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# On a Problem of Sakai in Unbounded Derivations (Operator Algebras and Their Applications )

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On a problem of Sakai in unbounded derivations

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as a quantization of spaces, especially  $n$ -dimensional real lines, Sakai [7] posed the following interesting problem : are there simple  $C^*$ -algebras  $\mathcal{O}$  and  $n^{th}$  family  $\{\delta_j\}_{j=1}^n$  of non approximately bounded pregenerators of  $\mathcal{O}$  such that given a  $*$ -derivation  $\delta$  of  $\mathcal{O}$  with  $D(\delta) = \sum_{j=1}^n D(\delta_j)$ , there exist  $k_1, k_2 \in \mathbb{R}$  and an approximately bounded  $*$ -derivation  $\delta_0$  of  $\mathcal{O}$  with the property that  $\delta = \sum_{j=1}^n k_j \delta_j + \delta_0$ .

In this note, we show that there is at least one model for two dimensional case. It is nothing but the irrational rotation algebra, namely the  $C^*$ -crossed product  $C(T) \times_{\theta} \mathbb{Z}$  of the  $C^*$ -algebra  $C(T)$  of all continuous functions on the one dimensional torus  $T$  by an irrational angle  $\theta$ . More precisely we have the following :

Theorem 1. Let  $\mathcal{O}_\theta$  be the irrational rotation algebra. Then there exist two non approximately bounded pregenerators  $\delta_1, \delta_2$  of  $\mathcal{O}_\theta$  such that any

\*-derivation  $\delta$  of  $\mathcal{O}_\alpha$  with  $D(\delta) = D(\delta_1) \cap D(\delta_2)$  can be expressed as  $\delta = k_1 \delta_1 + k_2 \delta_2 + \delta_0$  for some  $k_1, k_2 \in \mathbb{R}$  and an approximately bounded \*-derivation  $\delta_0$  of  $\mathcal{O}_\alpha$ .

Remark 1. Suppose  $D(\delta) = D(\bar{\delta}_j)$  ( $j=1$  or  $2$ ), then one can show that  $\delta = k \bar{\delta}_j + \delta_0$  for some  $k \in \mathbb{R}$ .

We now state our main theorem as follows:

Theorem 2. Let  $(\mathcal{O}, G, \alpha)$  be a  $C^*$ -dynamical system where  $\mathcal{O}$  is unital abelian,  $G$  is discrete abelian, and  $\alpha$  is effective. Suppose  $\beta_\epsilon = \exp t \delta_0$  ( $t \in T$ ) commuting with  $\alpha$ , and there exists an eigenunitary  $u$  for  $\beta$  which generates  $\mathcal{O}$ . Then for any \*-derivation  $\delta$  of  $\mathcal{O} \rtimes_\alpha G$  with  $D(\delta) = D(\tilde{\delta}_0) = D(\delta_0) \odot_G G$  there exist a  $k \in \mathbb{R}$ , two generators  $\delta_1$  and an approximately bounded \*-derivation  $\delta_2$  of  $\mathcal{O} \rtimes_\alpha G$  such that (i)  $D(\tilde{\delta}_j) = D(\delta)$  ( $j=1, 2$ ),  $\delta_1|_{\mathcal{O}} = 0$ ,  $\delta_1$  commutes with  $\tilde{\delta}_0$  (ii)  $\delta = k \tilde{\delta}_0 + \delta_1 + \delta_2$ , where  $D(\delta_0) \odot_G G$  is the set of all  $D(\delta_0)$ -valued function of  $G$  with finite support, and  $\tilde{\delta}_0(x)(g) = \delta_0[x(g)]$  ( $x \in D(\delta_0) \odot_G G$ ).

Remark 2. If  $G = \mathbb{Z}$ ,  $\delta_j = l \delta'_j$  for some  $l \in \mathbb{R}$  where  $\delta'_j$  is independent of  $\delta$ .

Let  $(\Omega, G, \alpha)$  and  $(\Omega, H, \beta)$  be two  $C^*$ -dynamical systems where  $\alpha, \beta$  commute. Then there is a  $C^*$ -dynamical system  $(\Omega \times_\alpha G, H, \tilde{\beta})$  such that  $\tilde{\beta}_x(x)(g) = \beta_x[x(g)]$  ( $x \in L^1(G; \Omega)$ ). Then we have the following proposition of fixed point type :

Proposition 3.  $(\Omega \times_\alpha G)^{\tilde{\beta}} = \Omega^{\beta} \times_\alpha G$

Proof. By definition,  $\Omega^{\beta} \times_\alpha G \subset (\Omega \times_\alpha G)^{\tilde{\beta}}$ .

