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ON A PROBABILISTIC GRAPH-THEORETICAL METHOD

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ABSTRACT. We introduce a method by means of which one can simply prove the existence of sparse hypergraphs with large chromatic number. Moreover this method gives the full solution of an Erdös-Ore problem.

Introduction. The existence of sparse (i.e. without short cycles) graphs with a large chromatic number is a classical combinatorial problem. It has been answered affirmatively by probabilistic means by Erdös [**1**] and for hypergraphs by Erdös and Hajnal [**2**] and a construction was provided by Lovász [**5**]. While in [**8**] the present authors suggested a different construction of these objects, the purpose of this note is to provide a new hopefully simpler probabilistic method.

This method allows to prove existence of sparse hypergraphs which contain a certain ordered subhypergraph for every ordering of its vertices. This has been asked by Erdös [**4**] in response to a theorem of Ore and Gallai (Corollary 3). The results may be strengthened so as to answer a question of Bollobás as well (Corollary 4).

Preliminaries. A graph G is a couple (V, E) where V is a set (of vertices) and $E \subseteq [V]^2 = \{e \subseteq V;\ |e| = 2\}$. A k-graph (k-uniform hypergraph) is a couple where V is a set and $E \subseteq [V]^k = \{e \subseteq V;\ |e| = k\}$. An embedding f: $(V, E) \to (V', E')$ is a 1-1 mapping which satisfies

(1) f is 1-1;

(2) $e \in E \Leftrightarrow \{f(x);\ x \in e\} \in E'$. A cycle of length s in a k-graph (V, E) is a sequence

$$x_0, e_1, x_1, e_2, \dots, e_s, x_s, e_0$$

which satisfies $x_i \in e_i$, $x_{i-1} \in e_i$ for $i \in \{1, 2, \dots, s\}$ and $x_s \in e_0$, $x_0 \in e_0$ and there are i, j with $e_i \neq e_j$.

Chromatic number of a k-graph is a minimal number of colours which are sufficient for colouring of vertices in such a way that no edge is monochromatic. It can be proved easily that a k-graph (V, E) does not contain 2-cycles iff $|e \cap e'| \leqslant 1$ for all $e \neq e'$, $e, e' \in E$ and generally a k-graph does not contain cycles of length $< s$ iff $|\bigcup E'| \geqslant (k-1)|E'| + 1$ for every $E' \subset E$, $|E'| < s$ (see e.g. [**2**], [**5**]).

The following lemma is a key fact for our method. The lemma may be

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easily proved by probabilistic argument (see e.g. [3]). We sketch a proof for completeness.

LEMMA. *For all positive integers k and s there exists a k-graph (X, E), $|X| = n$ without cycles of length $< s$ and with $|E| > n^{1+1/s}$ edges for all n sufficiently large.*

PROOF. Let us consider a set $\mathfrak{M}_x$ of all k-graphs $(X, E)_i |X| = n$ with $m = 2[n^{1+1/s}]$ edges. Then

$$|\mathfrak{M}_x| = \binom{\binom{n}{k}}{m}$$

and the average number of edges contained in cycles of length $< s$ is less than

$$\sum_{s=2}^{s-1} c(s,j) \binom{n}{(k-1)j} \frac{\binom{\binom{n}{k}-j}{m-j}}{\binom{\binom{n}{k}}{m}} = o(n)$$

where $c(s,j) > 0$ is a function of s, j which does not depend on n.

Consequently for all n sufficiently large there exists an example of a k-graph $G = (X, E)$, $|X| = n$, $|E| = 2[n^{1+1/s}]$ such that G contains at most $[n^{1+1/s}]$ edges contained in circuits of length $< s$. After deleting these edges we are left a k-graph with at least $[n^{1+1/s}]$ edges without cycles of length $< s$.

We find it convenient to use this lemma for constructing of special sparse graphs. This will be clear from below.

Results and applications.

THEOREM 1 ([1], [2]). *For all positive integers k, n, s there exists a k-graph (X, F) without cycles of length $< s$ with chromatic number $> n$.*

PROOF. Put $p = n(k-1) + 1$. Let (X, E) be a p-graph without cycles of length $< s$, $|X| = N$ and with $[N^{1+1/s}]$ edges. Let $\mathfrak{G}_x$ be the family of all k-graphs (X, F) which contain in each edge of E exactly one edge of F. The following holds:

(1) $|\mathfrak{G}_x| = \binom{p}{k}^{[N^{1+1/s}]}$ as (X, E) does not contain 2-cycles the choice of edges of F in edges of E is mutually independent.

(2) G does not contain cycles of length $< s$ for each $G \in \mathfrak{G}_x$.

However the number of k-graphs in the class $\mathfrak{G}_x$ which admit a given n-coloration is less then $(\binom{p}{k} - 1)^{[N^{1+1/s}]} + 1$. The theorem follows as

$$n^N \left(\binom{p}{k} - 1 \right)^{[N^{1+1/s}]} < |\mathfrak{G}_x| = \binom{p}{k}^{[N^{1+1/s}]}$$

for N sufficiently large.

