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On a posteriori Error Estimation in the Numerical Solution of Systems of Ordinary Differential Equations

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1. Introduction

Consider the boundary value problem

$$\frac{dx}{dt} = X(x, t), \qquad a \leqq t \leqq b, \tag{1.1}$$

$$f[x] = 0, \tag{1.2}$$

where x and $X(x, t)$ are real n-vectors, f is an operator from $D \subset C[J]$ into R^n which is continuously Fréchet differentiable in D, and $C[J]$ is the space of n-vector functions continuous on $J = [a, b]$.

In our previous paper [2], replacing (1.1) with an equivalent system of integral equations, we obtained *a posteriori* error estimates of continuous approximate solutions of (1.1), (1.2). In those estimates the fundamental matrix of a linear homogeneous system of differential equations plays an important role and its inverse matrix is also required. In many practical applications, however, exact fundamental matrices and their exact inverses are not available, so that the estimates are not always applicable.

The object of this paper is to give error estimates of approximate solutions in terms of approximate fundamental matrices and their approximate inverses. In Section 3 error estimates are obtained in the case where approximate fundamental matrices are continuous and also in the case where they are continuously differentiable. The results are illustrated with a numerical example.

In Section 4 we treat the case where the boundary condition depends on the fundamental matrices of the first variation equation of (1.1).

2. Notations and preliminaries

Let R^n denote a real n-space with any norm $\|\cdot\|$ and let $C[J]$ be the Banach space of all real n-vector functions $x(t)$ continuous on the interval $J = [a, b]$ with the norm $\|x\|_c = \sup_{t \in J} \|x(t)\|$. For any fixed $t_0 \in J$ let

$$C_0[J] = \{x \in C[J] \mid x(t_0) = 0\}.$$

Then $B_0 = C_0[J] \times R^n$ is a Banach space with the norm

$$\|y\|_b = \max(\|u\|_c, \|e\|) \qquad \text{for} \quad y = (u, e) \in B_0.$$

Let $M[J]$ denote the Banach space of all real $n \times n$ matrix functions $A(t)$ continuous on J with the norm $\|A\|_c = \sup_{t \in J} \|A(t)\|$.

The identity operator and the unit matrix are denoted by the same symbol I. The sum $F + G$ and the product FG of two operators F and G are defined in the usual manner.

For two Banach spaces X and Y, we denote by $L(X, Y)$ the set of all bounded linear operators from X into Y. When the operator $F: D \subset X \to Y$ is Fréchet differentiable at $x \in D$, we denote by $F'(x)$ the Fréchet derivative of F at x. A linear operator $K: Y \to X$ is said to be invertible if the equation $Ky = x$ has a unique solution $y \in Y$ for each $x \in X$. By the argument similar to the one used in [1, p. 50] we can show the following

LEMMA 1. *Let $L: X \to Y$ be a linear operator and $K: Y \to X$ be an invertible linear operator. Then L is invertible if*

$$\|I - LK\| < 1 \tag{2.1}$$

or

$$\|I - KL\| < 1. \tag{2.2}$$

Let $A = (a_1, a_2, \ldots, a_n) \in M[J]$ and for any $T \in L(C[J], C[J])$ define $TA \in M[J]$ by

$$TA = (Ta_1, Ta_2, \ldots, Ta_n).$$

Then we have

$$\|TA\|_c \leqq \|T\|_c \|A\|_c, \tag{2.3}$$

so that T can be considered to be an element of $L(M[J], M[J])$.

Let Ω' be a domain of the tx-space intercepted by two hyperplanes $t = a$ and $t = b$ such that the cross sections R_a and R_b at $t = a$ and $t = b$ make an open set in each hyperplane. Put $\Omega = R_a \cup \Omega' \cup R_b$ and let

$$D = \{x \in C[J] \mid (t, x(t)) \in \Omega \qquad \text{for all} \quad t \in J\}.$$

Let us consider the system of differential equations

$$\frac{dx}{dt} = X(x, t), \qquad t \in J \tag{2.4}$$

with the boundary condition

(2.5) $$f[x]=0,$$

where x and $X(x, t)$ are n-vectors, $X(x, t)$ is continuous in Ω and continuously differentiable with respect to x in Ω, and the operator $f: D\to R^n$ is continuously Fréchet differentiable in D. We assume that (2.4) has at least one solution in D.

Let $Q: D\to C_0[J]$ and $F: D\to B_0$ be the operators defined by

(2.6) $$Qx=x(t)-x(t_0)-\int_{t_0}^{t} X(x(s), s)ds \qquad \text{for} \quad x\in D,$$

(2.7) $$Fx=(Qx, f[x]) \qquad \text{for} \quad x\in D.$$

Then the boundary value problem (2.4), (2.5) is equivalent to the problem of finding a solution $x\in D$ of the equation

(2.8) $$Fx=0.$$

Let $X_x(x, t)$ be the Jacobian matrix of $X(x, t)$ with respect to x. Then $F'(x)h$ $(x\in D)$ is given by

(2.9) $$F'(x)h=(Q'(x)h, f'(x)h) \qquad \text{for} \quad h\in C[J],$$

where

(2.10) $$Q'(x)h=h(t)-h(t_0)-\int_{t_0}^{t} X_x(x(s), s)h(s)ds.$$

Let $L\in L(C[J], B_0)$ be the operator independent of x which approximates $F'(x)$ and is defined by

(2.11) $$Lh=(Ph, l[h]) \qquad \text{for} \quad h\in C[J],$$

where

(2.12) $$Ph=h(t)-h(t_0)-\int_{t_0}^{t} A(s)h(s)ds,$$

$A\in M[J]$ and $l\in L(C[J], R^n)$.

Let $\Phi(t)$ be the fundamental matrix of the system

$$\frac{dy}{dt}=A(t)y$$

with $\Phi(t_0)=I$ and put

(2.13) $$G=l[\Phi].$$

We denote by $\Phi_I(t)$ the inverse matrix of $\Phi(t)$ and put

(2.14) $$S = \Phi G^{-1}$$

if G is nonsingular.

Let E, E_1, S_1 and H be the elements of $L(C[J], C[J])$ defined by

(2.15) $$Eh = \int_{t_0}^{t} \Phi(t)\Phi_I(s)h(s)ds \qquad \text{for} \quad h \in C[J],$$

(2.16) $$E_1 = I + EA, \quad S_1 = I - Sl, \quad H = S_1 E$$

and let $T: D \to C[J]$ and $T_1: D \to L(C[J], C[J])$ be the operators such that

(2.17) $$Tx = X(x(t), t) - A(t)x(t) \qquad \text{for} \quad x \in D,$$

(2.18) $$T_1(x)h = \{X_1[x](t) - A(t)\}h(t) \qquad \text{for} \quad x \in D, \ h \in C[J],$$

where

(2.19) $$X_1[x] = X_x(x(t), t) \qquad \text{for} \quad x \in D.$$

In our previous paper [2] we have shown the following results: *L has an inverse operator L_I if and only if* $\det G \neq 0$. *If G is nonsingular, then*

(2.20) $$L_I y = S_1 E_1 u + Se \qquad \text{for} \quad y = (u, e) \in B_0,$$

(2.21) $$Kx = K_1 x \qquad \text{for} \quad x \in D,$$

where K and K_1 are the operators from D into $C[J]$ defined by

(2.22) $$K = I - L_I F,$$

(2.23) $$K_1 = HT + S(l - f).$$

THEOREM 1. *Let $x^{(0)} \in D$ be an approximate solution of* (2.8) *and suppose there exist an operator L, a positive number δ and nonnegative constants η, κ ($\kappa < 1$) such that*

(i) $\det G \neq 0$,

(ii) $D_\delta = \{x \in C[J] \mid \|x - x^{(0)}\|_c \leqq \delta\} \subset D$,

(iii) $\|Kx - Ky\|_c \leqq \kappa \|x - y\|_c$ *for all* $x, y \in D_\delta$,

(iv) $\|L_I F x^{(0)}\|_c \leqq \eta$,

(v) $\lambda = \eta/(1 - \kappa) \leqq \delta$.

Then the sequence $\{x^{(k)}\}$ defined by

(2.24) $$x^{(k+1)} = Kx^{(k)} \qquad (k = 0, 1, \ldots)$$

converges to $\hat{x} \in D_\delta$ *as* $k \to \infty$. $\hat{x}$ *is the unique solution of* (2.8) *in* D_δ, *and*

(2.25) $$\|\hat{x} - x^{(k)}\|_c \leqq \kappa^k \lambda \qquad (k = 0, 1, \ldots).$$

REMARK. Let κ be a constant satisfying

(2.26) $$\|H\|_c \mu_1 + \|S\|_c \mu_2 \leqq \kappa < 1,$$

where μ_1 and μ_2 are constants such that

(2.27) $$\|T_1(x)\|_c \leqq \mu_1 \qquad \text{for all} \quad x \in D_\delta,$$

(2.28) $$\|f'(x) - I\| \leqq \mu_2 \qquad \text{for all} \quad x \in D_\delta.$$

Then the condition (iii) is satisfied.

In this theorem the matrices $\Phi(t)$ and $\Phi_I(t)$ play important roles. But in practical applications we are often obliged to use the approximate fundamental matrices. In the next section we study how to modify this theorem in such a case.

3. Approximate fundamental matrices

Let $\tilde{\Phi}$ and $\tilde{\Phi}_I$ be the matrices that approximate Φ and Φ_I respectively. For any operator $R = R(\Phi, \Phi_I)$ depending on Φ and Φ_I we denote by $\tilde{R}$ the operator $R(\tilde{\Phi}, \tilde{\Phi}_I)$. Put

(3.1) $$\gamma(t) = \tilde{\Phi}(t)\tilde{\Phi}_I(t),$$

(3.2) $$\gamma_1(t) = I - \gamma(t),$$

(3.3) $$\rho = \max(b - t_0, t_0 - a).$$

We consider the following two cases for practical applications.

Case 1. $\tilde{\Phi}(t)$ and $\tilde{\Phi}_I(t)$ are continuous on J.

Case 2. $\tilde{\Phi}(t)$ and $\tilde{\Phi}_I(t)$ are continuously differentiable on J.

3.1. Case 1

Put

(3.4) $$r(t) = \tilde{\Phi}(t) - I - \int_{t_0}^{t} A(s)\tilde{\Phi}(s)ds,$$

(3.5) $$r_1(t) = \tilde{\Phi}_I(t) - I + \int_{t_0}^{t} \tilde{\Phi}_I(s)A(s)ds,$$

(3.6) $$r_2(t) = \gamma_1(t) + \tilde{\Phi}(t) r_1(t).$$

Let R, $R_1 \in L(C[J], C[J])$ and $R_2: D \to C[J]$ be defined by

(3.7) $$Rh = r_2(t)\int_{t_0}^{t} h(s)ds - \tilde{\Phi}(t)\int_{t_0}^{t} r_1(s)h(s)ds \qquad \text{for} \quad h \in C[J],$$

(3.8) $$R_1 h = r_2 h(t_0) + RAh \qquad \text{for} \quad h \in C[J],$$

(3.9) $$R_2 x = r_2 x(t_0) + RX[x] \qquad \text{for} \quad x \in D,$$

where $X[x] = X(x(t), t)$. Then we have the following

LEMMA 2. *$\tilde{L}_I$ exists and is invertible if*

(3.10) $$\det \tilde{G} \neq 0,$$

(3.11) $$\|\tilde{G}^{-1}\| \, \|l\| \, \|r\|_c \exp(\rho \|A\|_c) < 1,$$

(3.12) $$\|\tilde{S}_1 R_1\|_c < 1.$$

PROOF. By (3.10) $\tilde{L}_I$ can be defined.

Let $\alpha(t) = \tilde{\Phi}(t) - \Phi(t)$. Since

$$\Phi(t) - I - \int_{t_0}^{t} A(s)\Phi(s)ds = 0,$$

by (3.4) we have

$$\alpha(t) = r(t) + \int_{t_0}^{t} A(s)\alpha(s)ds,$$

which yields

$$\|\alpha(t)\| \leqq \|r\|_c + \left| \int_{t_0}^{t} \|A\|_c \|\alpha(s)\| ds \right|.$$

By Gronwall's inequality we have

(3.13) $$\|\tilde{\Phi} - \Phi\|_c \leqq \|r\|_c \exp(\rho \|A\|_c)$$

and by (3.11)

$$\|\tilde{G}^{-1}\| \, \|\tilde{G} - G\| \leqq \|\tilde{G}^{-1}\| \, \|l\| \, \|\alpha\|_c < 1.$$

Hence $\det G \neq 0$, and L is invertible.

We show next that

(3.14) $$\|I - \tilde{L}_I L\|_c < 1.$$

Let

(3.15) $$\beta(t) = \tilde{\Phi}_I(t) - \Phi_I(t), \quad q(t) = \int_{t_0}^t \beta(s)A(s)ds.$$

Then by (3.5)

(3.16) $$\beta(t) + q(t) = r_1(t),$$

because

$$\Phi_I(t) - I + \int_{t_0}^t \Phi_I(s)A(s)ds = 0.$$

For any $p \in C[J]$ let $u(t) = \int_{t_0}^t p(s)ds$. Since $\Phi_I' = -\Phi_I A$, the integration by parts shows that

(3.17) $$\int_{t_0}^t \Phi_I(s)A(s)u(s)ds = -\Phi_I(t)u(t) + \int_{t_0}^t \Phi_I(s)p(s)ds,$$

(3.18) $$\int_{t_0}^t \beta(s)A(s)u(s)ds = q(t)u(t) - \int_{t_0}^t q(s)p(s)ds.$$

By (3.15)–(3.18) we have

(3.19) $$\tilde{E}Au = \tilde{\Phi}(t)\left\{\int_{t_0}^t \Phi_I(s)A(s)u(s)ds + \int_{t_0}^t \beta(s)A(s)u(s)ds\right\}$$
$$= -u + \tilde{E}p + Rp.$$

From this and (3.8) it follows that

(3.20) $$\tilde{E}_1 Ph = h - \tilde{\Phi}h(t_0) - R_1 h \qquad \text{for} \quad h \in C[J].$$

Since by (2.13) and (2.14)

(3.21) $$\tilde{S}_1\tilde{\Phi} = (I - \tilde{\Phi}\tilde{G}^{-1}l)\tilde{\Phi} = 0,$$

by (2.20), (3.20) and (3.21) we have

$$(I - \tilde{L}_I L)h = h - \tilde{S}_1\tilde{E}_1 Ph - \tilde{S}l[h] = \tilde{S}_1 R_1 h.$$

Hence (3.14) is valid by (3.12), and $\tilde{L}_I$ is invertible by Lemma 1.

