On a Paper of Ramesh Gangolli

M. G. NADKARNI

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Introduction. In this note we construct an infinite vector-valued weakly stationary stochastic process such that

(i) it has an absolutely continuous spectrum,

(ii) it has a nontrivial deterministic part, and

(iii) its prediction error matrix is non-singular.

A special case of our example will show that certain results of [4] are in error. We shall state correct versions of these results. In [3], R. G. Douglas constructed a counter example showing that a certain result in [5] was in error. Our counter example was suggested after looking at his paper, but our method of construction differs from his.

Section 1. Let $\mathcal{H}$ be a Hilbert space which we take to be L_2 of some probability space $(\Omega, \mathcal{B}, P)$. We assume that $\mathcal{H}$ is infinite-dimensional. We shall need the following:

Lemma. *Let $\varphi, \varphi_1, \varphi_2, \cdots$ be a set of orthonormal vectors in $\mathcal{H}$. Then φ lies in the subspace spanned by $\{\varphi + \varphi_1, \varphi + \varphi_2, \varphi + \varphi_3, \cdots\}$.*

Proof.

$$\left|\left|\varphi - \frac{1}{n}\sum_{i=1}^{n}(\varphi + \varphi_k)\right|\right|^2 = \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty. \qquad \text{Q.E.D.}$$

Now we construct our example having properties (i), (ii) and (iii) stated in the introduction. Let $H, H_0, H_{-1}, H_1, H_{-2}, H_2, \cdots$ be infinite-dimensional mutually orthogonal subspaces of $\mathcal{H}$. In each H_k choose a one-dimensional purely non-deterministic weakly stationary stochastic process (Y_n^k), $-\infty < n < \infty$. Further for each k choose $||Y_0^k|| = 1$. Let f_k be the spectral density of (Y_n^k), $-\infty < n < \infty$. In H choose a one-dimensional weakly stationary stochastic process (Y_n), $-\infty < n < \infty$, with absolutely continuous spectrum. Let g be the spectral density of (Y_n), $-\infty < n < \infty$. Let (Y_n), $-\infty < n < \infty$, be so normalized that $||Y_n|| = 1$. Consider the infinite vector $\bar{X}_n = [Y_n^k + Y_{k+n}]$, $-\infty < k < \infty$. Then $(\bar{X}_n)$, $-\infty < n < \infty$, is an infinite-dimensional weakly stationary stochastic process.

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Theorem.

(*i*) $(\bar{X}_n)$, $-\infty < n < \infty$, *has an absolutely continuous spectrum.*

(*ii*) $(\bar{X}_n)$, $-\infty < n < \infty$, *has a non-trivial deterministic part.*

(*iii*) *The one step prediction error matrix of* $(\bar{X}_n)$, $-\infty < n < \infty$, *is a diagonal matrix with each term of the diagonal non-zero.*

Proof. (*i*) Due to the orthogonality of H with the H_k's we have

$$\begin{aligned}(X_n^k, X_0^l) &= (Y_n^k + Y_{k+n}, Y_0^l + Y_l) = (Y_n^k, Y_0^l) + (Y_{n+k}, Y_l)\\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{inu}\,\delta_{k,l} f_k(u)\,du + \frac{1}{2\pi}\int_0^{2\pi} e^{i(k+n-l)}g(u)\,du\\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{inu} f_{k,l}(u)\,du,\end{aligned}$$

where $f_{k,l}(u) = \delta_{k,l} f_k(u) + e^{i(k-l)u}g(u)$. Thus $(\bar{X}_n)$, $-\infty < n < \infty$, is a process with an absolutely continuous spectrum and its spectral density matrix $f(\cdot)$ has $(k, l)^{\text{th}}$ entry given by $f_{k,l}(u) = \delta_{k,l} f_k(u) + e^{i(k-l)u}g(u)$.

