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On a Problem of da Costa

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Abstract. The two main founders of paraconsistent logic, Stanisław Jaśkowski and Newton da Costa, built their systems on distinct grounds. Starting from different projects, they used different tools and ultimately designed quite different calculi to attend their needs. How successful were their enterprises? Here we discuss the problem of defining paraconsistent logics following the original instructions laid down by da Costa. We present a new approach to $\mathbf{P}^1$, the first full solution —proposed by Antônio Mário Sette— to the problem of da Costa, and argue in favor of yet another solution we shall study here: the logic $\mathbf{P}^2$. Both $\mathbf{P}^1$ and $\mathbf{P}^2$ constitute maximal 3-valued paraconsistent fragments of classical logic. Constructive completeness proofs are here presented for both logics.

1 Requisites to paraconsistent calculi

When proposing the first paraconsistent propositional system, in 1948, Jaśkowski expected it to enjoy the following properties (see [17]):

Jas1 when applied to inconsistent systems it should not always entail their trivialization;

Jas2 it should be rich enough to enable practical inferences;

Jas3 it should have an intuitive justification.

A few years later, in 1963, da Costa would independently tackle a similar problem, this time proposing a whole hierarchy of paraconsistent propositional calculi, known as C_n, for $0 < n < \omega$. His requisites to these calculi were the following (see [12]):

NdC1 in these calculi the principle of non-contradiction, in the form $\neg(A \wedge \neg A)$, should not be a valid schema;

NdC2 from two contradictory formulae, A and $\neg A$, it would not in general be possible to deduce an arbitrary formula B;

NdC3 it should be simple to extend these calculi to corresponding predicate calculi (with or without equality);

NdC4 they should contain the most part of the schemata and rules of the classical propositional calculus which do not interfere with the first conditions.

Some vagueness in the formulation of the above conditions does not necessarily represent an inconvenient, as it paves the way for the proposal of a myriad of different solutions to the general problem of paraconsistency (and such solutions continue to appear).

Comparing the two sets of clauses above in some detail, let us just observe that clause **NdC1** could not have worried Jaśkowski, for his first so-called *discussive* calculus had a non-adjunctive character, not allowing for the deduction of a 'self-contradictory' formula of the form $A \wedge \neg A$ from two given contradictory formulae, A and $\neg A$. At any rate, as it will be argued below, the somewhat fetishistic attention initially put on the status of the formula $\neg(A \wedge \neg A)$ has since long been distinguished from the more general problem of paraconsistency. We could next go on and notice some similarity between clauses **Jas1** and **NdC2**, and between clauses **Jas2** and **NdC4**. But then what could be said about the remaining clauses? Well, on the one hand, da Costa's preoccupation with the answer to **NdC3** gives one of the main reasons why he is often suggested by some authors to be the 'real founder of paraconsistent logic' (see, for instance, [2] or [16]), despite Jaśkowski's prior construction. On the other hand, perhaps the absence of **Jas3** from da Costa's set of requisites gives an excuse for the difficulty one may encounter in establishing intuitive interpretations for the calculi C_n. Semantics for these calculi, in terms of bivalent non-truth-functional valuations, were provided no earlier than 1977 (see [13, 20]). Doubting those semantics really explained the paraconsistent character of these calculi, Carnielli and the author have recently worked out some new interpretations to them, in terms of *possible-translations semantics*, splitting them into suitable three-valued logics (see [22] and [5]).

The problem of defining paraconsistent logics satisfying **Jas1**–**Jas3** was soon to be dubbed the *problem of Jaśkowski* (see, for instance, [14, 19]). But of course these clauses look much too vague. Jaśkowski himself assumed them to formulate, in general terms, 'the problem of logic of contradictory systems'. It is hard to believe in fact that any proposer of a paraconsistent logic would willingly acknowledge or aim his own logic *not* to satisfy any of those clauses. This way, the problem of Jaśkowski reveals itself as nothing but the most general problem of paraconsistency. Nevertheless, even from this bare characterization, many solutions to the 'problem of Jaśkowski' were proposed, ranging from the reinterpretation —by Błaszczuk and Dziobiak, and others— of some well-known modal logics, following the line initiated by Jaśkowski himself with the modal logic $S5$, to the investigation of some many-valued logics related to the hierarchy of Łukasiewicz —by D'Ottaviano and da Costa, and many others. Good related bibliographies can be found in [19] and [11].

The *problem of da Costa*, as we shall call the problem of defining paraconsistent logics respecting clauses **NdC1**–**NdC4**, looks much more determinate. Da Costa assumed the latter set of clauses to be 'natural', but admitted that **NdC3** and **NdC4** were somewhat vague. But we need not regard **NdC4** as vague. Under a quite reasonable account, we shall here interpret it as restricting our search

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space to *maximal* fragments of classical logic, i.e., those fragments for which any proper extension collapses into classical logic itself. So, from now on we will be studying some propositional logics in which: (**NdC1**) the schema $\neg(A \wedge \neg A)$ is not provable; (**NdC2**) the derivation $A, \neg A \vdash B$, for arbitrary A and B, is not allowed; (**NdC4**) the condition of *maximality* is fulfilled, that is, the addition to one of these logics of a new schema, unprovable in this logic but provable in classical logic, renders to this logic the full strength of classical logic.