LEMMA 3. *If* $\det\tilde{G} \neq 0$, *then*

(3.22) $$\tilde{K}x = \tilde{K}_1 x + \tilde{K}_2 x \qquad \textit{for} \quad x \in D,$$

where

(3.23) $$\tilde{K}_2 x = \tilde{S}_1 R_2 x \qquad \textit{for} \quad x \in D.$$

PROOF. For any $x \in D$ by (2.22), (2.20) and (2.7)

$$\tilde{K}x = (I - \tilde{L}_I F)x = x - \tilde{S}_1 \tilde{E}_1 Qx - \tilde{S}f[x]. \tag{3.24}$$

By (3.19) we have

$$\tilde{E}_1 Qx = x - \tilde{E}Tx - \tilde{\Phi}x(t_0) - R_2 x \tag{3.25}$$

and by (3.21)

$$\tilde{S}_1 \tilde{E}_1 Qx = x - \tilde{S}l[x] - \tilde{H}Tx - \tilde{S}_1 R_2 x.$$

Substitution of this into (3.24) yields (3.22) by (3.23).

We have the following

THEOREM 2. *Let $x^{(0)} \in D$ be an approximate solution of* (2.8) *and suppose there exist an operator $\tilde{L}_I$, a positive number δ and nonnegative constants η, κ, κ_1, κ_2 such that*

(i) *$\tilde{L}_I$ is invertible*;

(ii) $D_\delta = \{x \in C[J] \mid \|x - x^{(0)}\|_c \leqq \delta\} \subset D$;

(iii) $\kappa = \kappa_1 + \kappa_2 < 1$,

$$\|\tilde{H}\|_c \mu_1 + \|\tilde{S}\|_c \mu_2 \leqq \kappa_1, \tag{3.26}$$

$$\|\tilde{S}_1 R\|_c \mu_3 + \|\tilde{S}_1 r_2\|_c \leqq \kappa_2, \tag{3.27}$$

where μ_1, μ_2 and μ_3 are constants such that

$$\|T_1(x)\|_c \leqq \mu_1 \qquad \text{for all} \quad x \in D_\delta, \tag{3.28}$$

$$\|f'(x) - l\| \leqq \mu_2 \qquad \text{for all} \quad x \in D_\delta, \tag{3.29}$$

$$\|X_1[x]\|_c \leqq \mu_3 \qquad \text{for all} \quad x \in D_\delta; \tag{3.30}$$

(iv) $\|\tilde{L}_I F x^{(0)}\|_c \leqq \eta$;

(v) $\lambda = \eta/(1 - \kappa) \leqq \delta$.

Then the conclusion of Theorem 1 is valid with K replaced by $\tilde{K}$.

PROOF. For any x, $y \in D_\delta$ by the mean value theorem we have

$$\tilde{K}_1 x - \tilde{K}_1 y = \tilde{H}\int_0^1 T_1(y + \theta h)hd\theta + \tilde{S}\int_0^1 \{l - f'(y + \theta h)\}hd\theta, \tag{3.31}$$

$$R_2 x - R_2 y = R\int_0^1 X_1[y + \theta h]hd\theta + r_2 h(t_0), \tag{3.32}$$

where $h=x-y$. Since $y+\theta h \in D_\delta$, by (3.31), (3.28), (3.29) and (3.26)

$$\|\tilde{K}_1 x - \tilde{K}_1 y\|_c \leqq (\|\tilde{H}\|_c \mu_1 + \|\tilde{S}\|_c \mu_2)\|h\|_c \leqq \kappa_1 \|x-y\|_c,$$

and by (3.23), (3.32), (3.30) and (3.27)

$$\|\tilde{K}_2 x - \tilde{K}_2 y\|_c \leqq (\|\tilde{S}_1 R\|_c \mu_3 + \|\tilde{S}_1 r_2\|_c)\|h\|_c \leqq \kappa_2 \|x-y\|_c.$$

Hence by (3.22) and (iii)

$$\|\tilde{K}x - \tilde{K}y\|_c \leqq \kappa \|x-y\|_c.$$

Since $\kappa<1$, by the contraction mapping theorem [1, pp. 65–66] $\tilde{K}$ has a unique fixed point $\hat{x}$ in D_δ and the estimate (2.25) holds. From $\hat{x}=\tilde{K}\hat{x}$ it follows that $\tilde{L}_I F\hat{x}=0$, which is equivalent to $F\hat{x}=0$ by (i). Since any solution of $Fx=0$ is a fixed point of $\tilde{K}$, $\hat{x}$ is the unique solution of (2.8) in D_δ. This completes the proof.

Let α_0 and α_1 be constants such that

$$\rho(\|r_2\|_c\|A\|_c + \|\tilde{\Phi}\|_c\|r_1 A\|_c) \leqq \alpha_0, \tag{3.33}$$

$$\rho(\|r_2\|_c + \|\tilde{\Phi}\|_c\|r_1\|_c) \leqq \alpha_1. \tag{3.34}$$

Then by (3.7) for any $h \in C[J]$

$$\|RAh\|_c \leqq \alpha_0 \|h\|_c, \quad \|Rh\|_c \leqq \alpha_1 \|h\|_c.$$

Hence (3.12) and (3.27) can be replaced respectively by

$$\|\tilde{S}_1\|_c \alpha_0 + \|\tilde{S}_1 r_2\|_c < 1, \tag{3.35}$$

$$\|\tilde{S}_1\|_c \alpha_1 \mu_3 + \|\tilde{S}_1 r_2\|_c \leqq \kappa_2. \tag{3.36}$$

3.2. Case 2

Put

$$A_1(t) = \tilde{\Phi}'(t)\tilde{\Phi}^{-1}(t), \tag{3.37}$$

$$A_2(t) = -\tilde{\Phi}_I^{-1}(t)\tilde{\Phi}_I'(t). \tag{3.38}$$

Let $P_2, R_3, R_4 \in L(C[J], C[J])$ and $R_5: D \to C[J]$ be defined by

$$P_2 h = h(t) - h(t_0) - \int_{t_0}^{t} A_1(s)h(s)ds \qquad \text{for} \quad h \in C[J], \tag{3.39}$$

$$R_3 h = \tilde{E}(A - A_2)h + \gamma_1 h \qquad \text{for} \quad h \in C[J], \tag{3.40}$$

$$R_4 h = R_3(h - P_2 h) - \tilde{E}(A - A_1)h \qquad \text{for} \quad h \in C[J], \tag{3.41}$$

(3.42) $$R_5x = R_3(x - Qx) \quad \text{for} \quad x \in D.$$

Then we have the following

LEMMA 4. *Let*

(3.43) $$\det \tilde{G} \neq 0.$$

Then $\tilde{L}_I$ *is invertible if*

(3.44) $$\rho \|A_1 - \gamma A\|_c < 1$$

or

(3.45) $$\|\tilde{S}_1 R_4\|_c < 1.$$

PROOF. Let L_1 be the operator defined by

$$L_1 h = (P_2 h, l[h]) \quad \text{for} \quad h \in C[J].$$

Then it is invertible by (3.43).

For any $y = (u, e) \in B_0$ by (2.20)

(3.46) $$\tilde{L}_I y = \tilde{E}_1 u - \tilde{S}\{l[\tilde{E}_1 u] - e\}.$$

Since $P_2 \tilde{\Phi} = 0$ and $\tilde{G} = l[\tilde{\Phi}]$, we have

(3.47) $$P_2 \tilde{S} = (P_2 \tilde{\Phi}) \tilde{G}^{-1} = 0,$$

(3.48) $$l[\tilde{S}] = l[\tilde{\Phi}] \tilde{G}^{-1} = I.$$

Suppose first (3.44) holds. By (3.46) and (3.48)

(3.49) $$l[\tilde{L}_I y] = e.$$

By (3.46) and (3.47) the integration by parts yields

$$P_2 \tilde{L}_I y = P_2 \tilde{E}_1 u = u(t) - \int_{t_0}^{t} \{A_1(s) - \gamma(s) A(s)\} u(s) ds,$$

because $\tilde{\Phi}' = A_1 \tilde{\Phi}$ and $u \in C_0[J]$. By this and (3.49) we have

(3.50) $$(I - L_1 \tilde{L}_I) y = \left(\int_{t_0}^{t} \{A_1(s) - \gamma(s) A(s)\} u(s) ds, \, 0 \right).$$

Hence

$$\|(I - L_1 \tilde{L}_I) y\|_b \leqq \rho \|A_1 - \gamma A\|_c \|y\|_b,$$

and $\tilde{L}_I$ is invertible by (3.44) and Lemma 1.

We treat next the case where (3.45) is valid. For any $q \in C[J]$ let $u(t)=\int_{t_0}^{t} q(s)ds$. Since $\tilde{\Phi}_I' = -\tilde{\Phi}_I A_2$, by integration by parts we have

$$\tilde{E}Au = \tilde{E}(A - A_2)u - \gamma u + \tilde{E}q = -u + R_3 u + \tilde{E}q, \tag{3.51}$$

so that

$$\tilde{E}_1 P_2 h = h - \tilde{\Phi}\tilde{\Phi}_I(t_0)h(t_0) - R_4 h \qquad \text{for} \quad h \in C[J]. \tag{3.52}$$

Substitution of $u = P_2 h$ and $e = l[h]$ into (3.46) yields by (3.52) and (3.21)

$$(I - \tilde{L}_I L_1)h = \tilde{S}_1 R_4 h.$$

Hence $\tilde{L}_I$ is invertible by (3.45) and Lemma 1.

LEMMA 5. *If* $\det \tilde{G} \neq 0$, *then*

$$\tilde{K}x = \tilde{K}_1 x + \tilde{K}_2 x \qquad \textit{for} \quad x \in D, \tag{3.53}$$

where

$$\tilde{K}_2 x = \tilde{S}_1 R_5 x \qquad \textit{for} \quad x \in D. \tag{3.54}$$

PROOF. For any $x \in D$ by (3.51) we have

$$\tilde{E}_1 Qx = x - \tilde{E}Tx - \tilde{\Phi}\tilde{\Phi}_I(t_0)x(t_0) - R_5 x. \tag{3.55}$$

By (2.20) and (3.21)

$$\begin{aligned}\tilde{L}_I Fx &= \tilde{S}_1 \tilde{E}_1 Qx + \tilde{S}f[x] \\ &= x - \tilde{H}Tx - \tilde{S}(l[x] - f[x]) - \tilde{S}_1 R_5 x,\end{aligned}$$

from which (3.53) follows.

We have the following

THEOREM 3. *Suppose the assumptions of Theorem* 2 *are satisfied with* (3.27) *replaced by*

$$\|\tilde{S}_1 R_3\|_c (1 + \rho\mu_3) \leqq \kappa_2. \tag{3.56}$$

Then the conclusion of Theorem 2 *is valid.*

PROOF. For any x, $y \in D_\delta$ let $h = x - y$. Then by the mean value theorem

$$R_5 x - R_5 y = R_3 \int_0^1 \{I - Q'(y + \theta h)\} h d\theta. \tag{3.57}$$

Since $y + \theta h \in D_\delta$, we have by (3.30)

$$\|\{I - Q'(y + \theta h)\}h\|_c = \left\| h(t_0) - \int_{t_0}^{t} X_1[y + \theta h](s)h(s)ds \right\|_c$$
$$\leqq (1 + \rho\mu_3)\|h\|_c \qquad \text{for all} \quad \theta \in [0, 1],$$

and by (3.54), (3.57) and (3.56)

$$\|\tilde{K}_2 x - \tilde{K}_2 y\|_c \leqq \|\tilde{S}_1 R_3\|_c (1 + \rho\mu_3)\|h\|_c \leqq \kappa_2 \|x - y\|_c.$$

The proof is completed by the same argument as in the proof of Theorem 2.

Suppose $\|\gamma_1\|_c < 1$ and let $\sigma = 1/(1 - \|\gamma_1\|_c)$. Then since $\|\gamma^{-1}\|_c \leqq \sigma$ we have the following inequalities:

(3.58) $$\|A_1 - \gamma A\|_c \leqq \sigma \|\tilde{\Phi}' \tilde{\Phi}_I - \gamma A \gamma\|_c,$$

(3.59) $$\|A - A_1\|_c \leqq \sigma \|A\gamma - \tilde{\Phi}' \tilde{\Phi}_I\|_c,$$

(3.60) $$\|A - A_2\|_c \leqq \sigma \|\gamma A + \tilde{\Phi} \tilde{\Phi}_I'\|_c,$$

(3.61) $$\|A_1\|_c \leqq \sigma \|\tilde{\Phi}' \tilde{\Phi}_I\|_c.$$

For any constant α_2 such that

(3.62) $$\|\tilde{H}\|_c \|A - A_2\|_c + \|\tilde{S}_1\|_c \|\gamma_1\|_c \leqq \alpha_2,$$

we have

$$\|\tilde{S}_1 R_3 h\|_c \leqq \alpha_2 \|h\|_c \qquad \text{for} \quad h \in C[J],$$

so that

(3.63) $$\|\tilde{S}_1 R_4\|_c \leqq \alpha_2 (1 + \rho \|A_1\|_c) + \|\tilde{H}\|_c \|A - A_1\|_c,$$

(3.64) $$\|\tilde{S}_1 R_3\|_c (1 + \rho\mu_3) \leqq \alpha_2 (1 + \rho\mu_3).$$

Hence by (3.58)–(3.61), we can estimate the left sides of (3.44), (3.45) and (3.56) without computing $\tilde{\Phi}^{-1}$ and $\tilde{\Phi}_I^{-1}$.

3.3. Treatment in the original form

In this subsection we treat the boundary value problem (2.4), (2.5) directly without replacing (2.4) by a system of integral equations.