Suppose the inclusion is proper, then  $(\Omega^{\beta} \times_\alpha G) \times_{\tilde{\beta}} \hat{G} \subsetneq (\Omega \times_\alpha G)^{\tilde{\beta}} \times_{\tilde{\beta}} \hat{G}$  since  $\tilde{\beta}$  commutes with  $\hat{\alpha}$ . Since  $(\Omega \times_\alpha G)^{\tilde{\beta}} \times_{\tilde{\beta}} \hat{G} \subset ((\Omega \times_\alpha G) \times_{\tilde{\beta}} \hat{G})^{\tilde{\beta}}$ , it follows from duality [6,8] that  $\Omega^{\beta} \otimes C(L^2(G)) \subsetneq (\Omega \otimes C(L^2(G)))^{\beta \otimes \tilde{\beta}}$  which is a contradiction. Q.E.D.

Comment 1. We only consider locally compact abelian groups throughout this note.

In what follows, let  $\delta$  be a \*-derivation of  $\Omega \times_\alpha G$  such that  $D(\delta) = D(\delta_0)$  where  $\delta_0$  is a generator of  $\Omega$  commuting with  $\alpha$ . Suppose  $\delta$  commutes with  $\hat{\alpha}$ , and  $G$  is discrete. Then  $\delta(a) \in \Omega$  for  $a \in D(\delta_0)$ . Let  $(x_n)_n \subset D(\delta)$  with  $x_n \rightarrow 0$ ,  $\delta(x_n) \rightarrow y \in \Omega \times_\alpha G$ . Since  $x_n = \sum_k c_k^{(n)} \lambda(k)$  ( $c_k^{(n)} \in D(\delta_0)$ ), using the conditional expectation  $\mathbb{E}$  of  $\Omega \times_\alpha G$  onto  $\Omega$

one has  $\varepsilon(x_n \lambda(g)^*) \rightarrow 0$  and  $\varepsilon[(\delta(x_n) - y) \lambda(g)^*] \rightarrow 0$  for each  $g$  in  $G$ . Thus  $a_g^{(m)} \rightarrow 0$  and  $\varepsilon[\sum_k (\delta(a_k^{(m)}) \lambda(k-g) + a_k^{(m)} \delta(\lambda(k)) \lambda(g)^* - y_k \lambda(k-g))] \rightarrow 0$  where  $y = \sum_k y_k \lambda(k)$  be the Fourier expansion of  $y$  in  $\Omega \times_{\alpha} G$  ( $y_k \in \Omega$ ). Then  $a_g^{(m)} \rightarrow 0$  and  $\delta(a_g^{(m)}) \rightarrow y_g$  for all  $g$  in  $G$ . Since  $D(\delta|_{\Omega}) = D(\delta_0)$ , it follows from Batty's result [2] that  $\delta|_{\Omega}$  is closable. So  $y_g = 0$  for all  $g \in G$ . Consequently we have the following :

Lemma 4. If  $G$  is discrete, any \*-derivation  $\delta$  of  $\Omega \times_{\alpha} G$  such that (i)  $D(\delta) = D(\tilde{\delta}_0)$  and (ii)  $\delta$  commutes with  $\hat{\alpha}$  is closable.

Remark 3. In the above lemma, the conclusion is unclear unless the condition (ii) is added.

Now let  $\delta$  be a \*-derivation of  $\Omega \times_{\alpha} G$  with  $D(\delta) = D(\tilde{\delta}_0)$ . Define  $\mathcal{J} = \{x \in D(\delta_0) \mid a \mapsto \delta(ax)$  is continuous from  $D(\delta_0)$  into  $\Omega \times_{\alpha} G\}$ . Since  $\delta(a \lambda(g)b) = \delta(\lambda(g)) \alpha_g^{-1}(a) b + \lambda(g) \delta(\alpha_g^{-1}(a) b)$  and  $\delta_0$  commutes with  $\alpha$ , we have  $x \lambda(g) b = 0$  for all  $g \in G$  and  $b \in \mathcal{J}$  if  $a_n \in D(\delta_0) \rightarrow 0$  and  $\delta(a_n) \rightarrow x \in \Omega \times_{\alpha} G$ . Then  $\varepsilon(x \lambda(g)) b = 0$  where  $\varepsilon$  is the projection of norm one from  $\Omega \times_{\alpha} G$  onto  $\Omega$ . So  $\varepsilon(x \lambda(g)) \in L(\mathcal{J})$ , the left annihilator