2 Some non-solutions

2.1 Infinitely many, not enough

Immediately after stating the problem, da Costa presented the calculi C_n, $0 < n < \omega$. Each C_n was axiomatized by the following schemata:

($\mathbf{1_n}$) $A \to (B \to A)$

($\mathbf{2_n}$) $(A \to B) \to ((A \to (B \to C)) \to (A \to C)))$

($\mathbf{3_n}$) $A \to (B \to (A \wedge B))$

($\mathbf{4_n}$) $(A \wedge B) \to A$

($\mathbf{5_n}$) $(A \wedge B) \to B$

($\mathbf{6_n}$) $A \to (A \vee B)$

($\mathbf{7_n}$) $B \to (A \vee B)$

($\mathbf{8_n}$) $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$

($\mathbf{9_n}$) $A \vee \neg A$

($\mathbf{10_n}$) $\neg\neg A \to A$

($\mathbf{11_n}$) $B^{(n)} \to ((A \to B) \to ((A \to \neg B) \to \neg A))$

($\mathbf{12_n}$) $(A^{(n)} \wedge B^{(n)}) \to ((A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \to B)^{(n)})$

having only *Modus Ponens* (**MP**): $A, A \to B \vdash B$ as inference rule. In the above axioms, we let B° abbreviate the formula $\neg(B \wedge \neg B)$ (the one mentioned in **NdC1**), and we let B^n, for $0 \leq n < \omega$, be recursively defined by setting $B^0 \stackrel{def}{=\!=} B$ and $B^{n+1} \stackrel{def}{=\!=} (B^n)^\circ$. We then finally define $B^{(n)}$, for $0 < n < \omega$, by setting $B^{(1)} \stackrel{def}{=\!=} B^1$ and $B^{(n+1)} \stackrel{def}{=\!=} B^{(n)} \wedge B^{n+1}$.

If one added *reductio ad absurdum* to axioms ($\mathbf{1_n}$)–($\mathbf{10_n}$), one would obtain classical propositional logic. But axiom ($\mathbf{11_n}$) provides a qualified form of *reductio*, helping to prevent the validity of $B^{(n)}$ in C_n. For a given formula G, if $G^{(n)}$ is valid in a theory having C_n as its underlying logic, we say that G is n-consistent. We say, in that case, that axiom ($\mathbf{12_n}$) regulates the *propagation of n-consistency*. It is easy to check that n-consistency also propagates through negation, that is, that the schema $A^{(n)} \to (\neg A)^{(n)}$ is provable in C_n. Consequently, we can classically evaluate a given formula G in C_n if, and only if, all of its variables are n-consistent. From now on we shall drop the index n in 'consistent' and in 'consistency' whenever there is no risk of confusion.

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The semantical counterparts of the calculi C_n, mentioned in section 1, show that the binary connectives have their usual classical meanings. But the same does not happen with negation. Thus, for instance, no negated formula $\neg G$ may be a theorem of C_n, unless all of its variables are consistent. A classical negation for a given formula A may be defined in each C_n by setting $\sim A \stackrel{def}{=} \neg A \wedge A^{(n)}$. It might easily be checked that all formulae built using only the connectives $\wedge$, $\vee$, $\rightarrow$ and $\sim$ behave classically.

The logics in the sequence C_1, C_2, C_3, $\ldots$ are increasingly weaker, ultimately leading to a logic that is the deductive limit of the sequence (the logic C_{Lim}, presented in [8]). So, even if C_1 were maximal, no C_n with $n > 1$ could also be maximal. To show that clauses **NdC1** and **NdC2** are respected by the calculus C_1, the strongest calculus in the hierarchy, da Costa employed the following set of truth-tables, up to a renaming of the corresponding truth-values (see [12], p.499):

$\wedge$	T	t	F
T	T	T	F
t	T	T	F
F	F	F	F

$\vee$	T	t	F
T	T	T	T
t	T	T	T
F	T	T	F

$\rightarrow$	T	t	F
T	T	T	F
t	T	T	F
F	T	T	T

	$\neg$	$\sim$
T	F	F
t	T	F
F	T	T

where T and t are the designated values. It is an easy exercise to check that these truth-tables provide a sound, but not complete, semantics to each C_n, while neither $\neg(A \wedge \neg A)$ nor $A, \neg A \vdash B$ are satisfied if we pick the values t and F, respectively, for A and B. In what follows, the logic defined by these truth-tables will be called $\mathbf{P}^1$, following [26].

It is well-known that no C_n is characterizable by a finite-valued set of truth-tables (see [1] and [6]), and it is equally known that no C_n is a maximal fragment of classical logic. In fact, there are some surprisingly simple examples of classically valid schemata which are not provable by any C_n, but could be added to them without causing their collapse into classical logic. Let $B^\diamond$ abbreviate the formula $\neg(\neg B \wedge B)$, and let ${}^nB$, for $0 \leq n < \omega$, be recursively defined by setting ${}^0B \stackrel{def}{=} B$ and ${}^{n+1}B \stackrel{def}{=} ({}^nB)^\diamond$. We then define $B^{\langle n \rangle}$, for every $0 < n < \omega$, by setting $B^{\langle 1 \rangle} \stackrel{def}{=} {}^1B$ and $B^{\langle n+1 \rangle} \stackrel{def}{=} B^{\langle n \rangle} \wedge {}^{n+1}B$. Some new axioms that one might now consider adding to the calculi C_n are:

$(\mathbf{13_n})$ $B^{\langle n \rangle} \rightarrow B^{(n)}$

$(\mathbf{14_n})$ $A \rightarrow \neg\neg A$

The logics in the new hierarchy axiomatized by $(\mathbf{1_n})$–$(\mathbf{14_n})$ were called $C_n^{\neg\neg}$, for $0 < n < \omega$, and studied in [22] and [5]. For any given n, one can show each of these axioms to be independent of the others (see [22]). The bewildering fact that $B^{(n)}$ and $B^{\langle n \rangle}$ are not equivalent in C_n —and in particular $\neg(B \wedge \neg B)$ is not equivalent to $\neg(\neg B \wedge B)$— is one of the consequences of the local failure of the rule of intersubstitutivity of provable equivalents. Anyway, we must admit that we

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have no reason to believe that 'the principle of non-contradiction', mentioned in clause **NdC1**, should have only the form $\neg(A \wedge \neg A)$, but not the form $\neg(\neg A \wedge A)$. Thus we might expect these formulae to receive identical treatment, what does *not* happen in C_n.

