ON A POINT-SEGMENT DECOMPOSITION OF E^3 DEFINED BY McAULEY

LAWRENCE O. CANNON

1. **Introduction.** Two examples have recently been given of upper semicontinuous decompositions of E^3 into points and straight line segments. The first, by R. H. Bing, was described at a Summer Institute held at the University of Georgia in the summer of 1961 [3] and also published in 1962 [4]. The second example was announced in 1964 by L. F. McAuley [7] and presented again at the Topology Seminar at the University of Wisconsin in 1965 [8]. It has been conjectured that the decomposition spaces associated with both of these decompositions are not homeomorphic with E^3. In this paper it is shown that the decomposition space associated with McAuley's decomposition is homeomorphic with E^3.

Since McAuley's example is now available [8] we shall not reproduce his construction but shall use his notation and his figures whenever convenient. The basic procedure for constructing a homeomorphism between the decomposition space and E^3 is an iterated "shrinking" process of the type suggested first by Bing [2] and used for related work in [5] and [6]. The burden of this paper is a description of the shrinking process.

2. **Construction.** McAuley defines what he calls an *R_7-cube* which is topologically a 3-cell with seven handles. In the interior of the first R_7-cube A there are six more, $A_1, A_2, \cdots, A_6$, embedded as indicated in [8, Figures 3, 4, 5, 8]. At the second stage, six more R_7-cubes are embedded in each of the A_i and the process is continued. At the nth stage there are 6^n mutually exclusive R_7-cubes whose union McAuley denotes by A_n^*. Each component of the set $M=\bigcap_1^\infty A_n^*$ is a straight line segment, and the decomposition G consists of the components of M and the points of E^3-M.

It will be convenient to introduce some additional notation. We use *admissible n-sequences* to subscript the cells with handles used to define M. An admissible n-sequence, denoted by $n\alpha$, is an n-termed sequence each of whose terms is one of the digits $1, 2, \cdots, 6$. We also use $n\alpha i$ to designate the admissible $(n+1)$-sequence whose first n terms are $n\alpha$ and whose last term is i. With this convention, the six R_7-cubes in the interior of $A_{n\alpha}$ are $A_{n\alpha 1}, A_{n\alpha 2}, \cdots, A_{n\alpha 6}$. We also use

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$\sum A_{n\alpha}$ in place of A_n^* to denote the union of the 6^n R_7-cubes at the nth stage.

Each $A_{n\alpha}$ is topologically a cell with seven handles. In considering the nature of the embedding of the $A_{n\alpha i}$ in each $A_{n\alpha}$ it is helpful to examine the role of the handles. For this purpose we show the embedding in Figure 1. This figure may be compared with Figure 8 of [**8**]. Note that while the "straightness" is lost, it is much easier to picture the iteration of the construction. It may be noted that the diameter of the handles becomes very small in relation to the overall diameter, the "length" of the $A_{n\alpha}$. It should also be clear from Figure 1 that the construction may be done so that the handles at every stage following the first lie entirely in the spheres which have been expanded to show the detail of the linking of the handles. The reader in comparing our Figure 1 with McAuley's Figure 8 [**8**] will observe some differences in the linking of the top handles. McAuley has shown a much simpler linking than actually occurs in the construction of his example, as may be seen by examining [**8**, Figure 4]. The nature of the top linking is not important for the shrinking described in this paper. The exact form of the top linking may, however, be of considerable importance for the questions raised at the end of this paper.

3. **Shrinking.** There is a natural distinction between "upper" and "lower" handles of $A_{n\alpha}$ assuming the viewpoint of the reader looking at Figure 1. If $h(A_{n\alpha})$ is a homeomorphic image of $A_{n\alpha}$, we shall refer to the upper (lower) handles of $h(A_{n\alpha})$ as the image under the same homeomorphism of the upper (lower) handles of $A_{n\alpha}$. A set of planar disks $P_1, P_2, \cdots, P_k$ is said to be in *standard position relative to* $h(A_{n\alpha})$ if

(1) each component of $P_i \cap h(A_{n\alpha})$ is a disk in $h(A_{n\alpha})$ which separates the upper handles of $h(A_{n\alpha})$ from the lower handles of $h(A_{n\alpha})$ in $h(A_{n\alpha})$,

(2) no component of $P_i \cap h(A_{n\alpha})$ separates two components of $P_j \cap h(A_{n\alpha})$ in $h(A_{n\alpha})$ if $i \neq j$, and

(3) the planes are ordered from the upper to the lower handles of $h(A_{n\alpha})$ in the sense that $P_i \cap h(A_{n\alpha})$ separates the upper handles of $h(A_{n\alpha})$ from $P_j \cap h(A_{n\alpha})$ in $h(A_{n\alpha})$ if $1 \leqq i < j \leqq k$.

