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ON A PARAMETRIZATION OF MINIMAL IMMERSIONS OF R^2 INTO S^5

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1. **Introduction.** In [3], the author has studied minimal immersions of a surface into a space form and found some fundamental formulas for the Laplacian of scalar invariants of such immersions.

In [2], T. Itoh has constructed a 1-parameter family of minimal immersions of a Euclidean plane into S^5.

The purpose of this paper is to give a complete description for minimal immersions of the Euclidean plane and a flat torus in S^5. Let R^2 be the oriented Euclidean plane with the standard metric and S^5 the unit sphere in R^6. By $[\Psi]$ we denote an equivalence class of a minimal immersion $\Psi: R^2 \to S^5$ by isometries of S^5. We will prove the following theorems.

THEOREM 1. *There exists a* 1-1 *correspondence between the set of* $[\Psi]$*'s and a* 2*-dimensional sphere.*

The correspondence is given by (9) of § 3. Let O be the origin of R^6 and $\overrightarrow{Ox}$ the ray from O passing through a point $x \in R^6$. Then it is known that the cone

$$O\Psi(R^2) = \text{the union of } \overrightarrow{Ox} \quad \text{with} \quad x \in \Psi(R^2)$$

is also minimal in the Euclidean space R^6. By Hsiang [1], we shall call Ψ real algebraic if $O\Psi(R^2)$ is a real algebraic cone.

THEOREM 2. *Ψ induces the minimal immersion of the flat torus into S^5 if and only if Ψ is real algebraic. Moreover, there are infinite numbers of such immersions.*

2. **Preliminaries.** Terminologies and notations used in this paper are the same as in [3]. Let $x: M \to S^5$ be an isometric minimal immersion of some Riemann surface with a Riemannian metric into S^5. Let e_A, $1 \leqq A, B, \cdots \leqq 5$, be orthonormal frame fields in a neighborhood of M such that e_k, $1 \leqq k, l, \cdots \leqq 2$, are tangent to M. Let w_A, w_{AB} be the basic forms and the connection forms of S^5. Let $h_{\alpha kl}$, $3 \leqq \alpha, \beta, \cdots \leqq 5$, be the 2nd fundamental tensors for e_α. By $K_{(2)}$, $N_{(2)}$ and $f_{(2)}$, we denote the following non-negative scalar invariants on M:

$$(1)\qquad \begin{aligned} K_{(2)} &= \sum_\alpha (h^2_{\alpha 11} + h^2_{\alpha 12}) , \\ N_{(2)} &= \Big(\sum_\alpha h^2_{\alpha 11}\Big)\Big(\sum_\alpha h^2_{\alpha 12}\Big) - \Big(\sum_\alpha h_{\alpha 11} h_{\alpha 12}\Big)^2 , \\ f_{(2)} &= K^2_{(2)} - 4N_{(2)} . \end{aligned}$$

We can define the 3rd fundamental tensor h_{5ijk}. We set

$$(2)\qquad K_{(3)} = h^2_{5111} + h^2_{5112} .$$

Then $K_{(3)}$ is also a scalar invariant. In [3], we have proved the following formulas (3), (4) and (5):

$$(3)\qquad \Delta f_{(2)} = 2\{f_{(2)}K + |A_{(2)}|^2\} ,$$

$$(4)\qquad \frac{1}{4}\Delta K_{(2)} = -2N_{(2)} + KK_{(2)} + K_{(3)} + \sum_{\alpha=3}^{4} (h^2_{\alpha 11,1} + h^2_{\alpha 11,2}) ,$$

where K is the Gaussian curvature and $A_{(2)} = 2\sum_\alpha (h_{\alpha 11} + ih_{\alpha 12})(h_{\alpha 11,1} + ih_{\alpha 11,2})$. In a neighborhood of a point with $N_{(2)} \neq 0$, we get

$$(5)\qquad \frac{1}{2}\Delta K_{(3)} = 3KK_{(3)} + 2(h^2_{5111,1} + h^2_{5111,2}) .$$