Once again, we have built an increasingly weaker sequence of paraconsistent logics. Now, to show that clauses **NdC1** and **NdC2** are respected by the calculus $C_1^{\neg\neg}$, we could employ the same truth-tables and designated values of $\mathbf{P}^1$ above, except for the truth-table of negation, which should read as below.

	$\neg$
T	F
t	t
F	T

The logic defined by this new set of truth-tables will be called $\mathbf{P}^2$. Evidently, $\mathbf{P}^2$ is an upper deductive limit to both hierarchies C_n and $C_n^{\neg\neg}$.

Is $C_1^{\neg\neg}$ maximal? Nope. Let's facilitate the propagation of consistency, stipulating that any single given consistent component of a formula should be enough so as to guarantee the consistency of the whole formula, that is, let's substitute axiom $(\mathbf{12_n})$ for a stronger one:

$(\mathbf{12^\star_n})$ $(A^{(n)} \vee B^{(n)}) \to ((A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \to B)^{(n)})$

This axiom $(\mathbf{12^\star_n})$ is not provable in any $C_n^{\neg\neg}$, yet it is validated both by the truth-tables of $\mathbf{P}^1$ and those of $\mathbf{P}^2$. With this new axiom, one can now of course think of a new hierarchy of calculi, which we might here call $C_n^!$ (the first of these calculi was proposed and studied in [9] under the name **Ciboe**). In these calculi, a formula G is to be evaluated classically if at least one of its variables is consistent. Accordingly, no negated formula $\neg G$ will be a theorem of $C_n^!$ unless at least one of its variables is consistent. But, again, the strongest calculus of the hierarchy, $C_1^!$, is still not maximal...

To sum up with, so far we have described, along all the above hierarchies, no full solution to the problem of da Costa.

2.2 Through these many-valued eyes

An interesting sample of paraconsistent logic was studied by D'Ottaviano and da Costa in 1970 (see [14]). The three-valued logic J_3 was presented by these authors as a solution to the problem of Jaśkowski. Its set of truth-tables is:

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\to$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	1	1	1

	$\neg$	∇
1	0	1
$\frac{1}{2}$	$\frac{1}{2}$	1
0	1	0

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where 1 and $\frac{1}{2}$ are the designated values. Originally, J_3 was presented having as primitive connectives $\vee$, $\neg$, and the 'modal' ∇, the remaining connectives being defined by setting $A \wedge B \stackrel{def}{=\!=} \neg(\neg A \vee \neg B)$ and $A \rightarrow B \stackrel{def}{=\!=} \neg\nabla A \vee B$. It is not hard to see that the truth-tables definable by J_3 and by Łukasiewicz's $Ł_3$ coincide (but note that $Ł_3$ has only 1 as designated value): the negation, the conjunction and the disjunction of $Ł_3$ are identical to those of J_3, so one has to show but how to define the implication $\supset$ of $Ł_3$ inside J_3 —this could be done by setting $A \supset B \stackrel{def}{=\!=} (\nabla\neg A \vee B) \wedge (\nabla B \vee \neg A)$— and how to define the connective ∇ of J_3 in $Ł_3$ —that might come out as $\nabla A \stackrel{def}{=\!=} \neg A \supset A$.

Let's say that a truth-table is *expressible* by one of the above mentioned three-valued logics if it can be defined by a formula of their language, and let's call any such a truth-table *semi-classical* if the restriction of the inputs to $\{0,1\}$ (or $\{T,F\}$) in the variables of the formula that expresses it gives an output in $\{0,1\}$ (or $\{T,F\}$). Now it can be constructively shown that the unary and binary truth-tables expressible by J_3 (and $Ł_3$) are all and only the semi-classical three-valued truth-tables (see [22] or [10]). Moreover, this logic is also *functionally pre-complete*, i.e., adding any new truth-table to it will lead to the expressibility of all possible three-valued truth-tables (check [10] again, where J_3 appears under the name **LFI1**). It's worth noting that all logics in the hierarchies cited in section 2.1 above can be interpreted by way of possible-translations semantics based on the splicing of some three-valued logics whose truth-tables are expressible by J_3, but not by $\mathbf{P}^1$, nor by $\mathbf{P}^2$ (see [22], or [5, 6]).

The logic J_3 was first studied by Schütte, under the appellation $\mathbf{\Phi}_\mathrm{v}$ and a somewhat different signature (see [25]), and it constitutes a conservative extension of the logic $RM_3^{\tilde{\supset}}$ (obtained by the addition to the latter of the connective ∇) presented in 1986 by Avron (see [3]) as an extension of Dunn-McCall's relevant calculus **RM**. Now, Avron has shown, by algebraic means, that $RM_3^{\tilde{\supset}}$ is maximal with respect to classical logic. It was only natural to expect that J_3 should also be maximal with respect to some version of classical logic written in the right signature, and this maximality has in fact recently been checked directly (see [22] and [10]). So, J_3 satisfies **NdC4**. As J_3 is indeed a paraconsistent logic, satisfying **NdC2**, and its first-order extension, satisfying **NdC3**, may be found in [15] or [10], J_3 would seem to constitute a perfectly valid answer to the problem of da Costa, had it not been for the failure of **NdC1**: Yes, $\neg(A \wedge \neg A)$ *is* a theorem of J_3 (and this gives the reason why the qualified form of *reductio* given by axiom $(\mathbf{11_n})$ fails to be valid in J_3 —and it's the only axiom that fails among the above considered axioms). So it seems, again, that clause **NdC1** might not be such a 'natural' requisite to a paraconsistent logic, as it was supposed to be... Note that the paper [21] extends the work done in the present paper by providing a more general background and set of solutions to the problem of da Costa (see more about this in the final section). There, for historical reasons, only the logics conforming to **NdC1** are in fact called 'solutions' to the problem of da Costa

—the remaining ones are called 'semi-solutions' to that same problem.