LEMMA 1. *Suppose P_a, P_b, P_c, P_d are planar disks in standard position relative to $A_{n\alpha}$. Then there is a homeomorphism h of E^3 onto itself such that*

(1) *h is the identity on $E^3 - \operatorname{Int} A_{n\alpha}$,*

(2) *h is the identity between P_b and P_c, and*

(3) *h takes each of the thirty-six components of $\sum A_{(n+2)\alpha}$ onto a set which intersects at most one of the two disks P_a and P_d.*

PROOF. The proof of this lemma is contained in Figures 2, 3 and 4. We shall simply provide an outline to assist the reader in following the steps shown in these figures. With one exception we avoid having to picture handle linking by showing only the spheres containing the linking. In all cases the small spheres indicate linking handles as shown in the detail portions of Figure 1. Figure 2 shows the four disks P_a, P_b, P_c, P_d in standard position relative to $A_{n\alpha}$. It is clear that we may assume that the disks are also in standard position relative to $A_{m\alpha}$ for all $A_{m\alpha}$ contained in $A_{n\alpha}$.

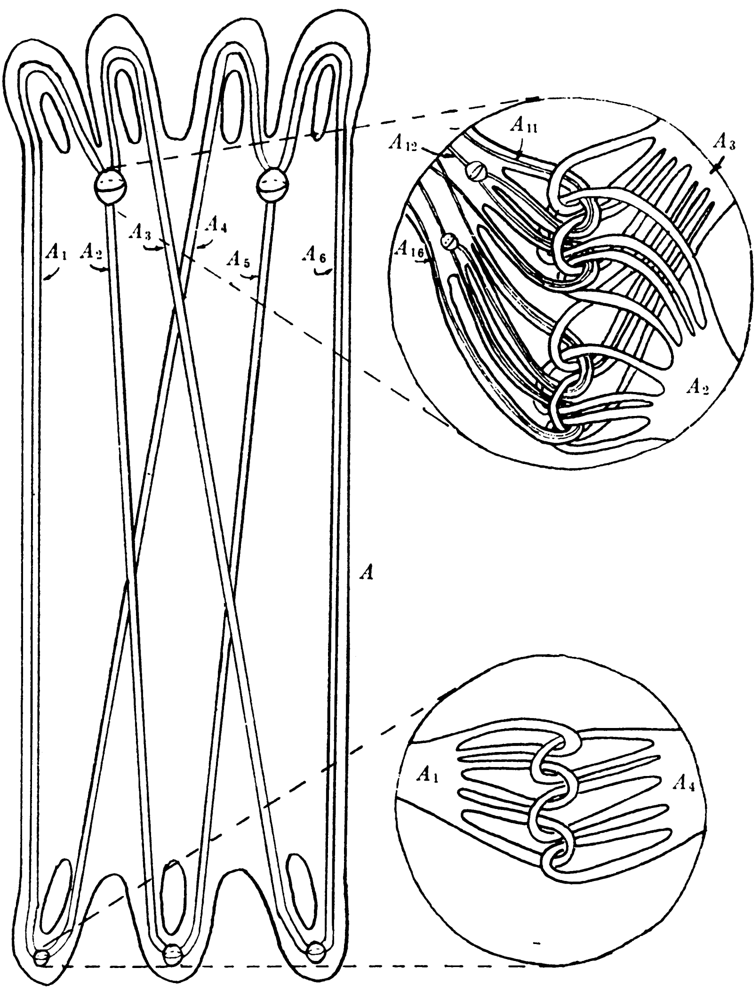

FIGURE 1

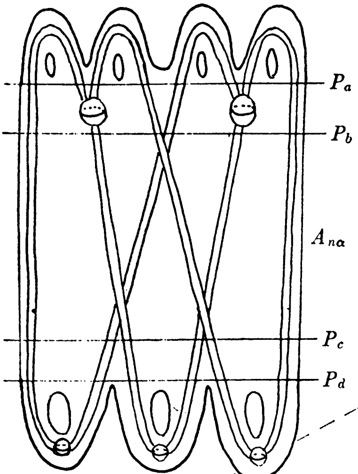

FIGURE 2

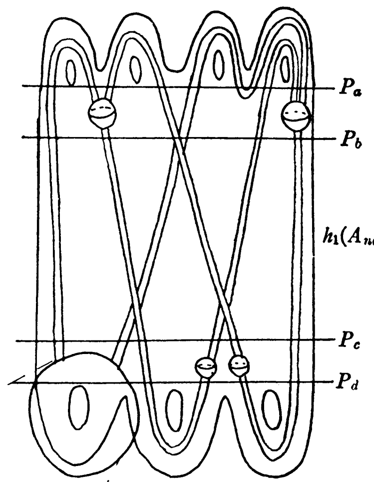

FIGURE 3

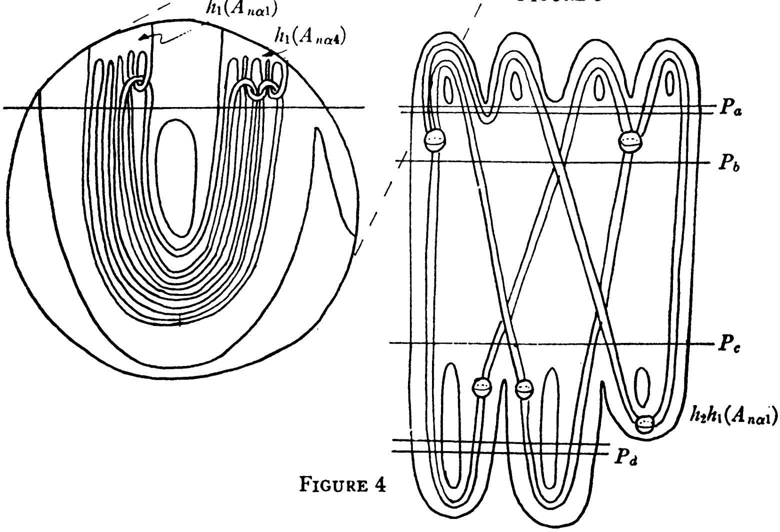

FIGURE 4