Let $C^1[J]$ denote the space of all real n-vector functions continuously differentiable on J with the norm $\|\cdot\|_c$ and let

$$D^1 = \{x \in C^1[J] \mid (t, x(t)) \in \Omega \quad \text{for all} \quad t \in J\}.$$

Let $B = C[J] \times R^n$ be a Banach space with the norm

$$\|y\|_b = \max(\|u\|_c, \|e\|) \qquad \text{for} \quad y = (u, e) \in B.$$

Let us consider the equation

$$(3.65)\qquad \mathscr{F}x \equiv \left(\frac{dx}{dt} - X(x, t), f[x]\right) = 0 \qquad \text{for} \quad x \in D^1$$

and introduce the linear operator $\mathscr{L}$ defined by

$$(3.66)\qquad \mathscr{L}h = \left(\frac{dh}{dt} - A(t)h,\ l[h]\right) \qquad \text{for} \quad h \in C^1[J].$$

The following results have been obtained in [4]: *If* $\det G \neq 0$, *then* $\mathscr{L}$ *has an inverse operator* $\mathscr{L}_I$, *which is defined by*

$$(3.67)\qquad \mathscr{L}_I y = Hu + Se \qquad \text{for} \quad y = (u, e) \in B.$$

Let $\mathscr{K}$ and $\mathscr{K}_1$ be the operators from D^1 into $C^1[J]$ defined by

$$(3.68)\qquad \mathscr{K}x = (I - \mathscr{L}_I\mathscr{F})x \qquad \text{for} \quad x \in D^1,$$

$$(3.69)\qquad \mathscr{K}_1 x = \mathscr{L}_I(\mathscr{L} - \mathscr{F})x \qquad \text{for} \quad x \in D^1.$$

Then

$$(3.70)\qquad \mathscr{K}x = \mathscr{K}_1 x = K_1 x \qquad \text{for} \quad x \in D^1.$$

Suppose $\tilde{\Phi}$ and $\tilde{\Phi}_I$ are continuously differentiable on J and let $\tilde{K}_2$ be the operator defined by

$$(3.71)\qquad \tilde{K}_2 x = \{\tilde{H}(A - A_2) + \tilde{S}_1\gamma_1\}x \qquad \text{for} \quad x \in D^1.$$

Then we have the following

THEOREM 4. *Let* $x^{(0)} \in D^1$ *be an approximate solution of* (3.65) *and suppose there exist an operator* $\tilde{\mathscr{L}}_I$, *a positive number* δ *and nonnegative constants* η, κ *such that*

(i) $\det \tilde{G} \neq 0$, $\|\gamma_1\|_c < 1$;

(ii) $D^1_\delta = \{x \in C^1[J] \mid \|x - x^{(0)}\|_c \leqq \delta\} \subset D^1$;

(iii) $\|\tilde{H}\|_c\mu_1 + \|\tilde{S}\|_c\mu_2 + \|\tilde{K}_2\|_c \leqq \kappa < 1$,

where μ_1 *and* μ_2 *are constants such that*

$$\|T_1(x)\|_c \leqq \mu_1 \qquad \textit{for all} \quad x \in D^1_\delta,$$

$$\|f'(x) - l\| \leqq \mu_2 \qquad \textit{for all} \quad x \in D^1_\delta;$$

(iv) $\|\tilde{\mathscr{L}}_I\mathscr{F}x^{(0)}\|_c \leqq \eta$;

(v) $\lambda = \eta/(1 - \kappa) \leqq \delta$.

Then the conclusion of Theorem 1 *is valid with* K *and* D_δ *replaced by* $\tilde{\mathscr{K}}$ *and* D_δ^1 *respectively.*

PROOF. Let $\mathscr{L}_1$ be the operator from $C^1[J]$ into B defined by

$$\mathscr{L}_1 h = \left(\frac{dh}{dt} - A_1(t)h,\ l[h]\right) \qquad \text{for} \quad h \in C^1[J].$$

Then by (i) $\mathscr{L}_1$ is invertible.

For any $y=(u, e)\in B$ we have by (3.67)

$$z = \tilde{\mathscr{L}}_I y = (I - \tilde{S}l)\tilde{E}u + \tilde{S}e = \tilde{E}u - \tilde{\Phi}\tilde{G}^{-1}(l[\tilde{E}u] - e) \tag{3.72}$$

and by (3.37)

$$\begin{aligned}\frac{dz}{dt} &= A_1(t)\tilde{\Phi}(t)\int_{t_0}^{t} \tilde{\Phi}_I(s)u(s)ds + \tilde{\Phi}(t)\tilde{\Phi}_I(t)u(t) \\ &\quad - A_1(t)\tilde{\Phi}(t)\tilde{G}^{-1}(l[\tilde{E}u] - e) \\ &= A_1 z + \gamma u.\end{aligned} \tag{3.73}$$

Since $\tilde{G}=l[\tilde{\Phi}]$, from (3.72) it follows that

$$l[z] = l[\tilde{E}u] - l[\tilde{\Phi}]\tilde{G}^{-1}(l[\tilde{E}u] - e) = e. \tag{3.74}$$

By (3.73) and (3.74) we have

$$(I - \mathscr{L}_1\tilde{\mathscr{L}}_I)y = (\gamma_1 u, 0),$$

so that

$$\|(I - \mathscr{L}_1\tilde{\mathscr{L}}_I)y\|_b \leqq \|\gamma_1\|_c\|y\|_b.$$

Hence $\tilde{\mathscr{L}}_I$ is invertible by (i) and Lemma 1.

For any $x\in D^1$ by (3.69) and (3.67) we have

$$\tilde{\mathscr{K}}_1 x = \tilde{\mathscr{L}}_I(\mathscr{L} - \mathscr{F})x = \tilde{H}Tx + \tilde{S}(l[x] - f[x]) = \tilde{K}_1 x \tag{3.75}$$

and by (3.38)

$$\tilde{E}\left\{\frac{dh}{dt} - A(t)h\right\} = \gamma h - \tilde{E}(A - A_2)h - \tilde{\Phi}\tilde{\Phi}_I(t_0)h(t_0).$$

Hence by (3.21)

$$(I - \tilde{\mathscr{L}}_I\mathscr{L})x = \{\tilde{H}(A - A_2) + (I - \tilde{S}l)\gamma_1\}x = \tilde{K}_2 x \tag{3.76}$$

and by (3.68), (3.75) and (3.76)

$$\tilde{\mathscr{K}} x = \tilde{\mathscr{L}}_I(\mathscr{L} - \mathscr{F})x + (I - \tilde{\mathscr{L}}_I\mathscr{L})x = \tilde{K}_1 x + \tilde{K}_2 x.$$

For any $x,\ y \in D_\delta^1$ by (3.31) and (iii) we have

$$\|\tilde{\mathscr{K}} x - \tilde{\mathscr{K}} y\|_c \leqq \{\|\tilde{H}\|_c \mu_1 + \|\tilde{S}\|_c \mu_2 + \|\tilde{K}_2\|_c\} \|x - y\|_c$$
$$\leqq \kappa \|x - y\|_c.$$

The proof is completed by the same argument as in the proof of Theorem 2.

3.4. A numerical example

We consider the two-point boundary value problem [3]

$$\text{(3.77)} \qquad \frac{dx}{dt} = X(x, t) \equiv \begin{pmatrix} x_2 \\ -x_1 - (x_1 - t)^3 + t + 0.1 \end{pmatrix} \qquad (-1 \leqq t \leqq 1),$$

$$\text{(3.78)} \qquad f[x] \equiv \begin{pmatrix} x_1(-1) + 0.9 \\ x_1(1) - 1.1 \end{pmatrix} = 0.$$

Let

$$\text{(3.79)} \qquad x_1^{(0)}(t) = t + 0.1, \qquad x_2^{(0)}(t) = 1$$

be an approximate solution of this problem, $t_0 = -1$,

$$\text{(3.80)} \qquad A(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and l be the operator defined by

$$\text{(3.81)} \qquad l[h] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} h(-1) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} h(1) \qquad \text{for} \quad h \in C[J].$$

Then

$$\text{(3.82)} \qquad X_x(x, t) - A(t) = \begin{pmatrix} 0 & 0 \\ -3(x_1 - t)^2 & 0 \end{pmatrix},$$

$$\text{(3.83)} \qquad f'(x) = l.$$

For simplicity we put

$$c(t) = \cos(1 + t), \quad s(t) = \sin(1 + t), \quad u(t) = \cos(1 - t),$$
$$v(t) = \sin(1 - t), \quad p(t) = 2s(t)c(t), \quad q(t) = 1 - 2s(t)^2,$$

$$m = 10^{-3}, \quad \sigma = (1 - c)/s, \quad s = \sin 2, \quad c = \cos 2.$$

For any constant ε ($|\varepsilon| < 1$) let $\mu = 1 + \varepsilon$, $\nu = 1 - \varepsilon$,

$$(3.84) \qquad \tilde{\Phi}(t) = \begin{pmatrix} \mu c(t) & \mu s(t) \\ -\mu s(t) & \nu c(t) \end{pmatrix}, \qquad \tilde{\Phi}_I(t) = \begin{pmatrix} c(t) & -\nu s(t) \\ s(t) & \mu c(t) \end{pmatrix}.$$

Then by (3.81) and (3.84) we have

$$(3.85) \qquad \tilde{G} = \mu \begin{pmatrix} 1 & 0 \\ c & s \end{pmatrix}, \qquad \tilde{S}(t) = (\mu s)^{-1} \begin{pmatrix} \mu v(t) & \mu s(t) \\ -C(t) & \nu c(t) \end{pmatrix},$$

$$(3.86) \qquad \tilde{H} h = \int_{-1}^{1} \tilde{H}(t, \tau) h(\tau) d\tau \qquad \text{for} \quad h \in C[J],$$

where

$$\tilde{H}(t, \tau) = s^{-1} \begin{pmatrix} \mu v(t) c(\tau) & -\mu \nu v(t) s(\tau) \\ -C(t) c(\tau) & \nu C(t) s(\tau) \end{pmatrix} \qquad (-1 \leqq \tau < t \leqq 1),$$

$$\tilde{H}(t, \tau) = -s^{-1} \begin{pmatrix} \mu s(t) u(\tau) & \mu s(t) D(\tau) \\ \nu c(t) u(\tau) & \nu c(t) D(\tau) \end{pmatrix} \qquad (-1 \leqq t \leqq \tau \leqq 1),$$

$$C(t) = u(t) - \varepsilon \cos(3 + t), \quad D(\tau) = v(\tau) + \varepsilon \sin(3 + \tau).$$

Hence

$$(3.87) \qquad \tilde{L}_I F x^{(0)} = m \begin{pmatrix} \mu\{1 - c(t) - \sigma s(t)\} \\ \nu\{s(t) - \sigma c(t)\} + \varepsilon\{(1 + t) q(t) + 2s(t) - p(t)\} \end{pmatrix},$$

$$(3.88) \qquad \tilde{K}_1 x^{(0)} = \begin{pmatrix} \alpha s(t) + \beta c(t) + \varepsilon \mu p(t) + \mu(1 - \varepsilon q(t))(t + 0.099) \\ \nu \alpha c(t)/\mu - \beta s(t) + \mu \nu \end{pmatrix},$$

$$(3.89) \qquad \tilde{\mathscr{L}}_I \mathscr{F} x^{(0)} = m \begin{pmatrix} \mu\{1 - \nu c(t) - \sigma_1 s(t) - \varepsilon q(t)\} \\ \nu\{\mu s(t) - \sigma_1 c(t)\} \end{pmatrix},$$

where

$$\alpha = m\sigma + \varepsilon\{\varepsilon(0.901c + 1.099)/s - 2\mu(1.099s + c)\},$$

$$\beta = m - 0.901\varepsilon^2, \quad \sigma_1 = \sigma + \varepsilon\sigma(1 + 2c).$$

We have

$$\gamma_1(t) = \varepsilon\begin{pmatrix} -1 & -\mu p(t) \\ p(t) & \varepsilon \end{pmatrix}, \tag{3.90}$$

$$r(t) = \varepsilon\begin{pmatrix} 1 & 2s(t) \\ 0 & 1-2c(t) \end{pmatrix}, \qquad r_1(t) = \varepsilon\begin{pmatrix} c(t)-1 & s(t) \\ -s(t) & c(t) \end{pmatrix}, \tag{3.91}$$

$$r_2(t) = \varepsilon\begin{pmatrix} \mu q(t) - c(t) - 1 & 0 \\ \mu s(t) & q(t) \end{pmatrix}, \tag{3.92}$$

$$\gamma A + \tilde{\Phi}\tilde{\Phi}_I' = \varepsilon\begin{pmatrix} -\mu p(t) & \mu q(t) \\ \varepsilon - q(t) & -p(t) \end{pmatrix}, \tag{3.93}$$

$$\tilde{\Phi}'\tilde{\Phi}_I - \gamma A \gamma = \varepsilon\begin{pmatrix} \mu(1+\mu)p(t) & \mu\{\varepsilon + q(t) + \varepsilon\mu p(t)^2\} \\ 1 - \varepsilon\mu - q(t) - \varepsilon p(t)^2 & \mu\nu(1+\mu)p(t) \end{pmatrix}. \tag{3.94}$$

Let us put $\varepsilon = 10^{-3}$ and use the infinity norm $\|\cdot\|_\infty$. Then

$$\|\tilde{H}\|_{\infty c} \leqq 2\max_{t,s\in J}\|\tilde{H}(t,s)\|_\infty \leqq 3.11104,$$

$$|\tilde{L}_I F x^{(0)}\|_{\infty c} \leqq \eta = 1.55712m.$$

By (3.82) and (3.83) we may choose

$$\mu_1 = 3(\delta + 0.1)^2, \quad u_2 = 0, \quad \mu_3 = 1 + \mu_1.$$

In the remainder of this subsection we omit the subscript ∞ for simplicity.

(i) The case where Theorem 2 is applied.

We have

$$\|\tilde{G}^{-1}\| \leqq 1.55586, \quad \|r\|_c \leqq 3.0m, \quad \|r_1\|_c \leqq 2.32544m, \quad \|r_2\|_c \leqq 2.12613m$$

$$\|\tilde{S}_1 r_2\|_c \leqq 2.81522m, \quad \|\tilde{S}_1\|_c \leqq 3.1995, \quad \|\tilde{\Phi}\|_c \leqq 1.41563,$$

$$\|\tilde{G}^{-1}\|\,\|l\|\,\|r\|_c \exp(2\|A\|_c) \leqq 0.0689,$$

$$2\|\tilde{S}_1\|_c(\|r_2\|_c + \|\tilde{\Phi}\|_c\|r_1\|_c) + \|\tilde{S}_1 r_2\|_c \leqq 0.0375,$$

so that $\tilde{L}_I$ is invertible by Lemma 2.

The choice $\delta = 1.8008m$ yields

$$\kappa = 0.13528, \quad \lambda = \eta/(1-\kappa) = 1.80074m = \lambda_1,$$

and an error estimate $\|\hat{x} - x^{(0)}\|_c \leqq \lambda_1$ is obtained.