of  $\mathcal{I}$ . Since  $\mathcal{I}$  is a two-sided ideal of  $D(\delta_0)$ , it follows from the same way as Longo [4] that  $L(\mathcal{I}) = 0$ . Thus  $\varepsilon(x\lambda(g)) = 0$  for all  $g \in G$ . Let  $x = \sum_g x_g \lambda(g)$  be the Fourier expansion of  $x$ . Then  $x_g = 0$ . So  $x = 0$ . Then  $\delta|_\alpha$  is closable from  $(D(\delta_0), \| \cdot \|_{\delta_0})$  into  $\alpha x_\alpha G$ . Therefore we have the following :

Lemma 5. Let  $\delta$  be a \*-derivation of  $\alpha x_\alpha G$  with  $D(\delta) = D(\tilde{\delta}_0)$ . Then  $\delta$  is relatively bounded on  $D(\delta_0)$  with respect to  $\delta_0$ , namely  $\|\delta(a)\| \leq K(\|a\| + \|\delta_0(a)\|)$  for all  $a \in D(\delta_0)$ , with some positive constant  $K$ .

Remark 4. Since  $\delta_0$  is a pregenerator, one can not directly apply Longo's result. However the crucial part of the above proof is due to his idea [4].

By the above lemma, let  $\beta_t = \exp t\delta_0$  ( $t \in \mathbb{R}$ ). Then there exist derivations  $\tilde{\beta}_f$  ( $f \in L^1(\mathbb{R})$ ) of  $\alpha x_\alpha G$  such that i)  $D(\tilde{\beta}_f) = D(\tilde{\delta}_0)$  and ii)  $\tilde{\beta}_f = \int_{\mathbb{R}} f(t) \tilde{\beta}_t \circ \delta \circ \tilde{\beta}_t dt$ . In fact, since  $\|\delta(a)\| \leq M(\|a\| + \|\delta_0(a)\|)$  for  $a \in D(\delta_0)$ ,  $\|\delta \circ \beta_t(a) - \delta \circ \beta_s(a)\| \leq M \{ \|\beta_t(a) - \beta_s(a)\| + \|\beta_t \circ \delta_0(a) - \beta_s \circ \delta_0(a)\| \}$ . So  $t \mapsto \delta \circ \beta_t(a)$  is continuous for each  $a \in D(\delta_0)$ . Thus  $t \mapsto \delta \circ \beta_t(x)$  is also continuous for  $x \in D(\tilde{\delta}_0)$  which gives

derivations  $\tilde{S}_f$  for  $f \in L^1(\mathbb{R})$  of  $\Omega X_\alpha G$  satisfying (i) and (ii). Similarly, for each  $g \in G$  one has a derivation  $\tilde{S}_g$  of  $\Omega X_\alpha G$  such that (i)  $D(\tilde{S}_g) = D(\tilde{\delta}_0)$  and (ii)  $\tilde{S}_g = \int_{\hat{G}} \langle \tilde{g}, P \rangle \tilde{\alpha}_p \circ \delta \circ \tilde{\alpha}_p^{-1} dp$ . Moreover suppose  $P_t = e^{it\delta_0}$  is periodic, then we have that  $(\tilde{S}_1)_0^\sim = (\tilde{S}_0)_1^\sim$  commutes with  $\tilde{\alpha}, \tilde{\beta}$ . In what follows we treat \*-derivations of  $\Omega X_\alpha G$  with the same domain as  $D(\tilde{\delta}_0)$  commuting with  $\tilde{\alpha}$  and  $\tilde{\beta}$ , which are denoted by  $\delta$ . Since it commutes with  $\tilde{\alpha}$ , it follows from Lemma 4 that it is closable. Hence one may assume that it is closed. Let  $x \in C^*(G)$ , and  $(x_i) \subset D(\delta)$  which converge to  $x$ . Put  $y_i = \int_{\mathbb{R}} \tilde{P}_t(x_i) dt \in \Omega X_\alpha G$ . Since  $\delta$  commutes with  $\tilde{P}$  and  $\delta$  is closed,  $y_i \in D(\delta) \cap (\Omega X_\alpha G)^{\tilde{P}}$  and  $y_i \rightarrow x$  since  $(\Omega X_\alpha G)^{\tilde{P}} = C^*(G)$  by Proposition 1. So  $\delta|_{C^*(G)}$  is a closed \*-derivation of  $C^*(G)$  since  $\delta(y_i) \in C^*(G)$ . Since  $\tilde{\alpha}_p \circ \delta \circ \tilde{\alpha}_p^{-1} = \delta$  for  $p \in \hat{G}$  and  $\mathcal{F} \circ \tilde{\alpha}_p \circ \mathcal{F}^{-1} = \tau_p$  on  $C(\hat{G})$ ,  $\tilde{\delta} = \mathcal{F} \circ \delta \circ \mathcal{F}^{-1}$  commutes with  $\tau$  on  $C(\hat{G})$  where  $\mathcal{F}$  is the Fourier isomorphism of  $C^*(G)$  onto  $C(\hat{G})$ , and  $\tau$  is the shift action of  $\hat{G}$  on  $C(\hat{G})$ . It follows from Goodman-Nakazato [3, 5] that there exists a one parameter subgroup  $(P_t)$  of  $\hat{G}$  such that  $\tilde{\delta}(f)(p) = \lim_{t \rightarrow 0} t^{-1} (f(P_t p))$

