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ON A PROBABILITY BOUND OF MARSHALL AND OLKIN

BY KUMAR JOGDEO

University of Illinois

A counterexample is given to a proposition of Marshall and Olkin (1974, *Ann. Statist.*) describing a probability bound. A theorem which gives the conditions under which mixtures of associated random variables remain associated, is stated. This provides a method to obtain the required bound.

1. Introduction. Among other results, Marshall and Olkin (1974) state the following proposition (as Proposition 5.1) which gives a useful lower bound.

PROPOSITION (Marshall and Olkin). *Let $\phi(u, z)$ be a real linear and increasing function in u for all z. Suppose $U_1, U_2, \cdots, U_n$ and Z are independent random variables, U_i having a common* log-*concave density and let*

$$X_i = \phi(U_i, Z)\,, \qquad i = 1, \cdots, n\,.$$

Then

$$P[X_i \leqq x_i, i = 1, \cdots, n] \geqq \prod_{i=1}^n P[X_i \leqq x_i]\,,$$
$$P[X_i > x_i, i = 1, \cdots, n] \geqq \prod_{i=1}^n P[X_i > x_i]\,.$$

Unfortunately, at a crucial stage in the proof of this proposition, it has been wrongly stated that the covariance equals the expectation of conditional covariance, making the proof invalid. In the following, via a counterexample, it will be shown that the proposition as stated is false. Further a relevant and more general theorem which will meet the goal is stated.

2. Example. Let Z be a *positive* nondegenerate random variable, U_1, U_2 be independent identically distributed random variables independent of Z. Suppose $EU_i = c < 0$ and

$$\int_0^\infty u\, dF_U(u) = c^+ > 0\,.$$

Let

$$\phi(u, z) = u|z|\,.$$

Then for every z, ϕ is linear and increasing in u. Define

$$X_i = \phi(U_i, Z)\,, \qquad i = 1, 2\,. \tag{2.1}$$

Then the conditions of Proposition 5.1 in [4] are satisfied if the density of U_i is chosen to be log-concave.

If as claimed,

$$P[X_i \leqq x_i, i = 1, 2] \geqq P[X_1 \leqq x_1]P[X_2 \leqq x_2]\,, \tag{2.2}$$

for every pair (x_1, x_2), then the random variables X_1, X_2 would be *positively quadrant dependent*. According to a theorem by Lehmann (1966) the relation (2.2)

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is equivalent to having

$$\text{(2.3)} \qquad \operatorname{Cov}[h(X_1), g(X_2)] \geqq 0 ,$$

for every pair of nondecreasing functions h, g. Choose

$$h(x) = x , \qquad g(x) = x \quad \text{if} \quad x > 0$$
$$= 0 \quad \text{otherwise.}$$

Recalling that Z is a *positive* random variable, the following relations hold with probability one,

$$E[h(X_1) \mid Z] = ZEU_1 = cZ ,$$

and

$$E[g(X_2) \mid Z] = Z \textstyle\int_0^\infty u \, dF_U(u) = c^+ Z .$$

Thus

$$\text{(2.4)} \qquad \operatorname{Cov}[h(X_1), g(X_2)] = E\{\operatorname{Cov}(h, g) \mid Z\} + \operatorname{Cov}[Eh(X_1) \mid Z, Eg(X_2) \mid Z] .$$

Given Z, the random variables X_1, X_2 being independent, the first term on the right side of (2.4) is zero, and it follows that

$$\operatorname{Cov}[h(X_1), g(X_2)] = \operatorname{Cov}[cZ, c^+Z]$$
$$= cc^+ \operatorname{Var} Z < 0 ,$$

contradicting (2.3).

3. A theorem about mixture. First we recall that a set of real random variables $\mathbf{Y} = (Y_1, \cdots, Y_n)$ is said to be 'associated' if for every pair of functions h^*, g^* defined on $R^n \to R$ which are nondecreasing in each of their n arguments

$$\operatorname{Cov}[h^*(\mathbf{Y}), g^*(\mathbf{Y})] \geqq 0 .$$

Esary, Proschan and Walkup (1966) who introduced the concept of association showed that a set of independent real random variables is associated. Further, the condition of association implies the inequalities of the proposition stated in Section 1.

Now Proposition 5.1 in [4] seeks to establish a result stating that a set of n random variables which is conditionally independent (hence associated) preserves association unconditionally. In other words, it seeks conditions on mixtures under which the property of association may be preserved. The following theorem is stated with this viewpoint. It generalizes a theorem by Khatri (1974) which is closer to the statement of Proposition 5.1 in [4].

DEFINITION. The distribution of an n-vector $\mathbf{X}$ is said to be a monotone mixture with an m-vector $\mathbf{W}$ if for every nondecreasing f (that is, nondecreasing in each of its arguments separately)

$$h(\mathbf{W}) = E[f(\mathbf{X}) \mid \mathbf{W}]$$

is nondecreasing in $\mathbf{W}$.

THEOREM. *Suppose, with probability one, the conditional distribution of* **X** *given* **W** *is associated. Further, let the distribution of* **X** *be a monotone mixture with* **W** *where* **W** *is associated. Then the vector* (**X**, **W**) *is associated.*

The proof follows easily from the expression for covariance in (2.4), where conditioning is done with **W**. Further, note that when **W** is a real random variable then the bound given by Khatri (1974) follows.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS 61801