(ii) The case where Theorem 3 is applied.
We have

$$\|\gamma_1\|_c \leqq 2.001m, \quad \|\tilde{\Phi}'\tilde{\Phi}_I - \gamma A\gamma\|_c \leqq 3.23516m. \tag{3.95}$$

Hence by (3.58) and Lemma 4 $\tilde{L}_I$ is invertible. The constant κ_2 is determined with the aid of (3.60) and (3.64). With the choice $\delta=1.79m$ we have

$$\kappa = 0.12982, \quad \lambda = 1.78942m = \lambda_2.$$

Now we consider two cases where Theorem 1 is applied incorrectly with $\tilde{\Phi}$ and $\tilde{\Phi}_I$ regarded as Φ and Φ_I respectively.

(iii) The case where Theorem 1 is applied with $\tilde{K}$ regarded as K.
The choice $\delta=1.724m$ yields

$$\kappa = 0.096576, \quad \lambda = 1.72357m = \lambda_3.$$

In this case $\tilde{K}_2$ is neglected, so that λ_3 is not necessarily a bound of $\|\hat{x}-x^{(0)}\|_c$.

(iv) The case where Theorem 1 is applied with $\tilde{K}_1$ regarded as K.
We have

$$\|\tilde{K}_1x^{(0)} - x^{(0)}\|_c \leqq \eta_1 = 1.07349m,$$

and the choice $\eta=\eta_1$ and $\delta=1.187m$ leads to

$$\kappa = 0.09556, \quad \lambda = 1.18691m = \lambda_4.$$

It is to be noted that λ_4 is a bound of $\|\hat{y}-x^{(0)}\|_c$ and is not always that of $\|\hat{x}-x^{(0)}\|_c$, where $\hat{y}$ is the limit of the sequence $y^{(k)}$ defined by $y^{(k+1)}=\tilde{K}_1y^{(k)}$ $(k=0, 1,\dots)$ with $y^{(0)}=x^{(0)}$. Hence the use of the iteration

$$x^{(k+1)} = K_1x^{(k)} \qquad (k = 0, 1,\dots)$$

is not recommended, though (2.21) is valid.

(v) The case where Theorem 1 is applied with $\varepsilon=0$.
In this case $\tilde{\Phi}$ and $\tilde{\Phi}_I$ are identical with Φ and Φ_I respectively. We have

$$\|H\|_c \leqq 3.11056, \quad \|L_IFx^{(0)}\|_c \leqq \eta_2 = 1.55741m$$

and the choice $\eta=\eta_2$ and $\delta=1.7239m$ yields

$$\kappa = 0.096561, \quad \lambda = 1.72387m = \lambda_5.$$

(vi) The case where Theorem 4 is applied.
Since $\det\tilde{G}=\mu s\neq 0$, by (3.95) the condition (i) of Theorem 4 is satisfied and $\tilde{\mathscr{L}}_I$ is invertible. From (3.62) and (3.71) it follows that $\|K_2\|_c\leqq\alpha_2$. We have

$$\|\tilde{\mathscr{L}}_l \mathscr{F} x^{(0)}\|_c \leqq \eta_3 = 1.55687m, \quad \alpha_2 = 10.81511m$$

and the choice $\eta=\eta_3$ and $\delta=1.745m$ leads to

$$\kappa = 0.107432, \quad \lambda = 1.744m = \lambda_6.$$

It seems that λ_1, λ_2 and λ_6 are not so greater than λ_5. It is to be noted that λ_3 differs slightly from λ_5 but λ_4 does much. It is seen that $\lambda_1>\lambda_2>\lambda_6>\lambda_5$. The same conclusions are valid also when the norms $\|\cdot\|_2$ and $\|\cdot\|_1$ are used. The results are listed in Table 1, where $\tilde{\eta}=\eta/m$, $\tilde{\delta}=\delta/m$, $\tilde{\kappa}=10\kappa$ and $\tilde{\lambda}=\lambda/m$.

Table 1.

norm	e \ c	i	ii	iii	iv	v	vi
$\|\cdot\|_\infty$	$\tilde{\eta}$	1.55712	1.55712	1.55712	1.07349	1.55741	1.55687
	$\tilde{\delta}$	1.80080	1.79000	1.72400	1.18700	1.72400	1.74500
	$\tilde{\kappa}$	1.35286	1.29820	0.96577	0.95560	0.96562	1.07432
	$\tilde{\lambda}$	1.80073	1.78942	1.72358	1.18691	1.72387	1.74426
$\|\cdot\|_2$	$\tilde{\eta}$	1.55712	1.55712	1.55712	1.25924	1.55741	1.55687
	$\tilde{\delta}$	1.72700	1.71300	1.67200	1.35100	1.67200	1.68500
	$\tilde{\kappa}$	0.97929	0.90973	0.68295	0.67864	0.68209	0.75700
	$\tilde{\lambda}$	1.72616	1.71295	1.67126	1.35092	1.67142	1.68437
$\|\cdot\|_1$	$\tilde{\eta}$	1.61719	1.61719	1.61719	1.77919	1.61745	1.61704
	$\tilde{\delta}$	1.91010	1.87200	1.79100	1.97100	1.79100	1.81600
	$\tilde{\kappa}$	1.53316	1.35658	0.96806	0.97149	0.96689	1.09490
	$\tilde{\lambda}$	1.91002	1.87101	1.79052	1.97063	1.79058	1.81586

4. A special boundary value problem

4.1. Problems and notations

Let $W[J]=C[J]\times M[J]$ be a Banach space with the norm

$$\|v\|_w = \max(p^{-1}\|v_0\|_c, q^{-1}\|v_1\|_c) \qquad \text{for} \quad v=(v_0, v_1)\in W[J],$$

where p and q are arbitrary positive constants. Let $B_0=C_0[J]\times M[J]\times R^n$ be a Banach space with the norm

$$\|\varphi\|_b = \max(\|u_0\|_c, \|u_1\|_c, \|e\|) \qquad \text{for} \quad \varphi=(u_0, u_1, e)\in B_0.$$

We assume that $X(x, t)$ is continuous in Ω and twice continuously differentiable with respect to x in Ω, and denote by $X_{xx}(x, t)$ the second Fréchet derivative of $X(x, t)$ with respect to x. For any $x\in D$ let $\Phi_{(x)}(t)$ be the fundamental matrix of the system

$$\frac{dz}{dt} = X_x(x(t), t)z \tag{4.1}$$

with $\Phi_{(x)}(t_0)=I$, and put $D^1=D\times U$, where $U=\{\Phi_{(x)}\in M[J] \mid x\in D\}$.

Let us consider the boundary value problem (2.4) and

$$f[y] = 0 \qquad \text{for} \quad y = (x, \Phi_{(x)}), \tag{4.2}$$

where the operator $f: D^1\to R^n$ is continuously Fréchet differentiable in D^1. For example, this problem arises from boundary value problems of the least squares type [2].

In the sequel the $C[J]$- and $M[J]$-components of any element of $W[J]$ are represented with subscripts 0 and 1 respectively, so that $x=(x_0, x_1)$.

Let $F: D^1\to B_0$ be defined by

$$Fx = (Qx_0, Q_1x, f[x]) \qquad \text{for} \quad x\in D^1, \tag{4.3}$$

where

$$Q_1x = x_1(t) - I - \int_{t_0}^{t} X_1[x_0](s)x_1(s)ds, \tag{4.4}$$

Qx_0 and $X_1[x_0]$ are given by (2.6) and (2.19) respectively. Then the problem (2.4), (4.2) is equivalent to that of finding a solution $x\in D^1$ of the equation

$$Fx = 0. \tag{4.5}$$

Let $\hat{x}\in D^1$ be the exact solution of (4.5), and $x^{(0)}\in D^1$ be an approximate solution. Then our object is to estimate the error of $x_0^{(0)}$ and that of $x_1^{(0)}$. We denote by $\lambda(p, q)$ an error bound of $x^{(0)}$ such that $\|\hat{x}-x^{(0)}\|_w\leqq\lambda(p, q)$. Since

$$\|\hat{x} - x^{(0)}\|_w = \max(p^{-1}\|\hat{x}_0 - x_0^{(0)}\|_c, q^{-1}\|\hat{x}_1 - x_1^{(0)}\|_c),$$

we have estimates

$$\|\hat{x}_0 - x_0^{(0)}\|_c \leqq \lambda(1, q), \quad \|\hat{x}_1 - x_1^{(0)}\|_c \leqq \lambda(p, 1).$$

The parameters p and q are introduced so as to make the bounds $\lambda(1, q)$ and $\lambda(p, 1)$ small.

Let

$$V = (v_1, v_2,\dots, v_n)\in M[J], \qquad h\in C[J].$$

For a bilinear operator N from $C[J]$ into $C[J]$ we define $N[h, V]$ by

$$N[h, V] = (N[h, v_1], N[h, v_2],\dots, N[h, v_n]).$$

For $Y_i \in L(C[J], C[J])$ $(i=1, 2,\ldots, n)$ let $Y \in L(C[J], M[J])$ be the operator defined by

$$Yh = (Y_1 h, Y_2 h,\ldots, Y_n h)$$

and let

$$YV = (Y_1 V, Y_2 V,\ldots, Y_n V).$$

For $x \in D^1$ the Fréchet derivative $F'(x)$ is defined by

$$F'(x)h = (Q'(x_0)h_0, Q_1'(x)h, f'(x)h) \qquad \text{for} \quad h \in W[J], \tag{4.6}$$

where $Q'(x_0)h_0$ is given by (2.10),

$$Q_1'(x)h = h_1 - \int_{t_0}^{t} X_1[x_0](s)h_1(s)ds - \int_{t_0}^{t} X_2(x_0)[h_0, x_1](s)ds, \tag{4.7}$$

$$f'(x)h = f_0(x)h_0 + f_1(x)h_1, \tag{4.8}$$

$$X_2(x_0)[h_0, x_1] = X_{xx}(x_0(t), t)[h_0(t), x_1(t)], \tag{4.9}$$

f_0 and f_1 are partial Fréchet derivatives of f with respect to x_0 and x_1 respectively.

Let $T_2: D^1 \to L(C[J], M[J])$ be the operator such that

$$T_2(x)h = X_2(x_0)[h, x_1] - Yh \qquad \text{for} \quad x \in D^1, \quad h \in C[J]. \tag{4.10}$$

Let $L \in L(W[J], B_0)$ be the operator independent of x which approximates $F'(x)$ and is defined by

$$Lh = (Ph_0, P_1 h, l[h]) \qquad \text{for} \quad h \in W[J], \tag{4.11}$$

where Ph_0 is given by (2.12),

$$P_1 h = h_1(t) - \int_{t_0}^{t} A(s)h_1(s)ds - \int_{t_0}^{t} [Yh_0](s)ds, \tag{4.12}$$

$$l[h] = l_0[h_0] + l_1[h_1], \tag{4.13}$$

$Y \in L(C[J], M[J])$, $l_0 \in L(C[J], R^n)$ and $l_1 \in L(M[J], R^n)$.

Let $l_2 \in L(C[J], R^n)$ be defined by

$$l_2 = l_0 + l_1 E Y \tag{4.14}$$

and put

$$G = l_2[\Phi]. \tag{4.15}$$

When $\det G \neq 0$, we define the operators S_j ($j=0, 1,\dots, 5$), H_0 and H_1 by the following formulas:

$$(4.16) \qquad S_0 = \Phi G^{-1}, \quad S_1 = EYS_0, \quad S_2 = S_0 l_1, \quad S_3 = S_1 l_1 - I,$$
$$S_4 = I - S_0 l_2, \quad S_5 = EYS_4, \quad H_0 = S_4 E, \quad H_1 = S_5 E.$$

In 4.2 an analogue of Theorem 1 is given for $x^{(0)}$ and in 4.3 the error of $x^{(0)}$ is estimated in terms of the approximate matrices of Φ and Φ_I.

4.2. Exact fundamental matrices

We have the following

LEMMA 6. *L has an inverse operator L_I if and only if*

$$(4.17) \qquad \det G \neq 0.$$

Suppose (4.17) *is satisfied. Then for any* $\varphi=(u_0, u_1, e)\in B_0$

$$(4.18) \qquad L_I\varphi = h,$$

where

$$(4.19) \qquad h_i = S_{i+4}E_1u_0 - S_{i+2}E_1u_1 + S_ie \qquad (i = 0, 1).$$

PROOF. By (4.11) the equation $Lh=\varphi$ is equivalent to the system

$$(4.20) \qquad Ph_0 = u_0,$$

$$(4.21) \qquad P_1h = u_1,$$

$$(4.22) \qquad l_0[h_0] + l_1[h_1] = e.$$

The general solution of (4.20) is given by

$$(4.23) \qquad h_0 = \Phi c + E_1u_0$$

with an arbitrary $c\in R^n$. The solution of (4.21) is

$$h_1 = E_1u_1 + EYh_0,$$

and substitution of (4.23) into this yields

$$(4.24) \qquad h_1 = E_1u_1 + EYE_1u_0 + EY\Phi c.$$

By (4.15), (4.23) and (4.24) from (4.22) it follows that

$$l_2[E_1u_0] + l_1[E_1u_1] + Gc = e.$$

Hence L_I exists and is unique if and only if c is determined uniquely for $\varphi \in B_0$, that is, $\det G \neq 0$.

If (4.17) holds, then

$$c = G^{-1}\{e - l_2[E_1 u_0] - l_1[E_1 u_1]\}.$$

Substituting this into (4.23) and (4.24) we have (4.18) and the proof is complete.

Let K and K_1 be the operators from D^1 into $W[J]$ defined by

$$K = I - L_I F, \tag{4.25}$$

$$K_1 x = y \quad \text{for} \quad x \in D^1, \tag{4.26}$$

where

$$y_i = H_i T x_0 - S_{i+2} E\{T_1(x_0)x_1 - Y x_0\} + S_i(l[x] - f[x]) - S_{i+2}\Phi \quad (i = 0, 1), \tag{4.27}$$

T and $T_1(x_0)$ are given by (2.17) and (2.18) respectively. The integration by parts yields

$$E_1 Q x_0 = x_0 - E T x_0 - \Phi x_0(t_0),$$

$$E_1 Q_1 x = x_1 - E T_1(x_0) x_1 - \Phi \quad \text{for} \quad x \in D^1.$$

Since $S_{i+4}\Phi = 0$ $(i=0, 1)$, by (4.3) and (4.19) we have

$$Kx = K_1 x \quad \text{for} \quad x \in D^1. \tag{4.28}$$

We have the following analogue of Theorem 1.