$- f(p)$ ) for all  $f \in D(\delta)$ . Since  $\langle g, \cdot \rangle \in D(\delta)$ , one has  $\delta(\lambda(g)) = \partial(g)\lambda(g)$  for all  $g \in G$  where  $\partial(g) = \lim_{t \rightarrow 0} (\langle g, p_t \rangle - 1)$ . Let  $\delta_1(a\lambda(g)) = \partial(g)a\lambda(g)$  for all  $a \in D(\delta_0)$  and  $g \in G$ . Then it is a pregenerator of  $\Omega \times_\alpha G$  such that  $D(\delta_1) = D(\tilde{\delta}_0)$  and  $\delta_1|_\Omega = 0$ ,  $\delta_1$  commutes with  $\tilde{\delta}_0$ . Since  $\delta$  is a closed \*-derivation of  $\Omega \times_\alpha G$  and  $\delta|_\Omega$  commutes with  $p_t = \exp t\delta_0$ , it follows from Batty [1] that  $\delta|_\Omega = k\delta_0$  for some  $k \in \mathbb{R}$ . Therefore we have that  $\delta(a\lambda(g)) = k\delta_0(a)\lambda(g) + a\partial(g)\lambda(g) = (k\tilde{\delta}_0 + \delta_1)(a\lambda(g))$ , which implies the following lemma :

Lemma 6. Let  $(\Omega, G, \alpha)$  be a  $C^*$ -dynamical system where  $\Omega$  is unital abelian and  $G$  is discrete abelian. Let  $\beta_t = \exp t\delta_0$  be a periodic action of  $\mathbb{R}$  on  $\Omega$ . Suppose  $\beta$  is ergodic, then given a \*-derivation  $\delta$  of  $\Omega \times_\alpha G$  with the property that (i)  $D(\delta) = D(\tilde{\delta}_0)$  and (ii)  $\delta$  commutes with  $\tilde{\delta}, \tilde{\beta}$ , there exist a  $k \in \mathbb{R}$  and a pregenerator  $\delta_1$  of  $\Omega \times_\alpha G$  such that  
(i)  $D(\delta_1) = D(\delta)$ ,  $\delta_1|_\Omega = 0$ ,  $\delta_1$  commutes with  $\tilde{\delta}_0$ , and  
(ii)  $\delta = k\tilde{\delta}_0 + \delta_1$  on  $D(\tilde{\delta}_0)$ .

Remark 5. The pregenerator  $\delta_1$  defined above would

be written as  $\delta_i = r\delta'_i$  for some  $r \in \mathbb{R}$  where  $\delta'_i$  is not depending on  $\delta$ . Actually if  $G = \mathbb{Z}$ , we have  $\delta'_i(a\lambda(n)) = i n a \lambda(n)$  for  $a \in D(\delta)$  and  $n \in \mathbb{Z}$ .