(*ii*) Let $\mathfrak{M}_{-1}$ be the subspace spanned by $\{X_n^k : -\infty < k < \infty, n \leq -1\}$. We show that each $Y_p \in \mathfrak{M}_{-1}$, $-\infty < p < \infty$. For each $l > 0$,

$$X_{-l}^{p+l} = Y_{-l}^{p+l} + Y_{p+l-l} = Y_p + Y_{-l}^{p+l} \in \mathfrak{M}_{-1}.$$

Further Y_p, Y_{-1}^{p+1}, Y_{-2}^{p+2}, $\cdots$ are orthonormal vectors. Hence by our lemma, Y_p lies in the closed subspace spanned by $\{X_{-l}^{p+l} : l > 0\} \subset \mathfrak{M}_{-1}$. Thus $Y_p \in \mathfrak{M}_{-1}$. The same reasoning shows that for every p, $Y_p \in \mathfrak{M}_n$ for every n. Hence $(\bar{X}_n)$, $-\infty < n < \infty$, has a non-trivial deterministic part.

(*iii*) It is easy to see that the prediction error matrix of $(\bar{X}_n)$, $-\infty < n < \infty$, is a diagonal matrix, with diagonal terms

$$\lambda_k = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log f_k(u)\,du\right). \qquad \text{Q.E.D.}$$

We now consider a special case.

Example A. In the above example choose (Y_n^k), $-\infty < n < \infty$, uncorrelated. Choose (Y_n), $-\infty < n < \infty$, also uncorrelated. Set $\bar{X}_n = [Y_n^k + Y_{k+n}]$, $-\infty < k < \infty$. Now the spectral density matrix $f(\cdot)$ of $(\bar{X}_n)$, $-\infty < n < \infty$, has entries $f_{k,l}(\cdot)$ given by $f_{k,l}(u) = \delta_{k,l} + e^{i(k-l)u}$. Clearly $\sum_{k=-\infty}^{\infty} |f_{k,l}(u)|^2 = \infty$ for every u. Hence $f(u)$ does not define any operator (bounded or unbounded) on l^2 with values in l^2. (Here l^2 is the Hilbert space of all bisequences (x_n), $-\infty < n < \infty$, such that $\sum |x_n|^2 < \infty$.) The prediction error matrix of this process is the identity matrix. It has a non-trivial deterministic part.

Example A contradicts certain results in [4], as we now show. Let $(\bar{X}_n)$, $-\infty < n < \infty$, be as in this example. For each $x \in l^2$, $x = (x_k)$, $-\infty < k < \infty$, define

$$F_n(x) = \sum_{k=-\infty}^{\infty} x_k X_n^k.$$

Then F_n, $-\infty < n < \infty$, is a weakly stationary sequence of distributions in the sense of [4]. According to considerations in paragraph two of page 903 of [4], the matrix $f(\cdot)$ with entries given by $f_{k,l}(u) = \delta_{k,l} + e^{i(k-l)u}$ defines a matrix of an operator (bounded or unbounded) on l^2 with values in l^2. This is obviously false. Hence all one can say is that the entity W' of paragraph two on page 903 of [4] is a non-negative definite matrix with summable entries such that $\int_0^{2\pi} W'(\theta)\, d\theta$ is a bounded operator.

Next, Example A shows that Theorem 5.3 of [4] is in error. This theorem asserts in effect that a full rank process (infinite-dimensional) with an absolutely continuous spectrum is purely non-deterministic. Our Example A contradicts this. Theorem 7.3 of [4] is also in doubt since the proof suggested for it relies on Theorem 5.3 of [4]. With the notation and terminology of [4] a correct version of Theorem 5.3 of [4] can be stated as follows:

Suppose that $(F_n)_{-\infty}^{\infty}$ has full rank. Let $W = W_p + W_Q$ be its spectral distribution function with W_P and W_Q as in Theorem 5.2 of [4]. *Let W_{abs} and W_{sing} be the absolutely continuous and singular parts of W. Assume that $W'(\theta)$ is a bounded and invertible operator for almost every θ. Then $W_P = W_{\text{abs}}$, $W_Q = W_{\text{sing}}$.*

We omit the proof of this theorem. It can be derived from a result on page 494 of [1], or it can be derived by extending method of Doob ([2], p. 597) where the analogous result is proved for $n \times n$ matrix-valued spectral distribution functions.

Remark. We would like to mention that [6] contains results with the help of which the condition of invertibility and boundedness of W' can be relaxed. [6] also contains an analysis of the situation where the prediction error matrix is not a bounded invertible matrix.

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Washington University

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