THEOREM 5. *Let $x^{(0)} \in D^1$ be an approximate solution of* (4.5) *and suppose there exist an operator L, a positive number δ and nonnegative constants η, κ such that*

(i) $\det G \neq 0$;

(ii) $D^1_\delta = \{x \in W[J] \mid \|x - x^{(0)}\|_w \leqq \delta\} \subset D^1$;

(iii) $\kappa = \max(p^{-1}\kappa_0, q^{-1}\kappa_1) < 1$,

where κ_0 and κ_1 are constants satisfying

$$p\|H_i\|_c \mu_1 + \|S_i\|_c \mu_2 + \|S_{i+2}E\|_c(q\mu_1 + p\mu_4) \leqq \kappa_i \quad (i = 0, 1), \tag{4.29}$$

and μ_1, μ_2 and μ_4 are constants such that

$$\|T_1(x_0)\|_c \leqq \mu_1 \quad \textit{for all} \quad x \in D^1_\delta, \tag{4.30}$$

(4.31) $$\|f'(x) - l\| \leqq \mu_2 \qquad \textit{for all} \quad x \in D^1_\delta,$$

(4.32) $$\|T_2(x)\|_c \leqq \mu_4 \qquad \textit{for all} \quad x \in D^1_\delta;$$

(iv) $\|L_I F x^{(0)}\|_w \leqq \eta$;

(v) $\lambda = \eta/(1-\kappa) \leqq \delta$.

Then the sequence $\{x^{(k)}\}$ *defined by*

(4.33) $$x^{(k+1)} = Kx^{(k)} \qquad (k = 0, 1, \ldots)$$

remains in D^1_δ *and converges to* $\hat{x} \in D^1_\delta$ *as* $k \to \infty$. $\hat{x}$ *is the unique solution of* (4.5) *in* D^1_δ, *and*

(4.34) $$\|\hat{x} - x^{(k)}\|_w \leqq \kappa^k \lambda \qquad (k = 0, 1, \ldots).$$

PROOF. For any $x, y \in D^1_\delta$ let $h = y - x$. Then by the mean value theorem

(4.35) $$Ty_0 - Tx_0 = \int_0^1 T_1(x_0 + \theta h_0) h_0 d\theta,$$

(4.36) $$T_1(y_0)y_1 - T_1(x_0)x_1 = T_1(y_0)h_1 + \int_0^1 X_2(x_0 + \theta h_0)[h_0, x_1] d\theta$$

$$= T_1(y_0)h_1 + \int_0^1 T_2(x(\theta))h_0 d\theta + Yh_0,$$

where $x(\theta) = (x_0 + \theta h_0, x_1)$. Since $x(\theta) \in D^1_\delta$ $(0 \leqq \theta \leqq 1)$, we have by (4.30) and (4.32)

(4.37) $$\|Ty_0 - Tx_0\|_c \leqq \mu_1 \|y_0 - x_0\|_c \leqq p\mu_1 \|y - x\|_w,$$

(4.38) $$\|T_1(y_0)y_1 - T_1(x_0)x_1 - Yh_0\|_c \leqq \mu_1 \|y_1 - x_1\|_c + \mu_4 \|y_0 - x_0\|_c$$

$$\leqq (q\mu_1 + p\mu_4) \|y - x\|_w,$$

and also by (4.31)

(4.39) $$\|l[y - x] - f[y] + f[x]\| = \left\| \int_0^1 \{l - f'(x + \theta h)\} h d\theta \right\|$$

$$\leqq \mu_2 \|y - x\|_w.$$

Let $u = Ky - Kx$. Then by (4.28) and (4.26)

$$u_i = H_i(Ty_0 - Tx_0) + S_i(l[y - x] - f[y] + f[x])$$

$$- S_{i+2}E\{T_1(y_0)y_1 - T_1(x_0)x_1 - Yy_0 + Yx_0\} \qquad (i = 0, 1),$$

so that by (4.37)–(4.39) and (4.29)

$$\|u_i\|_c \leqq \{p\|H_i\|_c\mu_1 + \|S_i\|_c\mu_2 + \|S_{i+2}E\|_c(q\mu_1 + p\mu_4)\} \|y - x\|_w$$
$$\leqq \kappa_i\|y - x\|_w \qquad (i = 0, 1).$$

Hence by (iii) we have

$$\|Ky - Kx\|_w = \max(p^{-1}\|u_0\|_c, q^{-1}\|u_1\|_c)$$
$$\leqq \max(p^{-1}\kappa_0, q^{-1}\kappa_1) \|y - x\|_w \leqq \kappa\|y - x\|_w.$$

The proof is completed by the same argument as in the proof of Theorem 2.

4.3. Approximate fundamental matrices

In this subsection error bounds of $x^{(0)}$ are given in terms of the approximate matrices $\tilde{\Phi}$ and $\tilde{\Phi}_I$.

4.3.1. Case 1

Let $R_6 \in L(W[J], M[J])$ and R_7: $D^1 \to M[J]$ be defined by

$$R_6 h = RAh_1 + RYh_0 \qquad \text{for} \quad h \in W[J], \tag{4.40}$$

$$R_7 x = r_2 + RX_1[x_0]x_1 \qquad \text{for} \quad x \in D^1, \tag{4.41}$$

where r_2 and R are given by (3.6) and (3.7) respectively. Then we have the following

LEMMA 7. *$\tilde{L}_I$ exists and is invertible if*

$$\det \tilde{G} \neq 0, \tag{4.42}$$

$$\|\tilde{G}^{-1}\| \{\beta_1\|\tilde{l}_2\| + \rho\beta_2\|l_1\| \|Y\|_c(\beta_1 + \|\tilde{\Phi}\|_c)\} < 1, \tag{4.43}$$

$$\max(p^{-1}\nu_0, q^{-1}\nu_1) < 1, \tag{4.44}$$

where β_1, β_2, ν_0 and ν_1 are constants such that

$$\beta = \exp(\rho\|A\|_c), \quad \beta_1 = \|r\|_c\beta, \quad \beta_2 = (\beta_1 + \|\tilde{\Phi}\|_c\|r_1\|_c)\beta, \tag{4.45}$$

$$p\|\tilde{S}_{i+4}R_1\|_c + \|\tilde{S}_{i+2}R_6\|_c \leqq \nu_i \qquad (i = 0, 1), \tag{4.46}$$

and R_1 is given by (3.8).

PROOF. We show first that

$$\|\tilde{G}^{-1}\| \|\tilde{G} - G\| < 1. \tag{4.47}$$

Let $\varphi(t, s) = \tilde{\Phi}(t)\tilde{\Phi}_I(s) - \Phi(t)\Phi_I(s)$. Then by (3.13)

$$\|\varphi(t, s)\| \leqq \|\tilde{\Phi}(t) - \Phi(t)\| + \|\tilde{\Phi}(t)r_1(s)\| + \left\|\int_{t_0}^{s} \varphi(t, \tau)A(\tau)d\tau\right\|$$

$$\leqq \beta_1 + \|\tilde{\Phi}\|_c\|r_1\|_c + \left|\int_{t_0}^{s} \|A\|_c\|\varphi(t, \tau)\|d\tau\right|,$$

because

$$\Phi(t)\{\Phi_I(s) - I + \int_{t_0}^{s} \Phi_I(\tau)A(\tau)d\tau\} = 0,$$

$$\tilde{\Phi}(t)\{\tilde{\Phi}_I(s) - I + \int_{t_0}^{s} \tilde{\Phi}_I(\tau)A(\tau)d\tau\} = \tilde{\Phi}(t)r_1(s).$$

Gronwall's inequality yields

$$\|\varphi(t, s)\| \leqq (\beta_1 + \|\tilde{\Phi}\|_c\|r_1\|_c)\beta = \beta_2.$$

Since

$$(\tilde{E} - E)h = \int_{t_0}^{t} \varphi(t, s)h(s)ds \qquad \text{for} \quad h \in C[J],$$

we have

(4.48) $$\|\tilde{E} - E\|_c \leqq \rho\beta_2.$$

By (4.15)

$$\tilde{G} - G = l_0[\tilde{\Phi} - \Phi] + l_1[\tilde{E}Y\tilde{\Phi} - EY\Phi]$$

$$= \tilde{l}_2[\tilde{\Phi} - \Phi] + l_1[(\tilde{E} - E)Y\Phi],$$

and so by (3.13) and (4.48)

$$\|\tilde{G} - G\| \leqq \|\tilde{l}_2\|\beta_1 + \|l_1\|\rho\beta_2\|Y\|_c(\beta_1 + \|\tilde{\Phi}\|_c).$$

Hence by (4.43) we have (4.47), which implies $\det G \neq 0$, and L is invertible by Lemma 6.

We show next that

(4.49) $$\|I - \tilde{L}_I L\|_w < 1.$$

By (3.19) and (4.40) we have

(4.50) $$\tilde{E}_1 P_1 h = h_1 - \tilde{E}Yh_0 - R_6 h \qquad \text{for} \quad h \in W[J].$$

Since $\tilde{S}_{i+4}\tilde{\Phi} = 0$ $(i=0, 1)$ and

$$\tilde{S}_{i+4}h_0 - \tilde{S}_{i+2}(h_1 - \tilde{E}Yh_0) = h_i - \tilde{S}_i l[h] \qquad (i = 0, 1),$$

by (4.18), (3.20) and (4.50)

$$(I - \tilde{L}_I L)h = u,$$

where

$$u_i = \tilde{S}_{i+4} R_1 h_0 - \tilde{S}_{i+2} R_6 h \qquad (i = 0, 1).$$

By (4.46) we have

$$\|u_i\|_c \leqq \{p\|\tilde{S}_{i+4} R_1\|_c + \|\tilde{S}_{i+2} R_6\|_c\} \|h\|_w \leqq \nu_i \|h\|_w \qquad (i = 0, 1),$$

and it follows that

$$\|(I - \tilde{L}_I L)h\|_w = \max(p^{-1}\|u_0\|_c, q^{-1}\|u_1\|_c)$$
$$\leqq \max(p^{-1}\nu_0, q^{-1}\nu_1) \|h\|_w.$$

Hence (4.49) is valid by (4.44), and $\tilde{L}_I$ is invertible by Lemma 1. This completes the proof.

Let α_3 and α_4 be constants such that

(4.51) $$\|r_2\|_c + \alpha_0 \leqq \alpha_3, \quad q\alpha_0 + p\alpha_1 \|Y\|_c \leqq \alpha_4,$$

where α_0 and α_1 are given by (3.33) and (3.34) respectively. Then by (3.8) and (4.40)

$$\|R_1\|_c \leqq \alpha_3, \quad \|R_6\|_c \leqq \alpha_4.$$

Hence (4.46) can be replaced by

(4.52) $$p\|\tilde{S}_{i+4}\|_c \alpha_3 + \|\tilde{S}_{i+2}\|_c \alpha_4 \leqq \nu_i \qquad (i = 0, 1).$$

LEMMA 8. *If* $\det \tilde{G} \neq 0$, *then*

(4.53) $$\tilde{K}x = \tilde{K}_1 x + \tilde{K}_2 x \qquad \textit{for} \quad x \in D^1,$$

where

(4.54) $$\tilde{K}_2 x = u,$$

(4.55) $$u_i = \tilde{S}_{i+4} R_2 x_0 - \tilde{S}_{i+2} R_7 x \qquad (i = 0, 1)$$

and R_2 *is the operator given by* (3.9).

PROOF. By (4.25), (4.18) and (4.3) we have $\tilde{K}x = y$, where

(4.56) $$y_i = x_i - \tilde{S}_{i+4} \tilde{E}_1 Q x_0 + \tilde{S}_{i+2} \tilde{E}_1 Q_1 x - \tilde{S}_i f[x] \qquad (i = 0, 1).$$

By (3.19) and (4.41)

(4.57) $$\tilde{E}_1 Q_1 x = x_1 - \tilde{E} T_1(x_0) x_1 - \tilde{\Phi} - R_7 x.$$

Substitution of (3.25) and (4.57) into (4.56) yields

(4.58) $$y_i = \tilde{H}_i T x_0 - \tilde{S}_{i+2} \tilde{E}\{T_1(x_0) x_1 - Y x_0\} + \tilde{S}_i(l[x] - f[x]) - \tilde{S}_{i+2}\tilde{\Phi} + \tilde{S}_{i+4} R_2 x_0 - \tilde{S}_{i+2} R_7 x \qquad (i = 0, 1),$$

because $\tilde{S}_{i+4}\tilde{\Phi}=0$ $(i=0, 1)$ and

(4.59) $$x_i - \tilde{S}_{i+4} x_0 + \tilde{S}_{i+2} x_1 = \tilde{S}_i l[x] + \tilde{S}_{i+2} \tilde{E} Y x_0 \qquad (i = 0, 1).$$

Hence (4.53) follows from (4.58) by (4.26) and (4.54).

Now we prove the following

THEOREM 6. *Let $x^{(0)} \in D^1$ be an approximate solution of* (4.5) *and suppose there exist an operator $\tilde{L}_I$, a positive number δ and nonnegative constants η, κ, κ_j $(j=0, 1, 2, 3)$ such that*

(i) *$\tilde{L}_I$ is invertible;*

(ii) $D_\delta^1 = \{x \in W[J] \mid \|x - x^{(0)}\|_w \leqq \delta\} \subset D^1$;

(iii) $\kappa = \max(p^{-1}(\kappa_0 + \kappa_2), q^{-1}(\kappa_1 + \kappa_3)) < 1$,

(4.60) $$p\|\tilde{H}_i\|_c \mu_1 + \|\tilde{S}_i\|_c \mu_2 + \|\tilde{S}_{i+2}\tilde{E}\|_c (q\mu_1 + p\mu_4) \leqq \kappa_i \qquad (i = 0, 1),$$

(4.61) $$p\|\tilde{S}_{i+4} R\|_c \mu_3 + \|\tilde{S}_{i+2} R\|_c (q\mu_3 + p\mu_5) + p\|\tilde{S}_{i+4} r_2\|_c \leqq \kappa_{i+2} \qquad (i = 0, 1),$$

where μ_j $(j=1, 2, 3, 4, 5)$ are constants such that

(4.62) $$\|T_1(x_0)\|_c \leqq \mu_1 \quad \textit{for all} \quad x \in D_\delta^1,$$

(4.63) $$\|f'(x) - l\| \leqq \mu_2 \quad \textit{for all} \quad x \in D_\delta^1,$$

(4.64) $$\|X_1[x_0]\|_c \leqq \mu_3 \quad \textit{for all} \quad x \in D_\delta^1,$$

(4.65) $$\|T_2(x)\|_c \leqq \mu_4 \quad \textit{for all} \quad x \in D_\delta^1,$$

(4.66) $$\|X_2(x_0)[\cdot, x_1]\|_c \leqq \mu_5 \quad \textit{for all} \quad x \in D_\delta^1;$$

(iv) $\|\tilde{L}_I F x^{(0)}\|_w \leqq \eta$;

(v) $\lambda = \eta/(1 - \kappa) \leqq \delta$.