Now we shall construct another scalar invariant of the isometric minimal immersion x. Since M has the fixed orientation, the vector $e_1 + ie_2$ is defined up to the transformation $e_1 + ie_2 \to e_1^* + ie_2^* = e^{i\theta}(e_1 + ie_2)$, where θ is real. Under such a change, we have $h_{\alpha 11} + ih_{\alpha 12} \to h^*_{\alpha 11} + ih^*_{\alpha 12} = e^{2i\theta}(h_{\alpha 11} + ih_{\alpha 12})$ and $h_{5111} + ih_{5112} \to h^*_{5111} + ih^*_{5112} = e^{3i\theta}(h_{5111} + ih_{5112})$. Thus for the fixed vector field e_3, we can define the following scalar invariant:

$$(6)\qquad L = (h_{5111} + ih_{5112})^2(h_{311} - ih_{312})^3 .$$

We remark that the normal vector e_5 is defined up to the sign, the 3rd osculating space being the 1-dimensional space. Therefore the function L is independent upon e_5 and depends on the e_3 and the orientation of M.

3. Construction of minimal immersions. Let Σ be a portion of an ellipsoid such that

$$(7)\qquad \Sigma = \left\{(t, u, v) \in R^3 : 0 \leqq t \leqq \frac{1}{2}, 0 \leqq u, 0 \leqq v, u^2 + v^2 = 2t(1 - t)\right\} .$$

In this section we shall give various minimal immersions of R^2 into S^5. Let P and Q be the skew-symmetric matrices such that, for $(t, u, v) \in \Sigma$,

$$P=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \sqrt{t} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1-t} & 0 \\ 0 & -\sqrt{t} & 0 & 0 & 0 & \frac{u}{\sqrt{t}} \\ 0 & 0 & -\sqrt{1-t} & 0 & 0 & \frac{v}{\sqrt{1-t}} \\ 0 & 0 & 0 & -\frac{u}{\sqrt{t}} & \frac{-v}{\sqrt{1-t}} & 0 \end{pmatrix},$$

$$Q=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{1-t} & 0 \\ -1 & 0 & 0 & -\sqrt{t} & 0 & 0 \\ 0 & 0 & \sqrt{t} & 0 & 0 & \frac{v}{\sqrt{t}} \\ 0 & -\sqrt{1-t} & 0 & 0 & 0 & \frac{-u}{\sqrt{1-t}} \\ 0 & 0 & 0 & -\frac{v}{\sqrt{t}} & \frac{u}{\sqrt{1-t}} & 0 \end{pmatrix},$$

where we must read $u/\sqrt{t}=0$ and $v/\sqrt{t}=0$ if $t=0$. Then we have $PQ=QP$ by virtue of $u^2+v^2=2t(1-t)$. We denote the eigenvalues of P and Q by $\pm\sqrt{-1}\lambda_i$ and $\pm\sqrt{-1}\mu_i$, $i=1, 2, 3$, respectively. We remark that λ_i or μ_i may be zero. We can take an orthogonal matrix $T={}^t(v_0, v_1, \cdots, v_5)$ such that

$$(8)\qquad T^{-1}PT=\begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{pmatrix}, \qquad T^{-1}QT=\begin{pmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{pmatrix},$$

where

$$P_i=\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}, \qquad Q_i=\begin{pmatrix} 0 & \mu_i \\ -\mu_i & 0 \end{pmatrix}$$

and we can assume $\lambda_i \geqq 0$. Then $\mathfrak{G}=\{T^{-1}(xP+yQ)T: (x, y)\in R^2\}$ is an abelian Lie subalgebra of $\mathfrak{so}(6, R)$. We consider $G=\exp\mathfrak{G}$. G is the Lie subgroup of $SO(6)$ and isomorphic to $T^d\times R^{2-d}$, $0\leqq d\leqq 2$, where T^d is the d-dimensional torus. By an orbit of an action of G, we define a smooth map $\Psi_{(t,u,v)}$ of R^2 in R^6 as follows:

$$\Psi_{(t,u,v)}(x, y) = \left(\sum_{s=0}^{5} T_{0s}t_{sj}(x, y)\right), \tag{9}$$

where $v_0 = (T_{00}, \cdots, T_{05})$ and $(t_{sj}(x, y)) \in G$. It is clear that

$$(t_{sj}(x, y)) = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix}, \quad X_i = \begin{pmatrix} \cos(\lambda_i x + \mu_i y) & \sin(\lambda_i x + \mu_i y) \\ -\sin(\lambda_i x + \mu_i y) & \cos(\lambda_i x + \mu_i y) \end{pmatrix}.$$