THEOREM 2. *Let $G = ((V, \leqslant), E)$ be an ordered k-graph (i.e. (V, E) is a*

k-graph; $(V, \leqslant)$ is a totally ordered set) without cycles of length $< s$. Then there exists a k-graph (V', E') without cycles of length $< s$ such that for every ordering $(V', \preccurlyeq)$ there exists a monotone mapping $f\colon (V, \leqslant) \to (V', \preccurlyeq)$ which is an embedding $(V, E) \to (V', E')$.

Proof. Let (V, E) be a k-graph. Denote by m the number of k-graphs with the vertex set V which are isomorphic to (V, E). If $m = 1$ one may put $(V', E') = (V, E)$. (One can show easily that in this case either $E = \varnothing$ or $E = [V]^k$.)

Let us suppose that $m > 1$. Put $|V| = p$. Let (X, U) be a p-graph without cycles of length $< s$ with $[N^{1+1/s}]$ edges. Let $\mathfrak{G}_x$ be the class of all k-graphs $G = (X, F)$ which satisfy:

(1) $G|e = (e, F \cap [e]^k) \simeq (V, E)$ for each edge $e \in U$;

(2) $F = \bigcup_{e \in U}(F \cap [e]^k)$.

Then (i) G does not contain a cycle of length $< s$ for each $G \in \mathfrak{G}_x$;

(ii) $|\mathfrak{G}_x| = m^{[N^{1+1/s}]}$.

On the other hand for each ordering $\leqslant$ of the set X the number of those $G = (X, F) \in \mathfrak{G}_x$ for which there exists no monotone embedding $((V, \leqslant), E) \to ((X, \preccurlyeq), F)$ is less than $(m-1)^{[N^{1+1/s}]} + 1$.

Consequently the set of those $G \in \mathfrak{G}_x$ which admit an ordering $(X, \preccurlyeq)$ for which there is no monotone embedding

$$((V, \leqslant), E) \to ((X, \preccurlyeq), F)$$

is less than $N!(m-1)^{[N^{1+1/s}]}$. This proves the theorem as the last quantity is $o(m^{[N^{1+1/s}]})$.

Corollary 3. *For every s there exists a graph $G = (V, E)$ without cycles of length $< s$ which contains for every ordering $\preccurlyeq$ of its vertices a cycle of length s with the ordering given in the Figure* 1.

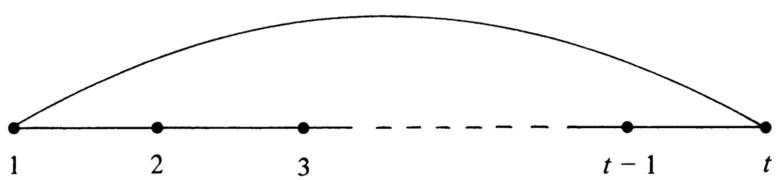

Figure 1

(For $s = 3$ the statement is evident, for $s = 4$ it was proved by Ore. Gallai showed that the Grötsch graph is an example for $s = 4$. For $s > 4$ this was asked by Erdös [**4**].)

Corollary 4. *For every s there exists a graph (V, E) without cycles of length $< s$ which is not a subgraph of a Hasse-diagram of a partially ordered set.*

This follows immediately from Corollary 3 and answers a question of B. Bollobás.

Theorem 2 is related to the concept of ordering property of a class of graphs and hypergraphs defined in [**6**]. Let $\mathfrak{K}$ be a class of hypergraphs. We say that $\mathfrak{K}$ has ordering property if for every hypergraph $G = (V, E) \in \mathfrak{K}$ there exists a hypergraph $G' = (V', E') \in \mathfrak{K}$ such that for every ordering $\leqslant$ of V and every ordering $\preccurlyeq$ of V' there exists a monotone mapping f: $(V, \leqslant) \to (V', \preccurlyeq)$ which is an embedding $G \to G'$.

The ordering property plays an important role in the study of partition properties of classes of hypergraphs. Using this concept one can reformulate the above Theorem 2.

THEOREM 5. *Let* $\mathfrak{A}$ *be a finite set of* 2*-connected graphs. Let* $\mathfrak{K} = \mathrm{Forb}(\mathfrak{A})$ *be the class of all finite graphs which do not contain any graph* $\in \mathfrak{A}$ *as an induced subgraph.* (*Thus* $G \in \mathrm{Forb}(\mathfrak{A})$ *iff there exists no embedding* $A \to G$ *for any* $A \in \mathfrak{A}$.) *Then* $\mathfrak{K}$ *has the ordering property.*

It is not a simple matter to prove the ordering property of a class $\mathfrak{K}$ constructively. This was done for the class of all finite graphs in [**11**], for the class of all finite graphs without K_k in [**9**], for the class of all finite graphs without cycles of length $3, 5, \dots 2k + 1$ in [**10**]. (The type representation of finite graphs was mainly used.)

Concluding remarks. 1. Theorem 5 has a hypergraph analogue.

2. The ordering property plays an important role in the study of partition properties of graphs. In fact this was the original motivation of this paper.

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