N_1k_1+N_2\gamma_1)\|u\|_X+(N_1k_2+N_2\gamma_2)\|v\|_Y+k_0e_1+\gamma_0e_2\\
&\le |\lambda_1|\delta_1+|\lambda_2\omega_1|\delta_2+(N_1k+N_2\gamma)\|u\|_X+(N_1k+N_2\gamma)\|v\|_Y+k_0e_1+\gamma_0e_2,\\
\|v\|_Y &\le |\lambda_2|\delta_1^*+|\lambda_1\nu_1|\delta_2^*+(N_1^*k_1+N_2^*\gamma_1)\|u\|_X+(k_2N_1^*+N_2^*\gamma_2)\|v\|_Y+k_0e_1^*+\gamma_0e_2^*\\
&\le |\lambda_2|\delta_1^*+|\lambda_1\nu_1|\delta_2^*+(N_1^*k+N_2^*\gamma)\|u\|_X+(kN_1^*+N_2^*\gamma)\|v\|_Y+k_0e_1^*+\gamma_0e_2^*,
\end{aligned}
$$

which imply that

$$
\begin{aligned}
\|u\|_X+\|v\|_Y \le{}& |\lambda_1|\delta_1+|\lambda_2\omega_1|\delta_2+|\lambda_2|\delta_1^*+|\lambda_1\nu_1|\delta_2^*+[k(N_1+N_1^*)+\gamma(N_2+N_2^*)]\|u\|_X\\
&+[k(N_1+N_1^*)+\gamma(N_2+N_2^*)]\|v\|_Y+k_0(e_1+e_1{}^*)+\gamma_0(e_2+e_2^*).
\end{aligned}
$$

Thus,

$$\|(u,v)\|_{X\times Y}\le\frac{|\lambda_1|\delta_1+|\lambda_2\omega_1|\delta_2+|\lambda_2|\delta_1^*+|\lambda_1\nu_1|\delta_2^*+k_0(e_1+e_1{}^*)+\gamma_0(e_2+e_2^*)}{1-[k(N_1+N_1^*)+\gamma(N_2+N_2^*)]}.$$

Hence, the set V is bounded. Thus, by Leray-Schauder alternative, we deduce that the operator $\mathcal{P}$ has at least one fixed point, which corresponds to the fact that the problem (1)–(2) has at least one solution on $[0, T]$. The proof is completed. □

4 Examples

This section is devoted to the illustration of the results derived in the previous section.

Example 4.1. Consider the system of fractional differential equations consisting of the equations given by

$$\begin{cases} D^{6/5}u(t) = \dfrac{1}{4\sqrt{1600+t}}(u(t) + \tan^{-1}v(t)) + \cos t, & t \in [0,1], \\ D^{7/4}v(t) = \dfrac{1}{\sqrt{2500+t}}(\sin u(t) + v(t)) + t\sin t, & t \in [0,1], \end{cases} \tag{33}$$

supplemented by the following boundary conditions

$$\begin{cases} D^{-4/5}u(0^+) + \dfrac{1}{4}D^{-4/5}u(1^-) = -4, \\ D^{1/5}u(0^+) - \dfrac{3}{2}D^{1/5}u(1^-) = -2I^{1/5}v\left(\dfrac{1}{4}\right) - 6v\left(\dfrac{1}{2}\right) - 4v\left(\dfrac{2}{3}\right), \\ D^{-1/4}v(0^+) - D^{-1/4}v(1^-) = 2, \\ D^{3/4}v(0^+) - \dfrac{3}{2}D^{3/4}v(1^-) = 3I^{3/4}u\left(\dfrac{1}{3}\right) + \dfrac{5}{3}u\left(\dfrac{11}{15}\right) + \dfrac{2}{5}u\left(\dfrac{3}{4}\right). \end{cases} \tag{34}$$

Here $\alpha = 6/5$, $\beta = 7/4$, $a_0 = 1/4$, $b_0 = -1/4$, $a_1 = -3/2$, $b_1 = -3/2$, $\nu = -2$, $\mu = 3$, $\lambda_1 = -4$, $\lambda_2 = 2$, $\mu_1 = -6$, $\mu_2 = -4$, $\sigma_1 = 5/3$, $\sigma_2 = 2/5$, $\eta_1 = 1/4$, $\xi_1 = 1/2$, $\xi_2 = 2/3$, $\eta_2 = 1/3$, $\zeta_1 = 11/15$, $\zeta_2 = 3/4$.