Then the conclusion of Theorem 5 is valid with K replaced by $\tilde{K}$.

PROOF. For any $x, y \in D_\delta^1$ let

$$h = y - x, \quad u = \tilde{K}_1 y - \tilde{K}_1 x, \quad v = \tilde{K}_2 y - \tilde{K}_2 x.$$

Then by (4.26), (4.37), (4.38), (4.39) and (4.60)

$$\|u_i\|_c \leqq \kappa_i \|y - x\|_w \qquad (i = 0, 1). \tag{4.67}$$

By (4.64), (4.66) and the mean value theorem we have

$$\begin{aligned} &\|X_1[y_0]y_1 - X_1[x_0]x_1\|_c \\ &= \|X_1[y_0]h_1 + \int_0^1 X_2(x_0 + \theta h_0)[h_0, x_1]d\theta\|_c \\ &\leqq \mu_3\|h_1\|_c + \mu_5\|h_0\|_c \leqq (q\mu_3 + p\mu_5)\|y - x\|_w, \end{aligned} \tag{4.68}$$

which yields by (4.41)

$$\|\tilde{S}_i R_7 y - \tilde{S}_i R_7 x\|_c \leqq \|\tilde{S}_i R\|_c (q\mu_3 + p\mu_5)\|y - x\|_w, \qquad (i = 2, 3). \tag{4.69}$$

Similarly by (3.32) and (4.64)

$$\|\tilde{S}_j R_2 y_0 - \tilde{S}_j R_2 x_0\|_c \leqq \{\|\tilde{S}_j R\|_c \mu_3 + \|\tilde{S}_j r_2\|_c\}\|y_0 - x_0\|_c, \qquad (j = 4, 5). \tag{4.70}$$

By (4.54), (4.69), (4.70) and (4.61) we have

$$\begin{aligned} \|v_i\|_c &\leqq \{p\|\tilde{S}_{i+4}R\|_c\mu_3 + \|\tilde{S}_{i+2}R\|_c(q\mu_3 + p\mu_5) + p\|\tilde{S}_{i+4}r_2\|_c\}\|y - x\|_w \\ &\leqq \kappa_{i+2}\|y - x\|_w \qquad (i = 0, 1). \end{aligned} \tag{4.71}$$

Let $z = \tilde{K}y - \tilde{K}x$. Then by (4.53), (4.67) and (4.71)

$$\|z_i\|_c = \|u_i + v_i\|_c \leqq (\kappa_i + \kappa_{i+2})\|y - x\|_w \qquad (i = 0, 1),$$

so that by (iii)

$$\begin{aligned} \|\tilde{K}y - \tilde{K}x\|_w &= \max(p^{-1}\|z_0\|_c, q^{-1}\|z_1\|_c) \\ &\leqq \max(p^{-1}(\kappa_0 + \kappa_2), q^{-1}(\kappa_1 + \kappa_3))\|y - x\|_w \leqq \kappa\|y - x\|_w. \end{aligned}$$

The proof is completed by the same argument as in the proof of Theorem 2.

4.3.2. Case 2

Let $P_3, R_8 \in L(W[J], M[J])$ and $R_9 \colon D^1 \to C[J]$ be defined by

$$P_3 h = h_1(t) - \int_{t_0}^t A_1(s)h_1(s)ds - \int_{t_0}^t [Yh_0](s)ds \qquad \text{for} \quad h \in W[J], \tag{4.72}$$

$$R_8 h = R_3(h_1 - P_3 h) - \tilde{E}(A - A_1)h_1 \qquad \text{for} \quad h \in W[J], \tag{4.73}$$

(4.74) $R_9x = R_3(x_1 - Q_1x) + \tilde{\Phi}(\tilde{\Phi}_I(t_0) - I)$ for $x \in D^1$,

where the matrix A_1 is given by (3.37) and the operator R_3 is given by (3.40). Then we have the following

LEMMA 9. *Let*

(4.75) $$\det \tilde{G} \neq 0.$$

Then $\tilde{L}_I$ is invertible if one of the following two conditions is satisfied:

(4.76) $$\nu = \rho\{\|A_1 - \gamma A\|_c + \|\gamma_1 Y\|_c \sigma\} < 1,$$

(4.77) $$\max(p^{-1}\nu_0, q^{-1}\nu_1) < 1,$$

where σ, ν_0 and ν_1 are constants such that

(4.78) $$\|\tilde{S}_4\tilde{E}_1\|_c + \|\tilde{S}_2\tilde{E}_1\|_c + \|\tilde{S}_0\|_c \leqq \sigma,$$

(4.79) $$p\|\tilde{S}_{i+4}R_4\|_c + \|\tilde{S}_{i+2}R_8\|_c \leqq \nu_i \qquad (i = 0, 1),$$

and R_4 is the operator given by (3.41).

PROOF. Let L_1 be the operator defined by

(4.80) $$L_1 h = (P_2 h_0, P_3 h, l[h]) \quad \text{for} \quad h \in W[J],$$

where P_2 is given by (3.39). Then it can be shown that L_1 is invertible by the same argument as in the proof of Lemma 6.

Suppose (4.76) holds. For any $\varphi = (u_0, u_1, e) \in B_0$ let $h = \tilde{L}_I \varphi$. Then by (4.18).

(4.81) $$h_i = \tilde{S}_{i+4}\tilde{E}_1 u_0 - \tilde{S}_{i+2}\tilde{E}_1 u_1 + \tilde{S}_i e \qquad (i = 0\ 1),$$

and in the same manner as for (3.50) we have

(4.82) $$(I - L_1\tilde{L}_I)\varphi = (v_0, v_1, 0),$$

where

(4.83) $$v_0(t) = \int_{t_0}^{t} \{A_1(s) - \gamma(s)A(s)\} u_0(s) ds,$$

(4.84) $$v_1(t) = \int_{t_0}^{t} \{A_1(s) - \gamma(s)A(s)\} u_1(s) ds + \int_{t_0}^{t} \gamma_1(s) [Y h_0](s) ds.$$

By (4.81), (4.78)

(4.85) $$\|h_0\|_c \leqq (\|\tilde{S}_4\tilde{E}_1\|_c + \|\tilde{S}_2\tilde{E}_1\|_c + \|\tilde{S}_0\|_c) \|\varphi\|_b \leqq \sigma \|\varphi\|_b,$$

and by (4.83), (4.84), (4.85) and (4.76) we have

$$\|v_0\|_c \leqq \rho\|A_1 - \gamma A\|_c\|u_0\|_c \leqq \nu\|\varphi\|_b,$$

$$\|v_1\|_c \leqq \rho\|A_1 - \gamma A\|_c\|u_1\|_c + \rho\|\gamma_1 Y\|_c\|h_0\|_c \leqq \nu\|\varphi\|_b,$$

so that by (4.82)

$$\|(I - L_1\tilde{L}_I)\varphi\|_b = \max(\|v_0\|_c, \|v_1\|_c) \leqq \nu\|\varphi\|_b.$$

Hence $\tilde{L}_I$ is invertible by (4.76) and Lemma 1.

We treat next the case where (4.77) is valid. By (3.51) we have

(4.86) $$\tilde{E}_1 P_2 v_0 = v_0 - \tilde{\Phi}\tilde{\Phi}_I(t_0)v_0(t_0) - R_4 v_0,$$

(4.87) $$\tilde{E}_1 P_3 v = v_1 - \tilde{E} Y v_0 - R_8 v \qquad \text{for} \quad v \in W[J].$$

Substituting $u_0 = P_2 v_0$, $u_1 = P_3 v$ and $e = l[v]$ into (4.81) and making use of (4.86) and (4.87), we obtain

$$(I - \tilde{L}_I L_1)v = (w_0, w_1),$$

where

$$w_i = \tilde{S}_{i+4} R_4 v_0 - \tilde{S}_{i+2} R_8 v \qquad (i = 0, 1).$$

Since by (4.79)

$$\|(I - \tilde{L}_I L_1)v\|_w = \max(p^{-1}\|w_0\|_c, q^{-1}\|w_1\|_c) \leqq \max(p^{-1}\nu_0, q^{-1}\nu_1)\|v\|_w,$$

$\tilde{L}_I$ is invertible by (4.77) and Lemma 1.

LEMMA 10. *If* $\det\tilde{G} \neq 0$, *then the conclusion of Lemma* 8 *is valid with* (4.55) *replaced by*

(4.88) $$u_i = \tilde{S}_{i+4} R_5 x_0 - \tilde{S}_{i+2} R_9 x \qquad (i = 0, 1),$$

where R_5 *is the operator given by* (3.42).

PROOF. By (3.51) we have

(4.89) $$\tilde{E}_1 Q_1 x = x_1 - \tilde{E} T_1(x_0)x_1 - \tilde{\Phi} - R_9 x \qquad \text{for} \quad x \in D^1.$$

Substitution of (3.55) and (4.89) into (4.56) completes the proof by the same argument as in the proof of Lemma 8.

THEOREM 7. *Suppose the assumptions of Theorem* 6 *are satisfied with* (4.61) *replaced by*

(4.90) $$p\|\tilde{S}_{i+4} R_3\|_c(1 + \rho\mu_3) + \|\tilde{S}_{i+2} R_3\|_c \rho(q\mu_3 + p\mu_5) \leqq \kappa_{i+2} \qquad (i = 0, 1).$$

Then the conclusion of Theorem 6 *is valid.*

PROOF. For any $x, y \in D_\delta^1$ we have by (3.57)

$$(4.91)\qquad \|\tilde{S}_i R_5 y_0 - \tilde{S}_i R_5 x_0\|_c \leqq \|\tilde{S}_i R_3\|_c (1 + \rho\mu_3) \|y_0 - x_0\|_c \qquad (i = 4, 5)$$

and by (4.74) and (4.68)

$$(4.92)\qquad \|\tilde{S}_j R_9 y - \tilde{S}_j R_9 x\|_c \leqq \|\tilde{S}_j R_3\|_c \rho(q\mu_3 + p\mu_5) \|y - x\|_w \qquad (j = 2, 3).$$

Let $v = \tilde{K}_2 y - \tilde{K}_2 x$. Then by (4.91), (4.92) and (4.90) it follows that

$$(4.93)\qquad \|v_i\|_c \leqq \{p\|\tilde{S}_{i+4} R_3\|_c (1 + \rho\mu_3) + \|\tilde{S}_{i+2} R_3\|_c \rho(q\mu_3 + p\mu_5)\} \|y - x\|_w$$
$$\leqq \kappa_{i+2} \|y - x\|_w \qquad (i = 0, 1).$$

The proof is completed by the same argument as in the proof of Theorem 6.

Let σ_0, σ_1 and σ_2 be constants such that

$$\|\gamma_1\|_c \leqq \sigma_0, \quad \|A - A_1\|_c \leqq \sigma_1, \quad \|A - A_2\|_c \leqq \sigma_2,$$

and let α_j ($j = 0, 1, 2, 3$) be constants satisfying

$$\|\tilde{S}_{i+4} R_3\|_c \leqq \|\tilde{H}_i\|_c \sigma_2 + \|\tilde{S}_{i+4}\|_c \sigma_0 \leqq \alpha_i \qquad (i = 0, 1),$$
$$\|\tilde{S}_j R_3\|_c \leqq \|\tilde{S}_j \tilde{E}\|_c \sigma_2 + \|\tilde{S}_j\|_c \sigma_0 \leqq \alpha_j \qquad (j = 2, 3).$$

Then (4.79) and (4.90) can be replaced respectively by

$$(4.94)\qquad p\alpha_i + \alpha_{i+4}\|A_1\|_c + p\alpha_{i+2}\rho\|Y\|_c + \sigma_1(p\|H_i\|_c + q\|S_{i+2}E\|_c) \leqq v_i \qquad (i = 0, 1),$$

$$(4.95)\qquad p\alpha_i + \alpha_{i+4}\mu_3 + p\alpha_{i+2}\rho\mu_5 \leqq \kappa_{i+2} \qquad (i = 0, 1),$$

where

$$\alpha_{i+4} = \rho(p\alpha_i + q\alpha_{i+2}) \qquad (i = 0, 1).$$

Hence by (3.58)–(3.61) we can estimate the left sides of (4.76) and (4.90) without computing $\tilde{\Phi}^{-1}$ and $\tilde{\Phi}_I^{-1}$.

4.3.3. Treatment in the original form

In this subsection we treat the boundary value problem (2.4), (4.2) directly without replacing (2.4) and (4.1) by systems of integral equations.

Let $C^1[J]$ be the space of all real n-vector functions continuously differentiable on J with the norm $\|\cdot\|_c$ and denote by $M^1[J]$ the space of all real $n \times n$ matrix functions continuously differentiable on J. Let $W^1[J] = C^1[J] \times M^1[J]$

be the space with the norm $\|\cdot\|_w$ and put $D^2=D^1 \cap W^1[J]$. Let $B=C[J]\times M[J]\times R^n \times M^n$ be a Banach space with the norm

$$\|\varphi\|_b = \max(\|u_0\|_c, \|u_1\|_c, \|e_0\|, \|e_1\|) \qquad \text{for} \quad \varphi=(u_0, u_1, e_0, e_1)\in B,$$

where $M^n=L(R^n, R^n)$.

We consider the equation

$$\mathscr{F}x = 0 \qquad \text{for} \quad x\in D^2, \tag{4.96}$$

where the operator $\mathscr{F}: D^2 \to B$ is defined by

$$\mathscr{F}x = \left(\frac{dx_0}{dt} - X(x_0, t),\ \frac{dx_1}{dt} - X_1[x_0]x_1, f[x],\ x_1(t_0) - I\right) \qquad \text{for} \quad x\in D^2. \tag{4.97}$$

Let $\mathscr{L}: W^1[J]\to B$ be the linear operator defined by

$$\mathscr{L}h = \left(\frac{dh_0}{dt} - A(t)h_0,\ \frac{dh_1}{dt} - A(t)h_1 - Yh_0,\ l[h],\ h_1(t_0)\right) \qquad \text{for} \quad h\in W^1[J]. \tag{4.98}$$

Then we have the following analogue of Lemma 6.