Let  $\delta$  be a linear mapping from a \*-subalgebra  $D(\delta)$  of  $\Omega$  into  $\Omega$  such that  $\delta(ab) = \delta(a)\alpha_g(b) + a\delta(b)$  for all  $a, b \in D(\delta)$  where  $g \neq e \in G$  is a fixed element. Suppose there is a unitary  $u$  of  $D(\delta)$  such that  $1 \notin \text{sp}(\alpha_g(u)u^*)$ , then we have by direct computation that  $\delta(u^n) = \sum_{k=0}^{n-1} \alpha_g(u^k)u^{-k}\delta(u)u^{n-1}$ . Since  $1 \notin \text{sp}(\alpha_g(u)u^*)$ , one has that  $\sum_{k=0}^{n-1} \alpha_g(u^k)u^{-k} = (\alpha_g(u^n)u^{-n} - 1)(\alpha_g(u)u^{n-1})^{-1}$ . So  $\delta(u^n) = \delta(u)u^*(\alpha_g(u)u^{n-1})^{-1}(\alpha_g - id)(u^n) = \delta(u)(\alpha_g(u) - u)^{-1}(\alpha_g - id)(u^n)$  for all  $n \in \mathbb{Z}$  since  $\delta(1) = 0$ . Put  $\alpha_g = \delta(u)(\alpha_g(u) - u)^{-1} \in \Omega$ . Since  $\alpha_g(\alpha_g - id)$  is bounded on  $\Omega$ , the conclusion follows. Namely we have the following :

Lemma 7. Suppose  $\Omega$  is unital abelian and  $G$  is discrete. Let  $\delta$  be a linear mapping of a \*-subalgebra  $D(\delta)$  of  $\Omega$  into  $\Omega$  such that  $\delta(ab) = \delta(a)\alpha_g(b) + a\delta(b)$  for a fixed  $g \neq e$ . Suppose there exists a unitary  $u \in D(\delta)$  such that  $1 \notin \text{sp}(\alpha_g(u)u^*)$ , then  $\delta = \alpha_g(\alpha_g - id)$  on  $D(\delta) \cap C^*(\Omega)$

for some  $\alpha_g \in \Omega$ .

Remark 6. By the above lemma, there is no unbounded  $\delta_g$ -cocycle closed \*-derivation if  $\alpha_g$  has an eigen unitary generating  $\Omega$ .

Now let  $\hat{\delta}_g$  ( $g \in G$ ) be a derivation of  $\Omega \times_{\alpha} G$  as in the previous way (following to Remark 4). Then it implies that  $\delta = \sum_g \hat{\delta}_g$  on  $D(\delta)$ . In fact, let  $\delta(a) = \sum_g \delta(a)(g) \lambda(g)$  and  $\delta(\lambda(h)) = \sum_g \delta(\lambda(h))(g) \lambda(g)$  be the Fourier expansion of  $\delta(a)$  and  $\delta(\lambda(h))$  respectively.

Then  $\hat{\delta}_g(a) = \delta(a)(g) \lambda(g)$  and  $\hat{\delta}_g(\lambda(h)) = \delta(\lambda(h))(g+h) \lambda(g+h)$ . Suppose  $\delta$  commutes with  $\tilde{\beta}$ , it follows from Lemma 6 that  $\hat{\delta}_e = k \tilde{\delta}_0 + \delta_e$  on  $D(\delta)$  where  $k, \delta_e$  are as in Lemma 6. Let  $\delta_g(a) = \hat{\delta}_g(a) \lambda(g)^*$  for  $a \in D(\delta_0)$  ( $g \neq e$ ). Then  $\delta_g$  satisfy the condition of Lemma 7. Suppose there exists a unitary  $u \in D(\delta_0)$  such that (i)  $1 \notin \text{Sp}(\delta_g(u)u^*)$  ( $g \neq e$ ) and (ii)  $\Omega = C^*(u)$ . Since  $\delta$  commutes with  $\tilde{\beta}$ , and  $\alpha$  commutes with  $\beta_t = \exp t \delta_0$  which is ergodic, we have  $\alpha_g \in \Omega$ . Then  $\hat{\delta}_g(a) = \alpha_g(\delta_g - id)(a) \lambda(g) = [a \lambda(g), a]$ . Hence  $\hat{\delta}_g(a \lambda(h)) = \hat{\delta}_g(a) \lambda(h) + a \hat{\delta}_g(\lambda(h)) = [a \lambda(g), a \lambda(h)] + a \hat{\delta}_g(\lambda(h))$ . Since  $\hat{\delta}_g - ad(a \lambda(h))$  is a derivation on  $D(\delta)$ , one has  $\hat{\delta}_g(\lambda(h)) = 0$  for  $h \in G$ .