3 The first solution: $\mathbf{P}^1$

Motivated by the inexistence until then of a semantics to the calculi C_n of da Costa, Sette studied in 1973 (see [26]) the three-valued calculus $\mathbf{P}^1$, whose truth-tables are exhibited above, in section 2.1. He showed $\mathbf{P}^1$ to be complete with respect to the following set of axioms:

(P1) $A \to (B \to A)$

(P2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C)))$

(P3) $(\neg A \to \neg B) \to ((\neg A \to \neg\neg B) \to A)$

(P4) $(A \to B) \to \neg\neg(A \to B)$

(P5) $\neg(A \to \neg\neg A) \to A$

having **(MP)** as the sole inference rule. Here, only $\to$ and $\neg$ are taken as primitive connectives; the others may be defined by setting $\sim A \overset{def}{=\!=} \neg(\neg A \to A)$, $A \wedge B \overset{def}{=\!=} \neg(A \to \sim B)$, and $A \vee B \overset{def}{=\!=} \sim A \to B$.

The axiomatization above seems to be rather economical, yet it could actually be made even more so, as axiom **(P5)** may be derived from the others (see [22]). Anyway, one may point a further defect in this axiomatization. We have already noticed above that $\mathbf{P}^1$ not only provides an upper deductive bound to the calculi C_n, that is, all theorems and inferences of C_n are theorems and inferences of $\mathbf{P}^1$, but $\mathbf{P}^1$ also satisfies clauses **NdC1** and **NdC2**. Sette has proven $\mathbf{P}^1$ to be maximal, so it also satisfies **NdC4**. But axioms **(P1)**–**(P4)** give no hints at all on the relation between the calculus $\mathbf{P}^1$ and the calculi C_n: Which schemata more should we add to the axiomatization of a given C_n in order to obtain $\mathbf{P}^1$?

Let us investigate that point. A quick look at the truth-tables of $\mathbf{P}^1$ will make the reader notice that the truth-value t disappears after any connective is applied to an atomic variable. Accordingly, one might imagine that in $\mathbf{P}^1$ any complex sentence behaves consistently, no matter the behavior of its component variables. And that's exactly what happens! As we shall see, an alternative axiomatization of $\mathbf{P}^1$ is obtained from that of any given C_n if we just add to the latter the following new schemata:

$(\mathbf{15_n})$ $(A \wedge B)^{(n)} \wedge (A \vee B)^{(n)} \wedge (A \to B)^{(n)}$

$(\mathbf{16_n})$ $(\neg B)^{(n)}$

What $(\mathbf{15_n})$ and $(\mathbf{16_n})$ are saying is exactly that formulae with binary connectives should behave consistently, and negated formulae are supposed to behave consistently as well. Evidently, when we add axiom $(\mathbf{15_n})$ to C_n, axioms $(\mathbf{12_n})$, $(\mathbf{12^\star_n})$ and also $(\mathbf{13_n})$ turn out to be derivable.

Lemma 3.1 *For any given n, axioms $(\mathbf{1_n})$–$(\mathbf{11_n})$ plus $(\mathbf{15_n})$, $(\mathbf{16_n})$ and* **(MP)** *axiomatize $\mathbf{P}^1$.*

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Proof To check soundness, it is easy to see that both $(\mathbf{15_n})$ and $(\mathbf{16_n})$ are validated by the truth-tables of $\mathbf{P}^1$. It is also obvious that axioms $(\mathbf{P1})$ and $(\mathbf{P2})$ are derivable from $(\mathbf{1_n})$ and $(\mathbf{2_n})$. Now, the sentences $(\neg B)^{(n)} \to (\mathbf{P3})$ and $(A \to B)^{(n)} \to (\mathbf{P4})$ may be derived from the axioms of C_n, or of $C_n^{\neg\neg}$ (to prove these facts the reader might prefer to use as efficient shortcuts the above mentioned semantics of these calculi). So, using $(\mathbf{15_n})$, $(\mathbf{16_n})$ and $(\mathbf{MP})$, $(\mathbf{P3})$ and $(\mathbf{P4})$ can be proven. ■

We have now multiple ways of axiomatizing $\mathbf{P}^1$, adding axioms to each given C_n. Just for convenience, let's fix an axiomatization for $\mathbf{P}^1$, from now on, as that of C_1 plus $(\mathbf{15_1})$ and $(\mathbf{16_1})$. Now the primitive connectives are all of $\wedge$, $\vee$, $\to$ and $\neg$. A direct and constructive proof of completeness of the truth-tables of $\mathbf{P}^1$ with regard to this new axiomatization may be obtained with the help of the following auxiliary lemmata:

Lemma 3.2 *The following schemata are derivable in C_1:*

3.2.1 $A^\circ \to (A \to \neg\neg A)$

3.2.2 $A^\circ \to (\neg A \to (A \to B))$

3.2.3 $B^\circ \to (A \to (\neg B \to \neg(A \to B)))$

3.2.4 $A^\circ \to (\neg A \to \neg(A \wedge B))$

3.2.5 $B^\circ \to (\neg B \to \neg(A \wedge B))$

3.2.6 $(A^\circ \wedge B^\circ) \to (\neg A \to (\neg B \to \neg(A \vee B))$

A constructive Kálmar-like lemma may then be sketched, *modifying* the one proposed in [26] (we advise the reader to wait until the next section to know the reason why we modified the original proofs of **Lemma 3.3** and of **Theorem 3.5**, which follow):