The homeomorphism of the lemma is the composition of two homeomorphisms h_1 and h_2. The effect of the first homeomorphism is shown in Figure 3. As may be seen from the figure, $h_1(A_{n\alpha 2})$ and $h_1(A_{n\alpha 6})$ intersect P_d but not P_a; $h_1(A_{n\alpha 3})$ and $h_1(A_{n\alpha 5})$ intersect P_a but not P_d. It may be seen from the detail section of the lower handle linking in Figure 1 or in [**8**, Figure 8], that each lower handle of $A_{n\alpha 1}$ links either one or two lower handles of $A_{n\alpha 4}$. The homeomorphism h_1 takes advantage of this to "shorten" one lower handle of $A_{n\alpha 1}$ and two lower handles of $A_{n\alpha 4}$. In the lower detail portion of Figure 1, these would be the top handle on the left and the bottom two on the right. This necessitates some stretching of the other handles of $A_{n\alpha 1}$ and $A_{n\alpha 4}$ as indicated in the detail section of Figure 3. Thus $h_1(A_{n\alpha 1})$ and $h_1(A_{n\alpha 4})$ intersect both P_a and P_d, but only two lower handles of $h_1(A_{n\alpha 1})$ and only one lower handle of $h_1(A_{n\alpha 4})$ intersect P_d. This first homeomorphism h_1 is to be such that P_b, P_c, P_d are in standard position relative to $h_1(A_{n\alpha 2})$ and $h_1(A_{n\alpha 6})$, and P_a, P_b, P_c are in standard position relative to $h(A_{n\alpha i})$, $i=1$, 3, 4, 5. It is obvious that all this may be accomplished without moving any point between P_b and P_c.