PROPOSITION 1. *$\Psi_{(t,u,v)}$ is an isometric minimal immersion of R^2 in S^5 such that $N_{(2)} = t(1-t)$, $K_{(3)} = u^2 + v^2$ and $L = t^{3/2}(u + iv)^2$ for some normal vector e_3. Moreover if $t > 0$, the image is not contained in any lower dimensional linear subspace of R^6.*

PROOF. We set $\Psi_0 = \Psi_{(t,u,v)}$ and $\Psi_A = (\sum_{s=0}^{5} T_{As}t_{sj}(x, y))$, where $v_A = (T_{A0}, \cdots, T_{A5})$, $0 \leqq A \leqq 5 \cdot \Psi_0(R^2) \subset S^5$ is clear. It is easily verified that

$$ {}^t(d\Psi_0, \cdots, d\Psi_5) = (Pdx + Qdy)\,{}^t(\Psi_0, \cdots, \Psi_5). \tag{10}$$

Therefore we can get $\partial\Psi_0/\partial x = \Psi_1$ and $\partial\Psi_0/\partial y = \Psi_2$. Hence Ψ_0 is an isometric immersion of R^2 into S^5. Since we can see $\{\Psi_\alpha, 3 \leqq \alpha \leqq 5\}$ are unit normal vectors on $\Psi_0(R^2)$, (10) is the Frenet-Borůvka formula for Ψ_0. It follows that we have

$$(h_{3ij}) = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & -\sqrt{t} \end{pmatrix}, \quad (h_{4ij}) = \begin{pmatrix} 0 & \sqrt{1-t} \\ \sqrt{1-t} & 0 \end{pmatrix}, \quad (h_{5ij}) = 0. \tag{11}$$

The formula (11) shows that Ψ_0 is a minimal immersion. If $t > 0$, since we have $N_{(2)} = t(1-t) > 0$ on R^2, $\Psi_0(R^2)$ is not contained in any lower dimensional linear subspace of R^6. From (10), we get

$$\begin{aligned} \sqrt{t}\, w_{35} &= uw_1 + vw_2, \\ \sqrt{1-t}\, w_{45} &= vw_1 - uw_2, \\ w_{12} &= w_{34} = 0. \end{aligned} \tag{12}$$

By (12) we get $K_{(3)} = u^2 + v^2$ and $L = t^{3/2}(u + iv)^2$ for e_3. This completes a proof of Proposition 1.

PROPOSITION 2. (i) *If $0 < t < 1/2$ and $uv \neq 0$, then we have $[\Psi_{(t,u,v)}] = [\Psi_{(\tilde{t},\tilde{u},\tilde{v})}]$ if and only if $(t, u, v) = (\tilde{t}, \tilde{u}, \tilde{v})$;*

(ii) *If $0 < t < 1/2$, then we have $[\Psi_{(t,0,u_0)}] = [\Psi_{(t,u_0,0)}]$, where $u_0 = \sqrt{2t(1-t)}$.*

(iii) *If $t = 0$, then we have the Clifford torus and if $t = 1/2$ then we have $[\Psi_{(1/2,u,v)}] = [\Psi_{(1/2,\sqrt{1/2},0)}]$.*

PROOF. Case (i): We suppose $[\Psi_{(t,u,v)}] = [\Psi_{(\tilde{t},\tilde{u},\tilde{v})}]$. Since we have