Using the given data, it is found that $\Lambda \simeq 70.9531$, $v_1 \simeq 2.9182$, $v_2 \simeq -6.3910$, $\omega_1 \simeq -15.3511$, $\omega_2 \simeq 15.2944$, $\delta_1 \simeq 0.7330$, $\delta_2 \simeq 0.0135$, $\delta_3 \simeq 0.3100$, $\delta_1^* \simeq 0.8213$, $\delta_2^* \simeq 0.0176$, $\delta_3^* \simeq 0.0869$, $N_1 \simeq 12.2414$, $N_2 \simeq 3.6569$, $N_1^* \simeq 1.1599$, $N_2^* \simeq 9.4324$, $L_1 = 1/160$, $L_2 = 1/50$, and

$$L_1(N_1 + N_1^*) + L_2(N_2 + N_2^*) \simeq 0.3455 < 1.$$

Thus, all the conditions of Theorem 3.1 are satisfied and hence the problem (33)–(34) has a unique solution on $[0,1]$.

Example 4.2. Let us consider the problem (33)–(34) with

$$\begin{aligned} F(t,u(t),v(t)) &= \frac{\sin^2 t}{\sqrt{2+t^3}} + \frac{\tan^{-1}u(t)}{2\sqrt{400+t}} + \frac{v(t)|u(t)|}{50(1+|u(t)|)}, \quad t \in [0,1], \\ G(t,u(t),v(t)) &= e^{-t} + \frac{|u(t)|\cos u(t)}{150+t} + \frac{v(t)}{640\sqrt{1+\sin^2 t}}, \quad t \in [0,1]. \end{aligned} \tag{35}$$

Clearly, $|F(t,u(t),v(t))| < \frac{1}{2} + \frac{1}{40}|u(t)| + \frac{1}{50}|v(t)|$, $G(t,u(t),v(t)) < 1 + \frac{1}{150}|u(t)| + \frac{1}{640}|v(t)|$, and so $k_0 = 1/2$, $k_1 = 1/40$, $k_2 = 1/50$, $\gamma_0 = 1$, $\gamma_1 = 1/150$, $\gamma_2 = 1/640$, $k = \max\{k_1,k_2\} = 1/40$, $\gamma = \max\{\gamma_1,\gamma_2\} = 1/150$. Moreover,

$$k(N_1 + N_1^*) + \gamma(N_2 + N_2^*) \simeq 0.4223 < 1.$$

Therefore, by Theorem 3.2, the problem (33)–(34) with F and G given by (35) has at least one solution on $[0,1]$.

5 Conclusion

We have investigated the existence and uniqueness of solutions for a nonlinear system of Riemann-Liouville fractional differential equations, equipped with nonseparated semi-coupled integro-multipoint boundary conditions. We apply Banach contraction mapping principle to establish the existence of a unique solution, while Leray-Schauder alternative is used to obtain the existence result for the problem at hand. We emphasize that the novelty of our results lies on the semi-coupled boundary conditions (2) and enrich the related literature on the topic. Our work also produces some special cases by fixing the parameters involved in the boundary conditions. For example, our results correspond to the ones for nonlocal semi-coupled fractional multipoint boundary conditions by fixing $\nu = 0 = \mu$ and the results for nonlocal semi-coupled fractional integral boundary conditions follow by taking all $\mu_i = 0$, $i = 1, \ldots, m$ and $\sigma_j = 0$, $j = 1, \ldots, n$.

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