In fact, since  $\hat{S}_g(\lambda(h)) = \delta(\lambda(h))(g+h)\lambda(g+h)$ , we have that  $\delta(\lambda(h+k))(h+k+g)u = \delta(\lambda(h))(h+g)\alpha_g(u) + \delta(\lambda(k))(k+g)u$  for all  $h, k \in G$ . Since  $1 \in D(\delta)$ , we have  $\delta(1)(g)=0$ . So  $\delta(\lambda(h))(h+g)=0$  for all  $h \in G$  or  $\alpha_g(u)=u$ . Since  $1 \notin \text{sp}(\alpha_g(u)u^*)$ , we have  $\delta(\lambda(h))(h+g)=0$  for all  $h \in G$ . Consequently  $\delta = k\tilde{\delta}_0 + \delta_1 + \sum_{g \in G} \text{ad}(\alpha_g\lambda(g))$  on  $D(\delta)$ . Let  $\delta_F = \text{ad}(\sum_{g \in F} \alpha_g\lambda(g))$  for a finite set  $F$  of  $G$ -elts with  $F = -F$ . Then  $\delta_F$  are bounded \*-derivations of  $\mathcal{O}\times_G G$  such that  $\delta_F(\lambda(h))=0$  and  $\delta_F$  converges to  $\delta_2$  pointwise on  $D(\delta)$  where  $\delta_2(a\lambda(h)) = \sum_{g \in G} [\alpha_g\lambda(g), a\lambda(h)] \subseteq (\delta - \hat{\delta}_e)(a)\lambda(h)$ . Then  $\delta = k\tilde{\delta}_0 + \delta_1 + \delta_2$  on  $D(\delta)$  and  $\delta_2(\lambda(g))=0$  for all  $g \in G$ , which implies the following proposition :

Proposition 8. Let  $(\mathcal{O}, G, \alpha)$  be a  $C^*$ -dynamical system where  $\mathcal{O}$  is unital abelian and  $G$  is discrete. Let  $\beta_t = \exp t\delta_0$  be an ergodic action of  $T$  on  $\mathcal{O}$  commuting with  $\alpha$ . Suppose there exists a unitary  $u \in D(\delta_0)$  such that (i)  $1 \notin \text{sp}(\alpha_g(u)u^*)$  ( $g \neq e$ ), (ii)  $\mathcal{O} = C^*(u)$ , then given a \*-derivation  $\delta$  of  $\mathcal{O} \times_G G$  such that (i)  $D(\delta) = D(\tilde{\delta}_0)$  and (ii)  $\delta$  commutes with  $\tilde{\beta}$ , there exist a  $k \in \mathbb{R}$ , a pregenerator  $\delta_1$  and an approximately bounded \*-derivation  $\delta_2$  of  $\mathcal{O} \times_G G$  such that (i)  $D(\delta_1) = D(\delta)$ ,  $\delta_1|_{\mathcal{O}} = 0$ ,  $\delta_1$  commutes with  $\tilde{\delta}_0$ ,

ii)  $D(\delta_2) = D(\delta)$ ,  $\delta_2(\lambda(g)) = 0$  for all  $g \in G$ , and iii)  $\delta = k\delta_0 + \delta_1 + \delta_2$ .

Remark 7. In the case of discrete abelian groups, the Fourier expansion of any element of  $\mathcal{D}X_\alpha G$  can be taken in the uniform sense. In fact, taking a net  $\{f_i\}$  of positive definite functions on  $G$  with finite support converging to 1, one can show that  $\sum_g f_i(g) a_g \lambda(g)$  converges to  $\sum_g a_g \lambda(g) \in \mathcal{D}X_\alpha G$  uniformly.