$f_{(2)} = \tilde{f}_{(2)}$, where $\tilde{f}_{(2)}$ denotes the quantity of $\Psi_{(\tilde{t},\tilde{u},\tilde{v})}$ corresponding to the $f_{(2)}$ of $\Psi_{(t,u,v)}$, we have $t(1-t) = \tilde{t}(1-\tilde{t})$ and $0 \leqq t, \tilde{t} \leqq 1/2$, hence we get $t = \tilde{t}$. If $0 < t < 1/2$, e_3 and e_4 in the 2nd osculating space satisfying (11) are determined up to the sign. In fact let e_A and $\tilde{e}_A$ be the frame fields such that (11) and (12) are satisfied. Let $\tilde{e}_1 + i\tilde{e}_2 = e^{-i\theta}(e_1 + ie_2)$. Then we have $\tilde{e}_3 + i\tilde{e}_4 = e^{-2i\theta}(e_3 + ie_4)$ and $\tilde{h}^2_{311} - \tilde{h}^2_{412} = \cos 4\theta(h^2_{311} - h^2_{412})$, where $\tilde{h}_{3ij}$ and $\tilde{h}_{4ij}$ are the components of the 2nd fundamental tensors for $\tilde{e}_A$. Since we have $\tilde{h}^2_{311} - \tilde{h}^2_{412} = h^2_{311} - h^2_{412} = 2t - 1 < 0$ in this case, we get $\theta = (k/2)\pi$ and k is an integer. It follows that we have $L = \pm \tilde{L}$ and so $(u + iv)^4 = (\tilde{u} + i\tilde{v})^4$. Making use of $u^2 + v^2 = \tilde{u}^2 + \tilde{v}^2 = 2t(1-t)$, $uv \neq 0$ and $u, \tilde{u}, v, \tilde{v} \geqq 0$, we have $u = \tilde{u}$ and $v = \tilde{v}$.

Case (ii): Let e_A be the frame field of $\Psi_{(t,0,u_0)}$ satisfying (11) and (12). We set $f_1 = e_2$, $f_2 = -e_1$, $f_3 = -e_3$, $f_4 = -e_4$ and $f_5 = e_5$. With respect to these new frame fields, we have $\Psi_{(t,u_0,0)}$.

Case (iii): If $t = 0$, then $\Psi_{(0,0,0)}$ is the Clifford torus. When $t = 1/2$, we have shown $[\Psi_{(1/2,u,v)}] = [\Psi_{(1/2,\sqrt{1/2},0)}]$ by the Theorem 3 of [3]. Thus we have proved the Propositoin 2.

4. **Parametrization of minimal immersions.** We shall prove the following proposition.

PROPOSITION 3. *Let $x\colon R^2 \to S^5$ be an isometric minimal immersion. Then there exists a $(t, u, v) \in \Sigma$ such that $x \in [\Psi_{(t,u,v)}]$.*

PROOF. Since $K = 0$, by the Gauss equation, we have $K_{(2)} = 1$. It follows that, by (1) and (3), we get $\Delta(-N_{(2)}) \geqq 0$, hence $-N_{(2)}$ is subharmonic on R^2 and non-positive. We claim $-N_{(2)} =$ constant on R^2. This is proved as follows: There exists a point $p_0 \in R^2$ such that $-N_{(2)}(p_0) \geqq -N_{(2)}(p)$ for all $p \in R^2$ by virtue of the maximum principle of the subharmonic functions and the boundedness of $-N_{(2)}$. Since Δ is an elliptic operator, by the well-known theorem, $-N_{(2)}$ must be a constant function on R^2. If $N_{(2)} = 0$, $x(R^2)$ is contained in a 3-dimensional space of constant curvature 1 in S^5 and therefore $x \in [\Psi_{(0,0,0)}]$. In the case of $N_{(2)} > 0$, vectors $\sum_\alpha h_{\alpha 11} e_\alpha$ and $\sum_\alpha h_{\alpha 12} e_\alpha$ are linearly independent on R^2. Let

$$(13) \qquad e^*_3 = \frac{\sum_\alpha h_{\alpha 11} e_\alpha}{\sqrt{\sum_\alpha h^2_{\alpha 11}}}, \qquad e^*_4 = \frac{\sum_\alpha h_{\alpha 12} e_\alpha - (\sum_\alpha h_{\alpha 12} e_\alpha, e^*_3) e^*_3}{\| \sum_\alpha h_{\alpha 12} e_\alpha - (\sum_\alpha h_{\alpha 12} e_\alpha, e^*_3) e^*_3 \|}$$

and e^*_5 is the unit normal vector field which span the 3rd osculating space. Then we have

$$(14) \qquad (h^*_{4ij}) = \begin{pmatrix} 0 & h^*_{412} \\ h^*_{412} & 0 \end{pmatrix}, \qquad h^*_{5ij} = 0,$$