LEMMA 11. *$\mathscr{L}$ has an inverse operator $\mathscr{L}_I$ if and only if*

$$\det G \neq 0. \tag{4.99}$$

Suppose (4.99) *is satisfied. Then for any* $\varphi=(u_0, u_1, e_0, e_1)\in B$

$$\mathscr{L}_I\varphi = h, \tag{4.100}$$

where

$$h_i = H_iu_0 - S_{i+2}Eu_1 + S_ie_0 - S_{i+2}\Phi e_1 \qquad (i=0, 1). \tag{4.101}$$

PROOF. By (4.98) the equation $\mathscr{L}h=\varphi$ is equivalent to the system

$$\frac{dh_0}{dt} - A(t)h_0 = u_0, \tag{4.102}$$

$$\frac{dh_1}{dt} - A(t)h_1 = u_1 + Yh_0, \tag{4.103}$$

$$l_0[h_0] + l_1[h_1] = e_0, \tag{4.104}$$

$$h_1(t_0) = e_1. \tag{4.105}$$

The general solution of (4.102) is given by

$$h_0 = \Phi c + Eu_0 \tag{4.106}$$

with an arbitrary $c \in R^n$. The solution of the initial value problem (4.103), (4.105) is

$$h_1 = \Phi e_1 + Eu_1 + EYh_0 \tag{4.107}$$

and substitution of (4.106) into (4.107) yields

$$h_1 = \Phi e_1 + Eu_1 + EYEu_0 + EY\Phi c. \tag{4.108}$$

By (4.15), (4.106) and (4.108) it follows from (4.104) that

$$l_2[Eu_0] + l_1[Eu_1] + l_1[\Phi]e_1 + Gc = e_0. \tag{4.109}$$

The proof is completed by the same argument as in the proof of Lemma 6.

Let $\mathscr{K}$ and $\mathscr{K}_1$ be the operators from D^2 into $W^1[J]$ defined by

$$\mathscr{K}x = (I - \mathscr{L}_I\mathscr{F})x \qquad \text{for} \quad x \in D^2, \tag{4.110}$$

$$\mathscr{K}_1 x = \mathscr{L}_I(\mathscr{L} - \mathscr{F})x \qquad \text{for} \quad x \in D^2. \tag{4.111}$$

Suppose $\tilde{\Phi}$ and $\tilde{\Phi}_I$ are continuously differentiable on J and let the operator $\tilde{K}_2$ be defined by

$$\tilde{K}_2 h = u \qquad \text{for} \quad h \in W^1[J], \tag{4.112}$$

where

$$u_i = \tilde{H}_i(A - A_2)h_0 - \tilde{S}_{i+2}\tilde{E}(A - A_2)h_1 + \tilde{S}_{i+4}\gamma_1 h_0 \tag{4.113}$$
$$- \tilde{S}_{i+2}\gamma_1 h_1 - \tilde{S}_{i+2}\tilde{\Phi}(\tilde{\Phi}_I(t_0) - I)h_1(t_0) \qquad (i = 0, 1).$$

Now we show the following

LEMMA 12. *$\tilde{\mathscr{L}}_I$ is invertible if*

$$\det \tilde{G} \neq 0, \tag{4.114}$$

$$\nu = \max(\|\gamma_1\|_c + \|\gamma_1 Y\|_c \sigma, \|I - \tilde{\Phi}(t_0)\|) < 1, \tag{4.115}$$

where σ is a constant such that

$$\|\tilde{H}_0\|_c + \|\tilde{S}_2\tilde{E}\|_c + \|\tilde{S}_0\|_c + \|\tilde{S}_2\tilde{\Phi}\|_c \leqq \sigma. \tag{4.116}$$

PROOF. Let L_1 be the operator defined by

(4.117) $$L_1 h = \left(\frac{dh_0}{dt} - A_1 h_0,\ \frac{dh_1}{dt} - A_1 h_1 - Y h_0,\ l[h],\ h_1(t_0)\right) \quad \text{for} \quad h \in W^1[J],$$

where A_1 is the matrix given by (3.37). Then it can be shown that L_1 is invertible by the same argument as in the proof of Lemma 11.

For any $\varphi = (u_0, u_1, e_0, e_1) \in B$ let $h = \tilde{\mathscr{L}}_I \varphi$. Then by (4.100)

(4.118) $$h_0 = \tilde{E} u_0 + \tilde{\Phi} \tilde{G}^{-1}(e_0 - \tilde{l}_2[\tilde{E} u_0] - l_1[\tilde{E} u_1] - l_1[\tilde{\Phi}] e_1),$$

(4.119) $$h_1 = \tilde{E}(u_1 + Y h_0) + \tilde{\Phi} e_1$$

and by (3.37)

(4.120) $$\frac{dh_0}{dt} = A_1 h_0 + \gamma u_0,$$

(4.121) $$\frac{dh_1}{dt} = A_1 h_1 + \gamma(u_1 + Y h_0).$$

Since $\tilde{G} = \tilde{l}_2[\tilde{\Phi}]$, by (4.13) and (4.14) we have

(4.122) $$\begin{aligned} l[h] &= \tilde{l}_2[h_0] + l_1[\tilde{E} u_1] + l_1[\tilde{\Phi}] e_1 \\ &= \tilde{l}_2[\tilde{E} u_0] + \tilde{l}_2[\tilde{\Phi}] \tilde{G}^{-1}\{e_0 - \tilde{l}_2[\tilde{E} u_0] - l_1[\tilde{E} u_1] - l_1[\tilde{\Phi}] e_1\} \\ &\quad + l_1[\tilde{E} u_1] + l_1[\tilde{\Phi}] e_1 = e_0. \end{aligned}$$

From (4.119) it follows that

(4.123) $$h_1(t_0) = \tilde{\Phi}(t_0) e_1.$$

By (4.120)–(4.123)

(4.124) $$(I - L_1 \tilde{\mathscr{L}}_I)\varphi = (\gamma_1 u_0,\ \gamma_1(u_1 + Y h_0),\ 0,\ (I - \tilde{\Phi}(t_0)) e_1).$$

By (4.100) and (4.116) we have

$$\|h_0\|_c \leqq (\|\tilde{H}_0\|_c + \|\tilde{S}_2 \tilde{E}\|_c + \|\tilde{S}_0\|_c + \|\tilde{S}_2 \tilde{\Phi}\|_c)\,\|\varphi\|_b \leqq \sigma \|\varphi\|_b,$$

so that by (4.124) and (4.115)

$$\begin{aligned} \|(I - L_1 \tilde{\mathscr{L}}_I)\varphi\|_b &= \max(\|\gamma_1 u_0\|_c,\ \|\gamma_1(u_1 + Y h_0)\|_c,\ \|(I - \tilde{\Phi}(t_0)) e_1\|) \\ &\leqq \max(\|\gamma_1\|_c + \|\gamma_1 Y\|_c \sigma,\ \|I - \tilde{\Phi}(t_0)\|)\,\|\varphi\|_b \leqq \nu \|\varphi\|_b. \end{aligned}$$

Hence $\tilde{\mathscr{L}}_I$ is invertible by (4.115) and Lemma 1.

We have the following

THEOREM 8. *Let $x^{(0)} \in D^2$ be an approximate solution of* (4.96) *and suppose there exist an operator $\tilde{\mathscr{L}}_I$, a positive number δ and nonnegative constants η, κ, κ_j ($j=0$, 1, 2, 3) such that*

(i) *$\tilde{\mathscr{L}}_I$ is invertible;*

(ii) $D_\delta^2 = \{x \in W^1[J] \mid \|x - x^{(0)}\|_w \leqq \delta\} \subset D^2$;

(iii) $\kappa = \max(p^{-1}(\kappa_0 + \kappa_2), q^{-1}(\kappa_1 + \kappa_3)) < 1$,

$$p\|\tilde{H}_i\|_c\mu_1 + \|\tilde{S}_i\|_c\mu_2 + \|\tilde{S}_{i+2}\tilde{E}\|_c(q\mu_1 + p\mu_4) \leqq \kappa_i \qquad (i = 0, 1), \tag{4.125}$$

$$\|A - A_2\|_c(p\|\tilde{H}_i\|_c + q\|\tilde{S}_{i+2}\tilde{E}\|_c) + \|\gamma_1\|_c(p\|\tilde{S}_{i+4}\|_c + q\|\tilde{S}_{i+2}\|_c) + q\|\tilde{S}_{i+2}\tilde{\Phi}\|_c\|\tilde{\Phi}_I(t_0) - I\| \leqq \kappa_{i+2} \qquad (i = 0, 1), \tag{4.126}$$

where μ_1, μ_2 and μ_4 are constants such that

$$\|T_1(x_0)\|_c \leqq \mu_1 \qquad \textit{for all} \quad x \in D_\delta^2, \tag{4.127}$$

$$\|f'(x) - l\| \leqq \mu_2 \qquad \textit{for all} \quad x \in D_\delta^2, \tag{4.128}$$

$$\|T_2(x)\|_c \leqq \mu_4 \qquad \textit{for all} \quad x \in D_\delta^2; \tag{4.129}$$

(iv) $\|\tilde{\mathscr{L}}_I \mathscr{F} x^{(0)}\|_w \leqq \eta$;

(v) $\lambda = \eta/(1 - \kappa) \leqq \delta$.

Then the conclusion of Theorem 5 is valid with K and D_δ^1 replaced by $\tilde{\mathscr{K}}$ and D_δ^2 respectively.

PROOF. For any $x \in D^2$ by (4.97) and (4.98) we have

$$(\mathscr{L} - \mathscr{F})x = (Tx_0,\ T_1(x_0)x_1 - Yx_0,\ l[x] - f[x],\ I)$$

and by (4.111) and (4.100)

$$\tilde{\mathscr{K}}_1 x = u, \tag{4.130}$$

where

$$u_i = \tilde{H}_1 Tx_0 - \tilde{S}_{i+2}\tilde{E}\{T_1(x_0)x_1 - Yx_0\} + \tilde{S}_i(l[x] - f[x]) - \tilde{S}_{i+2}\tilde{\Phi} \qquad (i = 0, 1). \tag{4.131}$$

For any $h \in W^1[J]$ let $v = \tilde{\mathscr{L}}_I \mathscr{L} h$. Since $\tilde{\Phi}_I' = -\tilde{\Phi}_I A_2$, by integration by parts we have

$$\tilde{E}\left\{\frac{dh_0}{dt} - A(t)h_0\right\} = \tilde{E}(A_2 - A)h_0 + \gamma h_0 - \tilde{\Phi}\tilde{\Phi}_I(t_0)h_0(t_0),$$

$$\tilde{E}\left\{\frac{dh_1}{dt} - A(t)h_1 - Yh_0\right\} = \tilde{E}(A_2 - A)h_1 - \tilde{E}Yh_0 + \gamma h_1 - \tilde{\Phi}\tilde{\Phi}_I(t_0)h_1(t_0)$$

and by (4.100)

$$v_i = \tilde{S}_{i+4}\{\tilde{E}(A_2 - A)h_0 + \gamma h_0 - \tilde{\Phi}\tilde{\Phi}_I(t_0)h_0(t_0)\} - \tilde{S}_{i+2}\{\tilde{E}(A_2 - A)h_1 - \tilde{E}Yh_0 + \gamma h_1 - \tilde{\Phi}\tilde{\Phi}_I(t_0)h_1(t_0)\} + \tilde{S}_i l[h] - \tilde{S}_{i+2}\tilde{\Phi}h_1(t_0) \quad (i = 0, 1). \tag{4.132}$$

Since $\tilde{S}_{i+4}\tilde{\Phi}=0$ $(i=0, 1)$ and

$$\tilde{S}_{i+2}\tilde{E}Yh_0 - \tilde{S}_{i+2}\gamma h_1 + \tilde{S}_i l[h] = -\tilde{S}_{i+4}h_0 + \tilde{S}_{i+2}\gamma_1 h_1 + h_1 \quad (i = 0, 1),$$

by (4.132) we have

$$(I - \tilde{\mathscr{L}}_I\mathscr{L})h = w, \tag{4.133}$$

where

$$w_i = \tilde{H}_i(A - A_2)h_0 - \tilde{S}_{i+2}\tilde{E}(A - A_2)h_1 - \tilde{S}_{i+4}\gamma_1 h_0 - \tilde{S}_{i+2}\gamma_1 h_1 - \tilde{S}_{i+2}\tilde{\Phi}(\tilde{\Phi}_I(t_0) - I)h_1(t_0) \quad (i = 0, 1). \tag{4.134}$$

Hence by (4.110), (4.111), (4.112) and (4.133)

$$\tilde{\mathscr{K}}x = \tilde{\mathscr{L}}_I(\mathscr{L} - \mathscr{F})x + (I - \tilde{\mathscr{L}}_I\mathscr{L})x = \tilde{\mathscr{K}}_1 x + \tilde{K}_2 x.$$

For any $x, y \in D_\delta^2$ let $u = \tilde{\mathscr{K}}_1 y - \tilde{\mathscr{K}}_1 x$. Then by (4.130)

$$u_i = \tilde{H}_i(Ty_0 - Tx_0) + \tilde{S}_i(l[y - x] - f[y] + f[x]) - \tilde{S}_{i+2}\tilde{E}\{T_1(y_0)y_1 - T_1(x_0)x_1 - Yy_0 + Yx_0\} \quad (i = 0, 1),$$

so that by (4.37)–(4.39) and (4.125)

$$\|u_i\|_c \leqq \{p\|\tilde{H}_i\|_c\mu_1 + \|\tilde{S}_i\|_c\mu_2 + \|\tilde{S}_{i+2}\tilde{E}\|_c(q\mu_1 + p\mu_4)\}\,\|y - x\|_w \leqq \kappa_i\|y - x\|_w \quad (i = 0, 1). \tag{4.135}$$

Let $v = \tilde{\mathscr{K}}_2 y - \tilde{\mathscr{K}}_2 x$. Then by (4.133) and (4.126) we have

$$\begin{aligned} \|v_i\|_c \leqq{}& \|\tilde{H}_i\|_c\|A - A_2\|_c\|y_0 - x_0\|_c + \|\tilde{S}_{i+2}\tilde{E}\|_c\|A - A_2\|_c\|y_1 - x_1\|_c \\ &+ \|\tilde{S}_{i+4}\|_c\|\gamma_1\|_c\|y_0 - x_0\|_c + \|\tilde{S}_{i+2}\|_c\|\gamma_1\|_c\|y_1 - x_1\|_c \\ &+ \|\tilde{S}_{i+2}\tilde{\Phi}\|_c\|\tilde{\Phi}_I(t_0) - I\|\,\|y_1 - x_1\|_c \\ \leqq{}& \kappa_{i+2}\|y - x\|_w \quad (i = 0, 1). \end{aligned} \tag{4.136}$$

Let $z=\tilde{\mathscr{K}}y-\tilde{\mathscr{K}}x$. Then by (4.135) and (4.136)

$$\|z_i\|_c = \|u_i + v_i\|_c \leqq (\kappa_i + \kappa_{i+2})\|y - x\|_w \qquad (i = 0, 1),$$

so that by (iii)

$$\|\tilde{\mathscr{K}}y - \tilde{\mathscr{K}}x\|_w = \max(p^{-1}\|z_0\|_c, q^{-1}\|z_1\|_c)$$

$$\leqq \max(p^{-1}(\kappa_0 + \kappa_2), q^{-1}(\kappa_1 + \kappa_3))\|y - x\|_w \leqq \kappa\|y - x\|_w.$$

The proof is completed by the same argument as in the proof of Theorem 5.