Lemma 3.3 *Let G be a formula whose set of atomic variables is $\{p_1, p_2, \ldots, p_n\}$. Given a valuation v in $\mathbf{P}^1$, let's define for each variable p_i, for $1 \le i \le n$, the following associated formulae:*

(i) $p_i^+ = p_i^\circ$ *and* $p_i^- = p_i$, *if* $v(p_i) = T$;

(ii) $p_i^+ = p_i$ *and* $p_i^- = \neg p_i^\circ$, *if* $v(p_i) = t$;

(iii) $p_i^+ = p_i^\circ$ *and* $p_i^- = \neg p_i$, *if* $v(p_i) = F$.

Let's denote by Δ_v the set $\{p_1^+, p_1^-, p_2^+, p_2^-, \ldots, p_n^+, p_n^-\}$, and define also:

(iv) $G_v = G$, *if* $v(G) = T$;

(v) $G_v = \neg G^\circ$, *if* $v(G) = t$;

(vi) $G_v = \neg G$, *if* $v(G) = F$.

So we state that the following holds: $\Delta_v \vdash G_v$.

Proof The proof is an induction on the complexity of G. The base step, when G is itself an atomic variable, is straightforward, from the definitions **(i)**–**(vi)**. If G is complex we must go through a very long chain of cases, subcases and subsubcases, checking one by one, using the new axiomatization of $\mathbf{P}^1$ and the induction hypothesis, **(IH)**. Here, at key points, **Lemma 3.2** comes in handy. The longest case is that of G having the form $A \vee B$, for some A and B less complex than G:

1. Suppose $v(A) \neq F$, and so $v(G) = T$.

1.1. Suppose A to be p_i, an atomic variable. If $v(A) = T$, we have, from **(i)**, $p_i^- = p_i$, and if $v(A) = t$ we have, from **(ii)**, $p_i^+ = p_i$. Besides that we have, from **(iv)**, $G_v = G = p_i \vee B$. But from axiom $(\mathbf{6_1})$ we know that $p_i \to (p_i \vee B)$ is a provable formula in $\mathbf{P}^1$, for any arbitrary B. So we conclude that $\Delta_v \vdash G_v$.

1.1. Suppose A to be non-atomic. So $v(A)$ must be T. From **(iv)** we have $A_v = A$, and by **(IH)** we conclude that $\Delta_v \vdash A$. But from **(iv)** we also have $G_v = G = A \vee B$, and so we proceed as in case **1.1.** above.

2. Suppose $v(B) \neq F$, and so $v(G) = T$. Proceed as in case **1.**, this time using axiom $(\mathbf{7_1})$.

3. Suppose both $v(A) = F$ and $v(B) = F$, and so $v(G) = F$.

3.1. Suppose A to be p_i and B to be p_j, both atomic variables. From **(iii)** we have that $p_i^+ = p_i^\circ$ and $p_i^- = \neg p_i$, $p_j^+ = p_j^\circ$ and $p_j^- = \neg p_j$. Now, from **Lemma 3.2.6** we know that the formula $(p_i^+ \wedge p_j^+) \to (p_i^- \to (p_j^- \to \neg(p_i \vee p_j)))$ is valid, so $\Delta_v \vdash G_v$.

3.2. Suppose A to be p_i and B to be non-atomic. From **(iii)** we have that $p_i^+ = p_i^\circ$ and $p_i^- = \neg p_i$, and from **(vi)** we have $B_v = \neg B$, and $G_v = \neg G = \neg(p_i \vee B)$. By **(IH)**, $\Delta_v \vdash \neg B$. But as B was supposed to be non-atomic, then B° holds good. From **Lemma 3.2.6**, $\Delta_v \vdash G_v$.

3.3. Suppose A to be non-atomic and B to be p_j. Proceed as in case **3.2.**, *mutatis mutandis.*

3.3. Suppose both A and B non-atomic. From **(vi)** we have $A_v = \neg A$, $B_v = \neg B$, and $G_v = \neg G = \neg(A \vee B)$. By **(IH)**, $\Delta_v \vdash \neg A$ and $\Delta_v \vdash \neg B$, and as A and B are non-atomic, A° and B° hold good. So, from **Lemma 3.2.6**, $\Delta_v \vdash G_v$. ■

Theorem 3.4 (Completeness)
Every tautology of $\mathbf{P}^1$ *is a theorem of this calculus.*

Proof Let G be a tautology whose atomic components are $p_1, p_2, \ldots, p_n$. So, using **Lemma 3.3**, and the fact that $\neg G^\circ \to G$ is derivable, given the axioms $(\mathbf{10_1})$ and $(\mathbf{4_1})$, we conclude that $\Delta_v \vdash G$, for any valuation v. Let's denote by Δ_v^1 the set $\Delta_v - \{p_1^+, p_1^-\}$. We now choose three distinct valuations, v_1, v_2 and v_3, differing exactly in p_1, i.e., such that $\Delta_{v_1}^1 = \Delta_{v_2}^1 = \Delta_{v_3}^1 = \Delta_v^1$, while: **(A)** $v_1(p_1) = T$; **(B)** $v_2(p_1) = t$; **(A)** $v_3(p_1) = F$.