where we may assume $h^*_{412} \geqq 0$. We take e^*_i such that

$$(h^*_{3ij}) = \begin{pmatrix} h^*_{311} & 0 \\ 0 & -h^*_{311} \end{pmatrix}, \tag{15}$$

where if necessary taking $-e^*_3$, we may assume $h^*_{311} \geqq 0$. As (14) is valid for any orthonormal system $\{e_i\}$, it follows that we have obtained frame fields $\{e^*_A\}$ such that (14) and (15) are valid at the same time. By the Gauss equation and the constancy of $N_{(2)}$, h^*_{311} and h^*_{412} are also positive constant and $h^{*2}_{311} + h^{*2}_{412} = 1$, hence we can set $h^*_{311} = \sqrt{t}$ and $h^*_{412} = \sqrt{1-t}$. If $t > 1/2$, we set $e_3 = -e^*_4$, $e_4 = e^*_3$, $e_1 = (1/\sqrt{2})(e^*_1 - e^*_2)$ and $e_2 = (1/\sqrt{2})(e^*_1 + e^*_2)$. For the new frame fields, we have $h_{311} = \sqrt{1-t}$. By virtue of (14) and (15), we have

$$\begin{aligned} \sqrt{t}\, w^*_{35} &= h^*_{5111} w^*_1 + h^*_{5112} w^*_2 , \\ \sqrt{1-t}\, w^*_{45} &= h^*_{5112} w^*_1 - h^*_{5111} w^*_2 , \\ w^*_{12} &= w^*_{34} = 0 . \end{aligned} \tag{16}$$

From the last formula of (16), we have $Dh^*_{\alpha ij} = 0$. It follows from (4) that we have

$$h^{*2}_{5111} + h^{*2}_{5112} = 2t(1-t) . \tag{17}$$

By (5), we get $h^*_{5ijk,l} = 0$. From the definition of Dh^*_{5ijk} and $w^*_{12} = 0$, h^*_{5ijk} are all constant. If necessary, taking $-e^*_5$, we may assume $h^*_{5111} \geqq 0$. By the same way as the proof of the case (ii) in the Proposition 2, we may also assume $h^*_{5112} \geqq 0$. Let $u = h^*_{5111}$ and $v = h^*_{5112}$. Thus we have $x = \Psi_{(t,u,v)}$ on some open set of R^2. Since x and $\Psi_{(t,u,v)}$ are real analytic, we have $x = \Psi_{(t,u,v)}$ on the whole plane R^2. q.e.d.

The proof of Theorem 1 now follows immediately from Propositions 1, 2 and 3. Thus (9) with (7) gives a parametrization of minimal immersions of R^2 into S^5. At the same time we have also

THEOREM 3. *Any isometric minimal immersion of R^2 into S^5 is an orbit of the action of an abelian Lie subgroup in $SO(6)$.*

5. **Proof of Theorem 2.** Let $\Psi = \Psi_{(t,u,v)}$. By definition Ψ can be represented by the following equations, for $(Y_1, Y_2, Y_3) \in C^3$,

$$\begin{cases} |Y_i|^2 = 1 , & i = 1, 2, 3 , \\ Y_1^{(\lambda_3\mu_2 - \lambda_3\mu_2)} Y_2^{(\lambda_1\mu_3 - \lambda_1\mu_3)} Y_3^{(\lambda_2\mu_1 - \lambda_2\mu_1)} = 1 , \end{cases} \tag{18}$$

where Ψ_{0j}, $0 \leqq j \leqq 5$, are the j-th component of Ψ_0 in R^6 and

$$Y_1 = \frac{\Psi_{00} + i\Psi_{01}}{T_{00} + iT_{01}} , \qquad Y_2 = \frac{\Psi_{02} + i\Psi_{03}}{T_{02} + iT_{03}} , \qquad Y_3 = \frac{\Psi_{04} + i\Psi_{05}}{T_{04} + iT_{05}} .$$

