ON A PERIODIC NEUTRAL LOGISTIC EQUATION

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1. Introduction. The oscillatory and asymptotic behaviour of the positive solutions of the autonomous neutral delay logistic equation

$$\dot{N}(t) = rN(t)\left[1 - \left(\frac{N(t-\tau) + c\dot{N}(t-\tau)}{K}\right)\right] \quad \text{where} \quad \dot{N}(t) = \frac{dN(t)}{dt} \tag{1.1}$$

with $r, c, \tau, K \in (0, \infty)$ has been recently investigated in [**2**]. More recently the dynamics of the periodic delay logistic equation

$$\frac{dN(t)}{dt} = r(t)N(t)\left[1 - \frac{N(t-m\tau)}{K(t)}\right] \tag{1.2}$$

in which r, K are periodic functions of period τ and m is a positive integer is considered in [**6**]. The purpose of the following analysis is to obtain sufficient conditions for the existence and linear asymptotic stability of a positive periodic solution of a periodic neutral delay logistic equation

$$\dot{N}(t) = r(t)N(t)\left[1 - \left(\frac{N(t-m\tau) + c(t)\dot{N}(t-m\tau)}{K(t)}\right)\right] \tag{1.3}$$

in which $\dot{N}$ denotes $\dfrac{dN}{dt}$ and r, K, c are positive continuous periodic functions of period τ and m is a positive integer. For the origin and biological relevance of (1.3) we refer to [**2**].

2. Existence of a periodic solution. We define positive constants $r_0, c_0, K_0, r^0, c^0, K^0$ as follows:

$$\left.\begin{aligned} 0 < r_0 &= \inf_{t\geq 0} r(t) \leq r(t) \leq \sup_{t\geq 0} r(t) = r^0, \\ 0 < K_0 &= \inf_{t\geq 0} K(t) \leq K(t) \leq \sup_{t\geq 0} K(t) = K^0, \\ 0 < c_0 &= \inf_{t\geq 0} c(t) \leq c(t) \leq \sup_{t\geq 0} c(t) = c^0. \end{aligned}\right\} \tag{2.1}$$

The literature concerned with the existence of periodic solutions of nonlinear neutral differential equations with periodic coefficients is scarce (see for instance the books by El'sgol'ts and Norkin [**1**] and Kolmanovskii and Nosov [**4**]).

We shall first consider the periodic ordinary differential equation

$$\dot{u}(t) = r(t)u(t)\left[1 - \left(\frac{u(t) + c(t)\dot{u}(t)}{K(t)}\right)\right], \tag{2.2}$$

which can be written as

$$\frac{du(t)}{dt} = r(t)u(t)\left[\frac{K(t) - u(t)}{K(t) + c(t)r(t)}\right]. \tag{2.3}$$

One can see from (2.3) that positive solutions of (2.3) satisfy

$$u(t)\left[\frac{r_0K_0}{K^0+c^0r^0}-\frac{r^0}{K_0+c_0r_0}u(t)\right]\leq\frac{du}{dt}\leq u(t)\left[\frac{r^0K^0}{K_0+c_0r_0}-\frac{r_0}{K^0+c^0r^0}u(t)\right] \tag{2.4}$$

and hence

$$\frac{r^0}{K_0+c_0r_0}[\alpha_0-u(t)]u(t)\leq\frac{du}{dt}\leq\frac{r_0}{K^0+c^0r^0}[\alpha^0-u(t)]u(t), \tag{2.5}$$

where

$$\alpha_0=\left(\frac{r_0}{r^0}\right)\frac{K_0+c_0r_0}{K^0+c^0r^0},\qquad \alpha^0=\left(\frac{r^0}{r_0}\right)\frac{K^0+c^0r^0}{K_0+c_0r_0}. \tag{2.6}$$

We can conclude from (2.5) that

$$\alpha_0<u(0)<\alpha^0\Rightarrow\alpha_0\leq u(t)\leq\alpha^0\quad\text{for all}\quad t\geq 0. \tag{2.7}$$

A positive solution $p(t)$ of (2.3) is said to be globally asymptotically stable if every other positive solution $x(t)$ satisfies

$$\lim_{t\to\infty}|x(t)-p(t)|=0. \tag{2.8}$$

LEMMA. *Suppose r, K, c are strictly positive continuous periodic functions of period τ. Then* (2.3) *has a globally asymptotically stable periodic solution of period τ.*