4.4. A numerical example

We consider the boundary value problem

$$(4.137) \qquad \frac{dx}{dt} = X(x, t) \equiv \begin{pmatrix} x_2 \\ -x_1 - (x_1 - t)^3 + t + 0.1 \end{pmatrix} \qquad (-1 \leqq t \leqq 1),$$

$$(4.138) \qquad f[y] \equiv (g'(x)[Z])^* g[x] = 0 \qquad \text{for} \quad y = (x, Z) \in D^1,$$

where a^* denotes the transpose of a matrix a, $Z(t)$ is the solution of the matrix equation

$$(4.139) \qquad \frac{dZ}{dt} = X_1[x]Z \equiv \begin{pmatrix} 0 & 1 \\ -1 - 3(x_1 - t)^2 & 0 \end{pmatrix} Z$$

with $Z(t_0)=I$, and

$$(4.140) \qquad g[x] = \begin{pmatrix} g_1[x] \\ g_2[x] \\ g_3[x] \end{pmatrix} \equiv \begin{pmatrix} x_1(-1) + 0.9 \\ \alpha(x_2(0)^2 - \beta) \\ x_1(1) - 1.1 \end{pmatrix},$$

$$(4.141) \qquad t_0 = -1, \quad \alpha = 0.1, \quad \beta = 1.1.$$

The condition (4.138) arises from the boundary value problem of the least squares type (4.137) and $(g[x])^* g[x] = \min.$ [2].

We denote by $y^{(0)} = (x^{(0)}, Z^{(0)})$ an approximate solution of this problem with

$$(4.142) \qquad x_1^{(0)}(t) = t + 0.1, \quad x_2^{(0)}(t) = 1, \quad Z^{(0)}(t) = \Phi(t),$$

where $\Phi(t)$ is the solution of the problem

$$\frac{d\Phi}{dt} = A(t)\Phi, \quad \Phi(-1) = I,$$

(4.143) $$A(t) \equiv X_1[x^{(0)}](t) = \begin{pmatrix} 0 & 1 \\ -\mu^2 & 0 \end{pmatrix}, \qquad \mu = \sqrt{1.03}.$$

With the notations

(4.144) $$s(t) = \sin\mu(t+1), \quad c(t) = \cos\mu(t+1), \quad \nu = 1/\mu,$$

$\Phi(t)$ and $\Phi_I(t)$ can be written as follows:

(4.145) $$\Phi(t) = \begin{pmatrix} c(t) & \nu s(t) \\ -\mu s(t) & c(t) \end{pmatrix}, \qquad \Phi_I(t) = \begin{pmatrix} c(t) & -\nu s(t) \\ \mu s(t) & c(t) \end{pmatrix}.$$

Let the operators N_i: $R^2 \to R^1$ ($i=0, 1$) and the matrices C_j ($j=1, 2, 3$) be defined by

(4.146) $$N_i h = h_i \qquad (i = 1, 2) \qquad \text{for} \quad h = (h_1, h_2)^* \in R^2,$$

(4.147) $$C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then for $y=(x, Z) \in D^1$ and $v=(h, U) \in W[J]$ we have

(4.148) $$X_2(x)[h, Z] = -6(x_1 - t)(C_3 Z C_2, C_3 Z C_3)h,$$

(4.149) $$f_0(y)h = Z(-1)^* C_2 h(-1) + Z(1)^* C_2 h(1) + 2\alpha^2(3x_2(0)^2 - \beta)Z(0)^* C_1 h(0),$$

(4.150) $$f_1(y)U = (g_1[x]N_1[U(-1)] + g_3[x]N_1[U(1)] + 2\alpha x_2(0) g_2[x] N_2[U(0)])^*.$$

We choose the operators Y, l_0 and l_1 as follows:

(4.151) $$Yh = X_2(x^{(0)})[h, Z^{(0)}], \quad l_0 = f_0(y^{(0)}), \quad l_1 = f_1(y^{(0)}).$$

For simplicity put

(4.152) $$m = 10^{-3}, \quad m_1 = -0.1m\nu^3, \quad a = 2\alpha^2(3-\beta), \quad a_1 = 2\alpha^2(1-\beta),$$

$$a_2 = \mu a, \quad a_3 = -0.2\nu a_1, \quad B_1 = \Phi(0)^* C_1, \quad B_2 = \Phi(1)^* C_2,$$

$$C_4 = \mu C_2 + \nu C_1, \quad C_5 = \nu C_2 + \mu C_1, \quad C_6 = \mu C_3 - \nu C_3^*,$$

$$e = (1, 1)^*, \quad u_1(t) = 1 + 2c(t), \quad u_2(t) = 1 - c(t),$$

$$u_3(t) = 1 - c(t)^3, \quad u_4(t) = \mu(t+1), \quad u_5(t) = 2 + c(t)^2,$$

$$u_6(t) = 2 + s(t)^2, \quad V(t) = \mu\Phi(t)^* C_2 \Phi(t),$$

$$V_1(t) = \nu \begin{pmatrix} -u_3(t) & -\nu s(t)^3 \\ \mu s(t) u_5(t) & u_3(t) \end{pmatrix},$$

$$V_2(t) = \nu^2 \begin{pmatrix} -s(t)^3 & \nu\{c(t)u_6(t) - 2\} \\ \mu u_3(t) & s(t)^3 \end{pmatrix},$$

$$V_3(t) = (V_1(t)^*, V_2(t)^*) N_2[\Phi(0)]^*,$$

$$V_4(t) = 3\nu\{C_5 V(t) + u_4(t) C_6 - C_2\}.$$

Then by (4.145) and (4.151)

$$G = C_2 + \nu V(1) + a_2(C_4 - V(0)) + a_3 V_3(0), \tag{4.153}$$

$$S_0(t) = \Phi(t) G^{-1}, \quad S_1(t) = -0.2\nu\Phi(t)(V_1(t), V_2(t)) G^{-1}, \tag{4.154}$$

$$S_2 EU = a_1 S_0(t) \int_{-1}^{0} N_2[\Phi(0)\Phi_I(\tau) U(\tau)]^* d\tau \qquad \text{for} \quad U \in M[J], \tag{4.155}$$

$$S_3 EU = -\Phi(t) \int_{-1}^{t} \Phi_I(\tau) \{0.6 C_3 \Phi(\tau) N_1[S_2 EU](\tau) + U(\tau)\} d\tau \qquad \text{for} \quad U \in M[J], \tag{4.156}$$

$$H_0 h = \int_{-1}^{1} H_0(t, \tau) h(\tau) d\tau \qquad \text{for} \quad h \in C[J], \tag{4.157}$$

where

$$H_0(t, \tau) = \begin{cases} \Phi(t)(I - M_1(\tau))\Phi_I(\tau), & \tau < t, \\ -\Phi(t) M_1(\tau) \Phi_I(\tau), & \tau \geqq t \qquad (-1 \leqq \tau < 0), \end{cases} \tag{4.158}$$

$$H_0(t, \tau) = \begin{cases} \Phi(t)(I - M_2)\Phi_I(\tau), & \tau < t, \\ -\Phi(t) M_2 \Phi_I(\tau), & \tau \geqq t \qquad (0 \leqq \tau \leqq 1), \end{cases} \tag{4.159}$$

$$M_1(\tau) = G^{-1}\{a B_1 \Phi(0) + a_3(V_3(0) - V_3(\tau))\} + M_2,$$

$$M_2 = G^{-1} B_2 \Phi(1).$$

By (4.142) and (4.151) we have

$$Q x^{(0)} = m(0, t+1)^*, \quad Q_1 y^{(0)} = 0, \tag{4.160}$$

(4.161) $f[y^{(0)}] = a_1(-\mu s(0), c(0))^*, \quad E_1Qx^{(0)} = mv^2(u_2(t), \mu s(t))^*,$

(4.162) $$l_2[E_1Qx^{(0)}] = mv\{as(0)B_1 + vu_2(1)B_2\}e + m_1a_1\begin{pmatrix} s(0)(1-4c(0)) + 3\mu c(0) \\ 3s(0) - 2vu_1(0)u_2(0) \end{pmatrix}.$$

Let $b=(b_1, b_2)^*$, b_3 and b_4 be defined by

(4.163) $$b = G^{-1}(f[y^{(0)}] - l_2[E_1Qx^{(0)}]),$$

(4.164) $$b_3 = 2(\mu^2 b_1/m - 1), \quad b_4 = 2\mu^2 b_2/m.$$

Then we have

(4.165) $$L_IFy^{(0)} = (h, U),$$

where

(4.166) $$h(t) = \Phi(t)b + [E_1Qx^{(0)}](t),$$

(4.167) $$U(t) = m_1\Phi(t)\{V_4(t) + b_3V_1(t) + b_4V_2(t)\}.$$

Now let us use the infinity norm $\|\cdot\|_\infty$ and apply Theorem 5 to our problem. Then by (4.153)–(4.159) we have the estimates

(4.168) $$\|S_0\|_{\infty c} \leqq 2.50387, \quad \|S_1\|_{\infty c} \leqq 2.32728, \quad \|S_2E\|_{\infty c} \leqq 0.45284m,$$
$$\|S_3E\|_{\infty c} \leqq 2.85033, \quad \|H_0\|_{\infty c} \leqq 3.18136,$$
$$\|H_1\|_{\infty c} \leqq \|EY\|_{\infty c}\|H_0\|_{\infty c} \leqq 5.35949.$$

For any $p>0$ and $q>0$ by (4.143)–(4.151) we may choose

(4.169) $$\mu_1 = 3p\delta(p\delta + 0.2), \quad \mu_2 = p\mu_{20} + q\mu_{21}, \quad \mu_4 = 6\delta(pq\delta + 0.1q + \sigma),$$

where

(4.170) $$\mu_{20} = 2q\delta\{1 + 3\alpha^2 p(p\delta + 2)(\delta + \sigma_1) + \alpha^2|3 - \beta|\},$$

(4.171) $$\mu_{21} = 2p\delta\{1 + \alpha^2(p\delta + 2)(\delta + 1) + \alpha^2|1 - \beta|\},$$

(4.172) $$\sigma = p\max_{t\in J}(|c(t)| + \mu|s(t)|), \quad \sigma_1 = q^{-1}\mu s(0).$$

By (4.165)–(4.167)

(4.173) $$\|L_IFy^{(0)}\|_{\infty w} \leqq \max(p^{-1}\eta_0, q^{-1}\eta_1) = \eta,$$

where

(4.174) $\|h\|_{\infty c} \leqq 1.65924m = \eta_0, \quad \|U\|_{\infty c} \leqq 0.63151m = \eta_1.$

The choice $p=1$, $q=1$ and $\delta=1.79003m$ yields

$$\kappa = 0.073058, \quad \lambda = \eta/(1-\kappa) = 1.79002m = \lambda(1, 1),$$

and we have estimates

(4.175) $\|\hat{x} - x^{(0)}\|_{\infty c} \leqq \lambda(1, 1), \quad \|\hat{Z} - Z^{(0)}\|_{\infty c} \leqq \lambda(1, 1).$

With the choice $p=1$, $q=q_1=2.2603$ and $\delta=1.73539m$ we have

$$\kappa = 0.043875, \quad \lambda = 1.73538m = \lambda(1, q_1).$$

The choice $p=p_1=2.6274$, $q=1$ and $\delta=0.76346m$ yields

$$\kappa = 0.172819, \quad \lambda = 0.76345m = \lambda(p_1, 1).$$

Hence we have error estimates

(4.176) $\|\hat{x} - x^{(0)}\|_{\infty c} \leqq \lambda(1, q_1), \quad \|\hat{Z} - Z^{(0)}\|_{\infty c} \leqq \lambda(p_1, 1).$

From (4.175) and (4.176) it is seen that the parameters p and q have been introduced with effect. The same conclusion is valid also when the norms $\|\cdot\|_2$ and $\|\cdot\|_1$ are used. The results are listed in Table 2, where $\tilde{\eta}=\eta/m$, $\tilde{\delta}=\delta/m$, $\tilde{\kappa}=10\kappa$ and $\tilde{\lambda}(p, q)=\lambda(p, q)/m$.

Table 2.

norm	p	q	$\tilde{\eta}$	$\tilde{\delta}$	$\tilde{\kappa}$	$\tilde{\lambda}(p, q)$
	1.0000	1.0000	1.65924	1.79003	0.73058	1.79002
$\|\cdot\|_\infty$	1.0000	2.2603	1.65924	1.73539	0.43875	1.73538
	2.6274	1.0000	0.63151	0.76346	1.72819	0.76345
	1.0000	1.0000	1.66331	1.73074	0.38957	1.73073
$\|\cdot\|_2$	1.0000	1.8981	1.66331	1.70952	0.27024	1.70951
	3.0018	1.0000	0.55411	0.61113	0.93303	0.61112
	1.0000	1.0000	1.77546	1.94034	0.84970	1.94033
$\|\cdot\|_1$	1.0000	1.6347	1.77546	1.89334	0.62256	1.89333
	2.5486	1.0000	0.69664	0.86199	1.91816	0.86198

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