We describe the effect of the homeomorphism h_2 in detail only on the interior of $h_1(A_{n\alpha 1})$. This homeomorphism is the identity outside $h_1(A_{n\alpha 1}\cup A_{n\alpha 4})$ and between P_b and P_c, and its action in $h_1(A_{n\alpha 4})$ is very similar to its action in $h_1(A_{n\alpha 1})$. In Figure 4 we have drawn $h_1(A_{n\alpha 1})$ relatively straight and have indicated the two components of $P_a\cap h_1(A_{n\alpha 1})$ and two subdisks of P_d which are intersected by $h_1(A_{n\alpha 1})$. Before applying h_2, the embedding of the $h_1(A_{n\alpha 1i})$ in $h_1(A_{n\alpha 1})$ is the same as the embedding of the $A_{n\alpha i}$ in $A_{n\alpha}$ shown in Figure 1 and Figure 2. After applying h_2, which just moves three of the spheres containing linking through handles of $h_1(A_{n\alpha 1})$, the sets $h_2h_1(A_{n\alpha 12})$, $h_2h_1(A_{n\alpha 13})$, $h_2h_1(A_{n\alpha 14})$ and $h_2h_1(A_{n\alpha 16})$ intersect P_a and not P_d; the sets $h_2h_1(A_{n\alpha 11})$ and $h_2h_1(A_{n\alpha 15})$ intersect P_d and not P_a. A similar transformation is effected in $h_1(A_{n\alpha 4})$ but may be done even more simply because only one lower handle of $h_1(A_{n\alpha 4})$ intersects P_d. An important observation is that the homeomorphism of the lemma, $h=h_2h_1$, may be taken to be such that P_a, P_b, P_c, P_d are in standard position relative to each of the thirty-six components of $h(\sum A_{(n+2)\alpha})$ in $A_{n\alpha}$ (although no component intersects more than three of P_a, P_b, P_c, P_d).

The purpose of Lemma 2 is the establishment of Armentrout's Condition B, from which it follows that McAuley's decomposition space is homeomorphic with E^3. See [1, §4].

Lemma 2. *If U is an open set containing M and ϵ is a positive number, there is a homeomorphism h of E^3 onto E^3 such that*

(1) *h is the identity on E^3-U and*

(2) *for any element g of the decomposition G,* diam $[h(g)]<\epsilon$.

Proof. Since U is open and contains the nondegenerate elements of G, there is a stage j such that $(\sum A_{j\alpha})\subset U$. There is an integer n greater than j and a finite set of planar disks $P_1, P_2, \cdots, P_k$ which are in standard position relative to $A_{m\alpha}$ for $m\geqq n$ and such that each component of $(\sum A_{n\alpha})-(\bigcup_{i=1}^{k} P_i)$ has diameter less than $\epsilon/4$.

The shrinking required in Lemma 2 is virtually the same as is done in the proof of Lemma 2.1 of [**5**]. If follows from Lemma 1 that there is a homeomorphism h_1 of E^3 onto itself which is fixed outside $\sum A_{n\alpha}$ and between P_2 and P_{k-1} which takes each component of $\sum A_{(n+2)\alpha}$ onto a set which intersects at most one of P_1 and P_k. Lemma 1 may be applied again to each component of $h_1(\sum A_{(n+2)\alpha})$, using P_1, P_2, P_{k-2}, P_{k-1} or P_2, P_3, P_{k-1}, P_k for the disks P_a, P_b, P_c, P_d of Lemma 1. In a finite number of steps we reach a stage, say s, at which, using the composed homeomorphism h, each component of $h(\sum A_{s\alpha})$ intersects at most three of $P_1, P_2, \cdots, P_k$ and hence has diameter less than ϵ. Since every nondegenerate element of G is a subset of $\sum A_{s\alpha}$, the second part of the lemma is satisfied, and the first part is a consequence of the fact that h is fixed outside $\sum A_{n\alpha}$.

4. **Questions.** McAuley's construction may be modified in several ways to obtain other point-segment decompositions of E^3. In constructing his R_7-cubes, McAuley used two points near the top of a disk and three points near the bottom of the same disk (see [**8**, Figure 1]). We call this a *McAuley* (2, 3)-*construction.* Basically the same construction can be carried out using more points either at the top or at the bottom. The handle of any defining set never links more than two handles of any other defining set at the same stage.

We did not have to use the linking of the upper handles at all in our shrinking. A process very similar to the shrinking done in Lemmas 1 and 2 can be done for a McAuley (2, 4)-construction, again without using the upper handle linking. We have been unable to do the same thing for a McAuley (2, 5)-construction without using the upper handle linking. Shrinking is possible for a McAuley (3, 3)-construction but is more lengthy and requires strong use of both upper and lower handle linking. Three stages are required to accomplish the shrinking done in two stages in our Lemma 1, but the decomposition space is still topologically E^3. Among several obvious questions, we pose two:

QUESTION 1. Are there integers m and n such that the McAuley (m, n)-construction yields a decomposition G for which the decomposition space E^3/G is not homeomorphic with E^3?

QUESTION 2. How are the techniques and conclusion of this paper related, if at all, to Bing's point-segment decomposition [**3**], [**4**]?

ADDED IN PROOF. Mrs. Edythe P. Woodruff has communicated to the author a method for shrinking a McAuley (3, 3)-construction in only two stages using both upper and lower handle linking.

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Utah State University