From case **(A)** we have that $p_1^+ = p_1^\circ$ and $p_1^- = p_1$, so:

$$\Delta_v^1, p_1^\circ, p_1 \vdash G \tag{1}$$

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But from case **(C)** we have that $p_1^+ = p_1^\circ$ and $p_1^- = \neg p_1$, so:

$$\Delta_v^1, p_1^\circ, \neg p_1 \vdash G \tag{2}$$

From (1) and (2), by axioms $(\mathbf{8_1})$ and $(\mathbf{9_1})$ —the proof by cases— we have:

$$\Delta_v^1, p_1^\circ \vdash G \tag{3}$$

Now from case **(B)** we have that $p_1^+ = p_1$ and $p_1^- = \neg p_1^\circ$, so:

$$\Delta_v^1, \neg p_1^\circ, p_1 \vdash G \tag{4}$$

We note that, from axioms $(\mathbf{10_1})$ and $(\mathbf{4_1})$, the formula $\neg p_1^\circ \to p_1$ is derivable, then from (4) we have:

$$\Delta_v^1, \neg p_1^\circ \vdash G \tag{5}$$

Using proof by cases once more, now on (3) and (5), we conclude:

$$\Delta_v^1 \vdash G \tag{6}$$

These steps helped us 'eliminating' the variable p_1. We can now recursively define the set Δ_v^i as the set $\Delta_v^{i-1} - \{p_i^+, p_i^-\}$, for $1 < i \leq n$, and then repeat the above steps $n-1$ times. In the end of this process we obtain the empty set Δ_v^{n+1}, and the proof is completed. ■

We now prove maximality, once more *modifying* the proof proposed by Sette (again, the next section will explain the reason for the modification):

Theorem 3.5 $\mathbf{P}^1$ *is a maximal fragment of classical logic.*

Proof Let $g(p_1, p_2, \ldots, p_n)$ be a valid formula of the classical propositional calculus that is not a theorem of $\mathbf{P}^1$ and is written over the variables $p_1, p_2, \ldots, p_n$. We may suppose, without loss of generality, that for any given v we have that $v(g(p_1, p_2, \ldots, p_n)) = F$ iff $v(p_i) = t$ for all p_i, $1 \leq i \leq n$. Indeed, if a given formula $h(p_1, p_2, \ldots, p_n, p_{n+1})$ assumes the value F when $v(p_i) = t$ for all $1 \leq i \leq n$ and $v(p_{n+1}) \neq t$, then we may fix the value of the variable p_{n+1} substituting it for $p_1 \to p_1$ in case $v(p_{n+1}) = T$ or for $\neg(p_1 \to p_1)$ in case $v(p_{n+1}) = F$, obtaining this way a formula of the form $g(p_1, p_2, \ldots, p_n)$.

Now consider the formula D_n, defined by setting $D_n \stackrel{def}{=\!=} p_1^\circ \vee p_2^\circ \vee \ldots \vee p_n^\circ$. Then, for any valuation v, it is easy to see from the truth-tables of $\mathbf{P}^1$ that $v(D_n) = T$ iff $v(p_i) \neq t$ for some p_i, $1 \leq i \leq n$. Otherwise, $v(D_n) = F$. We claim that the sentence $g(p_1, p_2, \ldots, p_n) \to D_n$ is a tautology of $\mathbf{P}^1$. Indeed, for any given v there are two possible situations:

(i) $v(D_n) = T$;

(ii) $v(D_n) = F$, what occurs only if $v(p_i) = t$ for all p_i, $1 \leq i \leq n$. But in this case we know that $v(g(p_1, p_2, \ldots, p_n)) = F$.

In both situations, however, we may conclude that $v(g(p_1, p_2, \ldots, p_n) \to D_n) = T$, for any arbitrary v. By the completeness of $\mathbf{P}^1$, the sentence inside parentheses is provable. Consequently, in $\mathbf{P}^1$ plus $g(p_1, p_2, \ldots, p_n)$, by $(\mathbf{MP})$, the sentence D_n is provable. In particular, fixing an arbitrary atomic variable p, the iterated disjunction $\overset{n}{\vee}\, p^\circ$ is provable, and so p° is a valid schema. From axiom $(\mathbf{11_1})$ and $(\mathbf{MP})$, *reductio ad absurdum* turns out to hold without restrictions, and the classical calculus thus obtains. ■

4 Yet another solution: $\mathbf{P}^2$

In all logics presented above, the binary connectives, $\wedge$, $\vee$ and $\to$ all had a classical behavior, but the negation had some of its usual properties not recognized, so that it was allowed to behave paraconsistently. Notwithstanding, the classical interrelation between negation and the binary connectives attained variable degrees of accomplishment. So, for instance, J_3 was the only logic where all De Morgan laws could be checked to hold, while no forms of contraposition, such as $(A \to B) \to (\neg B \to \neg A)$, are valid in none of the above paraconsistent calculi. In the case of the hierarchies in section 2.1, the relation between the consistency of the variables and the consistency of the complex formula built over them is regulated by the axioms of propagation of consistency. In an extreme case, that of $\mathbf{P}^1$, the consistency of complex formulae is automatically guaranteed, either if these formulae contained binary connectives or if it contained negations.

In one way or other, the non-classical behavior here has been associated with negation. Negation was allowed to have a somewhat 'wild' behavior, and our paraconsistent systems were designed to cope with that. So it may seem surprising that in $\mathbf{P}^1$ an atomic proposition starts to behave consistently only *because* it has received a negation sign —or maybe *in spite of that*. From then on, by a sort of miracle, all formulae built over this negated variable shall be evaluated classically. No doubt this feature has seemed unsatisfactory to many logicians working with $\mathbf{P}^1$. What if we wanted to bring paraconsistent behavior into some complex propositions, while still looking for a solution to the problem of da Costa? Perhaps at least a *purely negated formula*, that is, one whose only connective is negation, should never be considered consistent by default. Can we build a logic that realizes such an intent?