Proof. The existence of at least one positive periodic solution of (2.3) is a consequence of (2.7) and Theorem 15.3 on p. 164 of Yoshizawa [**5**]. We shall show that if u and v are any two positive solutions of (2.3) then

$$\lim_{t\to\infty}|u(t)-v(t)|=0. \tag{2.9}$$

The uniqueness and the global asymptotic stability of the periodic solution will then follow from (2.9). We define w such that

$$w(t)=\ln[u(t)]-\ln[v(t)] \tag{2.10}$$

and derive

$$\frac{dw(t)}{dt}=-\left(\frac{r(t)}{K(t)+c(t)r(t)}\right)[u(t)-v(t)]. \tag{2.11}$$

We observe

$$w(t)=\ln[u(t)]-\ln[v(t)]=[u(t)-v(t)]\left(\frac{1}{\theta(t)}\right), \tag{2.12}$$

where $\theta(t)$ lies between $u(t)$ and $v(t)$ and hence $\theta(t)$ is strictly positive and is bounded away from zero. Together, (2.11) and (2.12) imply

$$\frac{d}{dt}[w^2(t)]=-2\left[\frac{r(t)\theta(t)}{K(t)+c(t)r(t)}\right]w^2(t),$$

from which one can derive

$$\lim_{t\to\infty} w(t) = 0,$$

and so (2.9) follows. This completes the proof.

3. Periodic neutral logistic equation. Let $p(t)$ denote the unique positive periodic solution of (2.3). It can be verified that p satisfies

$$\frac{dp(t)}{dt} = r(t)p(t)\left[1 - \left(\frac{p(t) + c(t)\dot{p}(t)}{K(t)}\right)\right] \tag{3.1}$$

and hence p also satisfies

$$\dot{p}(t) = r(t)p(t)\left[1 - \left(\frac{p(t-m\tau) + c(t)\dot{p}(t-m\tau)}{K(t)}\right)\right]. \tag{3.2}$$

Thus the existence of a periodic solution of (1.3) is resolved easily. Furthermore we note that

$$K_0 \le p(t) \le K^0 \tag{3.3}$$

and

$$-d^0 \le \dot{p}(t) \le d^0, \tag{3.4}$$

where

$$d^0 = \frac{r^0K^0}{K_0 + c_0r_0} K^0(K^0 - K_0). \tag{3.5}$$

We let

$$x(t) = \ln[N(t)] - \ln[p(t)], \tag{3.6}$$

so that

$$N(t) = p(t)e^{x(t)}. \tag{3.7}$$

We derive from (1.3) that x is governed by

$$\begin{aligned} \dot{x}(t) &= \frac{\dot{N}(t)}{N(t)} - \frac{\dot{p}(t)}{p(t)} \\ &= -a(t)[e^{x(t-m\tau)} - 1] - b(t)e^{x(t-m\tau)}\dot{x}(t-m\tau), \end{aligned} \tag{3.8}$$

where

$$\left.\begin{aligned} \alpha(t) &= \frac{r(t)}{K(t)}[p(t-m\tau) + c(t)\dot{p}(t-m\tau)] \\ b(t) &= \frac{r(t)}{K(t)} c(t)p(t-m\tau). \end{aligned}\right\} \tag{3.9}$$

The linear variational system corresponding to the solution p of (1.3) can be obtained from (3.8) by neglecting the nonlinear terms in (3.8). This leads to the linear variational equation

$$\dot{y}(t) = -a(t)y(t-m\tau) - b(t)\dot{y}(t-m\tau). \tag{3.10}$$

Definition. The positive periodic solution p of (1.3) is said to be locally asymptotically stable in the C^1 metric if every solution of (3.10) satisfies

$$\lim_{t\to\infty} y(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \dot{y}(t) = 0. \tag{3.11}$$

If the convergence in (3.11) is exponential, then the solution p of (1.3) is said to be exponentially asymptotically stable in the C^1 metric.

We remark that there are no routine techniques developed in the literature for the verification of asymptotic stability in the C^1 metric for neutral differential equations.

Theorem. *Suppose the delay* $m\tau$ *and the neutral coefficient* $c(t)$ *are small enough to satisfy*

$$\left.\begin{aligned} c^0 d^0 < K_0, \qquad a^0 m\tau + b^0 < 1, \\ \frac{a^0(a^0 m\tau + b^0)}{1-(a^0 m\tau + b^0)} < a_0, \end{aligned}\right\} \tag{3.12}$$

where

$$\left.\begin{aligned} a^0 &= \frac{r^0}{K_0}[K^0 + c^0 d^0], \qquad a_0 = \frac{r_0}{K^0}[K_0 - c^0 d^0], \\ b_0 &= \frac{r_0}{K^0} c_0 K_0, \qquad b^0 = \frac{r^0}{K_0} c^0 K^0. \end{aligned}\right\} \tag{3.13}$$