In fact, since we have $Y_i = \exp\sqrt{-1}(\lambda_i x + \mu_i y)$, (18) is directly verified. It follows from (18) that Ψ is real algebraic if and only if there exist integers m_i such that

$$\begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix} : \begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_3 & \mu_3 \end{vmatrix} : \begin{vmatrix} \lambda_2 & \mu_2 \\ \lambda_3 & \mu_3 \end{vmatrix} = m_1 : m_2 : m_3 \,. \tag{19}$$

If Ψ induces a minimal immersion of the flat torus into S^5, there is a set of points (a, c), (b, d) such that

$$\begin{aligned} \lambda_i a + \mu_i c &= p_i \,, \\ \lambda_i b + \mu_i d &= q_i \,, \qquad i = 1, 2, 3 \,, \end{aligned} \tag{20}$$

where $ad - bc \neq 0$ and $\{p_i, q_i\}$ are integers. By the direct calculation, we get (19) with $m_1 = (p_1 q_2 - q_1 p_2)$, $m_2 = (p_1 q_3 - q_1 p_3)$ and $m_3 = (p_2 q_3 - q_2 p_3)$.

We shall study the converse problem. We may assume $\lambda_j \mu_i - \lambda_i \mu_j \neq 0$ for some $i < j$. For the simplicity, we set $i = 1$, $j = 2$ and $\lambda_1 x + \mu_1 y = \theta$ and $\lambda_2 x + \mu_2 y = \tau$. Then if we have (19), we get $\lambda_3 x + \mu_3 y = -(m_3/m_1)\theta + (m_2/m_1)\tau$ and hence Ψ induces a minimal immersion of the flat torus into S^5. The proof of the former part of the Theorem 2 completes and the latter half follows from the following section. q.e.d.

THE ANOTHER PROOF OF THEOREM 2. G is the closed Lie subgroup of $SO(6)$ if and only if the condition (19) is satisfied. Therefore by the Hsiang's Theorem [1] and the Theorem 3, Theorem 2 follows.

6. The case of $u = v$ or $v = 0$. At the last section, we shall give explicitly constructed 1-parameter families of minimal immersions.

(i) In the case of $u = v$, we can get the following 1-parameter family $\Psi^+_{(t)} = \Psi_{(t, \sqrt{t(1-t)}, \sqrt{t(1-t)})}$:

$$\begin{aligned} \Psi^+_{(t)}(x, y) = \frac{1}{\sqrt{2(2-t)}} &(\exp\sqrt{-1}(\sqrt{1+k^2}\,x + \sqrt{1-k^2}\,y), \\ &\exp\sqrt{-1}(\sqrt{1-k^2}\,x + \sqrt{1+k^2}\,y), \sqrt{2(1-t)} \\ &\times \exp\sqrt{-1}(x - y)) \,, \quad \text{where} \quad k^2 = \sqrt{t(2-t)} \,. \end{aligned} \tag{21}$$

(ii) If $v = 0$, we set $\Psi^0_{(t)}(x, y) = \Psi_{(t, \sqrt{2t(1-t)}, 0)}$. Then we get

$$\begin{aligned} \Psi^0_{(t)}(x, y) = \frac{1}{\sqrt{2(1+t)}} &(\exp\sqrt{-1}(\sqrt{1-t}\,x + \sqrt{1+t}\,y), \\ &\exp\sqrt{-1}(\sqrt{1-t}\,x - \sqrt{1+t}\,y), \sqrt{2t}\exp\sqrt{-1}\sqrt{2}\,x) \,. \end{aligned} \tag{22}$$

We remark that $\Psi^+_{(t)}$ and $\Psi^0_{(t)}$ were constructed by T. Itoh [2]. If we set $\sqrt{1+k^2}\,x + \sqrt{1-k^2}\,y = \theta$ and $\sqrt{1-k^2}\,x + \sqrt{1+k^2}\,y = \tau$, then (21) is simply represented by

$$\frac{1}{\sqrt{2(2-t)}}(e^{i\theta},\ e^{i\tau},\ \sqrt{2(1-t)}e^{i\frac{\theta-\tau}{\sqrt{2t}}})\ .$$

Thus $\Psi^{+}_{(t)}$ is the algebraic minimal immersion of $S^1 \times S^1$ into S^5 if and only if $\sqrt{2t}$ is a rational number, and so there exist infinitely many algebraic minimal tori.

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