Look at what happens when we add axiom $(\mathbf{14_n})$: $A \to \neg\neg A$ to C_n. It is easy to show that in this case consistency starts to propagate backwards through negation, that is, the schema $(\neg A)^{(n)} \to A^{(n)}$ is provable: Now a negated formula is consistent only if it was already consistent before negation was applied to it. Therefore, a purely negated formula would only be consistent if the variable in its kernel were somehow classical. That's exactly how the truth-tables of the logic $\mathbf{P}^2$ work, and we now suggest to axiomatize them with the help of axioms $(\mathbf{1_n})$–$(\mathbf{11_n})$ plus $(\mathbf{14_n})$, $(\mathbf{15_n})$ and the rule $(\mathbf{MP})$ (of course, once more, axioms $(\mathbf{12_n})$, $(\mathbf{12^\star_n})$ and $(\mathbf{13_n})$ are easily derivable). For convenience, from now on, we will fix $n = 1$

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in this axiomatization. The logic $\mathbf{P}^2$ was thoroughly studied on its own right in [22], having first been axiomatized by Mortensen in [24], under the name $C0.2$. Unfortunately, the author of [24] insisted that the three-valued truth-tables of $\mathbf{P}^2$ should have only one designated value. As a consequence, the proposed logic then fails to be paraconsistent and the non-constructive completeness proof presented by Mortensen for $\mathbf{P}^2$ still holds good, while modus ponens fails, together with the corresponding soundness proof.

The connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$ are all taken as primitive, but once more we might define $\sim$, $\wedge$ and $\vee$ in terms of $\rightarrow$ and $\neg$, exactly as in the case of $\mathbf{P}^1$ (recall section 3). Moreover, the very negation of $\mathbf{P^1}$ can also be defined in $\mathbf{P^2}$, by considering $A \rightarrow \neg A$, while the negation of $\mathbf{P^2}$ is clearly *not* definable in $\mathbf{P^1}$. We now prove:

Lemma 4.1 *The* **Lemma 3.3** *holds for* $\mathbf{P}^2$.

Proof Here is the reason why we have modified the statement of **Lemma 3.3**, regarding $\mathbf{P}^1$, as proposed by Sette: Now the same result, with minimal modifications, apply also to $\mathbf{P}^2$. The proof is much similar, but we have to review the case of G being a negated formula, and the cases where the mere supposition of a sentence A being non-atomic has led us to conclude that A° is a theorem —this holds good for $\mathbf{P}^1$, but is no longer true in the case of $\mathbf{P}^2$, at least not in the case where A is a purely negated formula. Let's compare the longest case to the same case as treated by **Lemma 3.3**, namely, the case of G having the form $A \vee B$:

1. Suppose $v(A) \neq F$, and so $v(G) = T$.

- **1.1.** Suppose A to be p_i, an atomic variable. Proceed as in **Lemma 3.3**.
- **1.2.** Suppose A to be non-atomic. Now $v(A)$ could be T, and so we proceed as in **Lemma 3.3**, but $v(A)$ could also be t, in the case of A having the form $\neg_n p_i$, where p_i is an atomic variable and $\neg_n$ represents the negation, $\neg$, iterated n times. Evidently, $v(\neg_n p_i) = t$ iff $v(p_i) = t$, and from **(ii)** and **(iv)** we have $p_i^- = \neg p_i^\circ$, and $G_v = G = A \vee B = \neg_n p_i \vee B$. Now, if n is even, i.e., if it has the form $2m$, then we derive $p_i \rightarrow \neg_n p_i$, by m applications of the new axiom $(\mathbf{14_1})$ and rule $(\mathbf{MP})$. If n is odd, having the form $2m+1$, then $\neg p_i \rightarrow \neg_n p_i$ can be derived, by m applications of $(\mathbf{14_1})$ (if $m = 0$, this is also true, for $D \rightarrow D$ is a valid schema). In both cases we derive then $p_i^- \rightarrow A$, by axioms $(\mathbf{10_1})$, $(\mathbf{4_1})$ and $(\mathbf{5_1})$, so $\Delta_v \vdash A$, and so, from axiom $(\mathbf{6_1})$, $\Delta_v \vdash G_v$ follows.

2. Suppose $v(B) \neq F$, and so $v(G) = T$. Proceed as in case **1.**, *mutatis mutandis.*

3. Suppose both $v(A) = F$ and $v(B) = F$, and so $v(G) = F$.

- **3.1.** Suppose A to be p_i and B to be p_j, both atomic variables. Proceed as in **Lemma 3.3**.

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3.2. Suppose A to be p_i, and B to be non-atomic. From **(iii)** we have that $p_i^+ = p_i^\circ$ and $p_i^- = \neg p_i$, and from **(vi)** we have $B_v = \neg B$, and $G_v = \neg G = \neg(p_i \vee B)$. By **(IH)**, $\Delta_v \vdash B_v$. If B is not a purely negated formula, B° holds good, and we proceed as in **Lemma 3.3**. If B is $\neg_n p_j$, for some variable p_j, then $B_v = \neg_{n+1} p_j$. As $v(A) = v(\neg_n p_j) = F$, we conclude that $v(p_j) \neq t$.

If n has the form $2m$, then $v(p_j) = F$, and from **(iii)** we have that $p_j^+ = p_j^\circ$ and $p_j^- = \neg p_j$. Now we remember that $D^\circ \to (\neg D)^\circ$ is a provable schema even in C_1, and if one applies it n times, together with **(MP)**, one obtains $p_j^+ \to B^\circ$. Moreover, as in case **1.2.**, m applications of $(\mathbf{14_1})$ will give us $p_j^- \to B_v$. But, from **Lemma 3.2.6**, we have that $(p_i^+ \wedge B^\circ) \to (p_i^- \to (B_v \to G_v))$. So, $\Delta_v \vdash G_v$.