Then the positive periodic solution p of (1.3) *is* (*locally*) *exponentially asymptotically stable in the* C^1 *metric.*

Proof. It is enough to show that the trivial solution of (3.10) is exponentially asymptotically stable in the C^1 metric. We first rewrite (3.10) in the form

$$\dot{y}(t) = -a(t)y(t) + a(t)\int_{t-m\tau}^{t} \dot{y}(\xi)\,d\xi - b(t)\dot{y}(t-m\tau). \tag{3.14}$$

By the variation of constants formula we have from (3.14)

$$y(t) = y(t_0)\exp\left[-\int_{t_0}^{t} a(u)\,du\right] + \int_{t_0}^{t}\left[\left(a(s)\int_{s-m\tau}^{s}\dot{y}(\xi)\,d\xi\right) - b(s)\dot{y}(s-m\tau)\right]\exp\left[-\int_{t_0}^{t} a(u)\,du\right]\exp\left[\int_{t_0}^{s} a(u)\,du\right]ds. \tag{3.15}$$

Supplying y from (3.15) in (3.14),

$$\begin{aligned}\dot{y}(t) = &-a(t)y(t_0)\exp\left[-\int_{t_0}^{t} a(u)\,du\right]\\ &-a(t)\left[\int_{t_0}^{t}\left(a(s)\int_{s-m\tau}^{s}\dot{y}(\xi)\,d\xi - b(s)\dot{y}(s-m\tau)\right)\right.\\ &\left.\times\exp\left[-\int_{t_0}^{t} a(u)\,du\right]\exp\left[\int_{t_0}^{s} a(v)\,dv\right]ds\right]\\ &+a(t)\int_{t-m\tau}^{t}\dot{y}(\xi)\,d\xi - b(t)\dot{y}(t-m\tau), \end{aligned} \tag{3.16}$$

and estimating the terms of (3.16),

$$\begin{aligned}|\dot{y}(t)| \le\ & a^0\,|y(t_0)|\exp\left[-\int_{t_0}^{t} a(u)\,du\right]\\ &+a^0(a^0m\tau+b^0)\int_{t_0}^{t}\sup_{u\le s}\left(|\dot{y}(u)|\exp\left[\int_{t_0}^{u} a(v)\,dv\right]\right)\exp\left[-\int_{t_0}^{t} a(u)\,du\right]ds\\ &+(a^0m\tau+b^0)\sup_{s\le t}|\dot{y}(s)|. \end{aligned} \tag{3.17}$$

Rearranging the terms in (3.17),

$$\begin{aligned}&[1-(a^0m\tau+b^0)]\sup_{s\le t}\left(|\dot{y}(s)|\exp\left[\int_{t_0}^{s} a(u)\,du\right]\right)\\ &\qquad\le a^0\,|y(t_0)| + a^0(a^0m\tau+b^0)\int_{t_0}^{t}\left(\sup_{u\le s}|\dot{y}(u)|\exp\left[\int_{t_0}^{u} a(v)\,dv\right]\right)ds. \end{aligned} \tag{3.18}$$

By the Gronwall-Bellman inequality, from (3.18) we derive

$$\sup_{s\le t}\left(|\dot{y}(s)|\exp\left[\int_{t_0}^{s} a(u)\,du\right]\right) \le \left[\frac{a^0\,|y(t_0)|}{1-(a^0m\tau+b^0)}\right]\exp\left(\frac{a^0(a^0m\tau+b^0)}{1-(a^0m\tau+b^0)}(t-t_0)\right). \tag{3.19}$$

It is easily seen from (3.19) that

$$|\dot{y}(t)| \le \left[\frac{a^0\,|y(t_0)|}{1-(a^0m\tau+b^0)}\right]\exp\left[\left(\frac{a^0(a^0m\tau+b^0)}{1-(a^0m\tau+b^0)} - a_0\right)(t-t_0)\right], \tag{3.20}$$

which implies by (3.12) that

$$\lim_{t\to\infty}\dot{y}(t) = 0, \tag{3.21}$$

the convergence in (3.21) being exponential. The fact that

$$\lim_{t\to\infty} y(t) = 0$$

follows from (3.21) and (3.10). This completes the proof.

We conclude with the following observation. If $c(t)=0$ and τ is sufficiently small then (1.3) becomes (1.2) and (1.2) has a globally attracting positive periodic solution (for details see [6]). The authors believe that if $|c(t)|$ is sufficiently small for all $t \geq 0$ then (1.3) will have a globally attracting periodic solution; we note that even in the autonomous case (see [2]) it has not been possible to establish the above intuitively expected global attractivity result. This aspect of our conjecture is open for further investigation.

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