If n has the form $2m - 1$, then $v(p_j) = T$, and from **(i)** we have that $p_j^+ = p_j^\circ$ and $p_j^- = p_j$. Once more we derive $p_j^+ \to B^\circ$, and m applications of $(\mathbf{14_1})$ give us $p_j^- \to B_v$. As above, we use **Lemma 3.2.6** to obtain $\Delta_v \vdash G_v$.

3.3. Suppose A to be non-atomic and B to be p_j. Proceed as in case **3.2.**, *mutatis mutandis*.

3.4. Suppose both A and B non-atomic. The result in this case comes as a mixture of the arguments in cases **3.2.** and **3.3.** ■

Theorem 4.2 (Completeness)
Every tautology of $\mathbf{P}^2$ *is a theorem of this calculus.*

Proof The proof follows step by step the completeness proof of $\mathbf{P}^1$ (**Theorem 3.4**), now making use of **Lemma 4.1** above. ■

Theorem 4.3 $\mathbf{P}^2$ *is a maximal fragment of classical logic.*

Proof Here the modification we introduced in the strategy chosen by Sette for proving the maximality of $\mathbf{P}^1$ is crucial. In our case, the proof of **Theorem 4.3** turns out to be *exactly the same* as the proof of **Theorem 3.5** —and that constituted, in fact, our main reason to present an alternative proof of maximality for $\mathbf{P}^1$.

One just has to notice that, according to the truth-tables of $\mathbf{P}^2$, it still holds that $v(D_n) = T$ iff $v(p_i) \neq t$, for some p_i, $1 \leq i \leq n$, and otherwise $v(D_n) = F$. Moreover, it is equally clear that the addition of a schema such as p° to the axioms of $\mathbf{P}^2$, once more, causes the classical calculus to obtain. ■

5 Comments

We have claimed that $\mathbf{P}^1$ and $\mathbf{P}^2$ *are* solutions to the problem of da Costa. But clause **NdC3** required that first-order extensions of these logics should be

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presented. That is the easiest part, though. To define adequate universal and existential quantifiers for these three-valued logics one could use, for instance, the idea of *distribution quantifiers*, investigated in [4]. Without entering into details, we claim in the present case that these quantifiers can be axiomatized by the addition to any of these calculi of the following schemata:

(17) $\forall x A(x) \to A(t)$

(18) $A(t) \to \exists x A(x)$

(19) $(\forall x A(x))^{\circ} \wedge (\exists x A(x))^{\circ}$

plus the rules **(R1)**: $A \to B(x) \vdash A \to \forall x B(x)$, and **(R2)**: $A(x) \to B \vdash \exists x A(x) \to B$. In **(17)** and **(18)**, as usual, the term t is required to be free for x in $A(x)$.

In [27], Sette and Carnielli introduced the logic $\mathbf{I}^1$, dual to $\mathbf{P}^1$, with paracomplete character. $\mathbf{I}^1$ was shown to be a maximal paracomplete fragment of classical logic. Now, one may obtain a logic $\mathbf{I}^2$ from $\mathbf{I}^1$ in the very same way as $\mathbf{P}^2$ has been obtained from $\mathbf{P}^1$, namely, by substituting the truth-table of negation from $\mathbf{I}^1$ for the truth-table of negation from $\mathbf{P}^2$, presented above, in section 2.1 (remembering that in $\mathbf{I}^1$ and $\mathbf{I}^2$ only the value T is a designated one). Once more, $\mathbf{I}^2$ may easily be axiomatized and proven to be maximal with respect to classical logic (check [21]).

Carnielli and Lima-Marques, in [7], have shown how $\mathbf{P}^1$ and $\mathbf{I}^1$ could be interpreted in terms of a *society semantics*, a semantics in which the interaction of a society of agents, each one reasoning with classical logic itself, works so as to characterize a non-classical behavior. The idea of society semantics seems very close to the original conception of Jaśkowski's (pre-)discussive logic (see [17, 6]), where inconsistencies are permitted to appear, for instance, in the circumstance of a discussion between people —if they only expressed themselves using somewhat vague terms. In [23] the reader will find a general and constructive way of converting the semantics of all the above mentioned many-valued logics, including $\mathbf{P}^2$ and $\mathbf{I}^2$, into bivalent non-truth-functional semantics, society semantics, possible-translations semantics and modal-like semantics. Axiomatizations by sequent systems and tableaux are also provided. We believe thus that the solutions to the problem of da Costa presented above can indeed be given completely 'intuitive' justifications, closely approximating, along these lines, the problem of da Costa to the original requisites from the problem of Jaśkowski.

A generalization of the hereby presented investigations on the problem of da Costa is offered by the author in [21], where conjunction, disjunction, implication and negation are allowed to vary so as to allow for the construction of a family of 8,192 maximal paraconsistent three-valued logics, among which one can find $\mathbf{P}^1$, $\mathbf{P}^2$, J_3, and other known paraconsistent logics, such as **LFI2**, from [10], constituting a series of solutions or semi-solutions (in the sense of section 2.2) to the problem of da Costa. The constructive Kálmar-like lemma used above turns out to be quite complicated to write down in the general case, and so some preference is given in [21] to non-constructive completeness proofs. The idea explored in those proofs, nevertheless, as well as in the related general maximality proofs, which follow the general pattern of the maximality proofs that can be found in the present paper, is the idea of taking advantage of the abstract notion of 'consistency', internalized in the object language, following the investigations reported in [9, 6]. A dual family of 1,024 maximal paracomplete three-valued logics, including the above mentioned $\mathbf{I^1}$ and $\mathbf{I^2}$, but also logics such as **LPF**, from [18], is also presented in [21], and is equally subject to the above comments.

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