Proof of Theorem 2: Since  $\beta$  commutes with  $\alpha$  and  $\beta$  is ergodic, we have  $\alpha_g(u)u^* \in C^*(u)$ . Since  $\mathcal{D} = C^*(u)$  and  $\alpha$  is effective, there are  $c_g \neq 1$  ( $\forall g$ ) such that  $\alpha_g(u) = c_g u$ . So  $1 \notin \text{sp}(\alpha_g(u)u^*)$  ( $\forall g$ ). Let  $\tilde{\beta}_n = \int_T e^{-int} \tilde{\beta}_t \cdot \delta \cdot \tilde{\beta}_t dt$  on  $\mathcal{D}(\delta)$  for  $n \in \mathbb{Z}$ . Since  $\tilde{\beta}_0$  commutes with  $\tilde{\beta}$ , it follows from Proposition 8 that  $\tilde{\beta}_0 = k\delta_0 + \delta'_1 + \delta'_2$  where  $\delta'_1, k$  are as in Proposition 8. Since  $\tilde{\beta}_t \cdot \tilde{\beta}_n \cdot \tilde{\beta}_t = e^{int} \tilde{\beta}_n$  ( $n \in \mathbb{Z}$ ),  $\tilde{\beta}_t \cdot \tilde{\beta}_n(\lambda(g)) = e^{int} \tilde{\beta}_n(\lambda(g))$ . Since  $\beta_t(u^n) = e^{int} u^n$ , we have that  $u^n \tilde{\beta}_n(\lambda(g)) \in (\mathcal{D}X_\alpha G)^{\tilde{\beta}} = C^*(G)$ . So there are  $b(n, g) \in C^*(G)$  such that  $\tilde{\beta}_n(\lambda(g)) = u^n b(n, g)$ . Let  $\delta(\lambda(g)) = \sum_h \delta(\lambda(g))(h) \lambda(h)$  and  $b(n, g) = \sum_h b(n, g)(h) \lambda(h)$  be the Fourier expansion of  $\delta(\lambda(g))$  and  $b(n, g)$  respectively.

Since  $\Omega = C^*(\mathcal{U})$  and  $\beta_t(\mathcal{U}) = e^{it}\mathcal{U}$ , we have that  $\delta(\lambda(g))(h)$   
 $= \alpha(0) + \sum_{n \neq 0} b(n, g)(h) u^n$  where  $\alpha(0)$  is the 0-component of  
the expansion of  $\delta(\lambda(g))(h)$  in  $\Omega$ . Since  $\tilde{\beta}_0 = k' \tilde{\delta}_0 + \delta'_1 + \delta'_2$ ,  
one has  $\tilde{\beta}_0(\lambda(g)) = \delta(g)\lambda(g)$ . By unicity,  $\int_T \beta_t(\delta(\lambda(g))(h)) dt$   
 $= \delta(g) \mathbb{1}(g \neq h), = 0$  (otherwise), which is nothing but  $\alpha(0)$ .  
Therefore we deduce that  $\delta(\lambda(g)) = \delta(g)\lambda(g) + \sum_h \sum_{n \neq 0} b(n, g)(h) u^n \lambda(h) = \delta(g)\lambda(g) + \sum_{n \neq 0} u^n b(n, g) = \delta(g)\lambda(g) + \sum_{n \neq 0} \tilde{\beta}_n(\lambda(g))$ .  
moreover  $\delta(a) = \sum_g \tilde{\beta}_g(a)$  for all  $a \in \mathcal{D}(\delta_0)$ . It follows  
from Lemma 7 that  $\tilde{\beta}_g(a) = f_g(\alpha_g - id)(a) \lambda(g)$  for some  
 $f_g \in \Omega$  ( $g \neq e$ ). So  $\tilde{\beta}_g(a) = [f_g \lambda(g), a]$  for all  $a \in \mathcal{D}(\delta_0)$ .  
Since  $\hat{\beta}_e$  commutes with  $\hat{\alpha}$ , we have  $\hat{\beta}_e(a) \in \Omega$  for all  
 $a \in \mathcal{D}(\delta_0)$ . Since  $(\hat{\beta}_e)_0^\sim$  commutes with  $\hat{\alpha}$  and  $\hat{\beta}$ , it means  
that  $(\hat{\beta}_e)_0^\sim = k \tilde{\delta}_0 + \delta'_1$  where  $k, \delta'_1$  are as in Lemma 6.  
Then  $\int_T e^{-it} \beta_t \circ \hat{\beta}_e(u) dt = k \delta_0(u)$ . Since  $\beta_t(u) = e^{it}u$ ,  
we have  $\delta_0(u) = iu$ . Let  $\hat{\beta}_e(u) = \sum_n a_n u^n$  ( $a_n \in \mathbb{C}$ ). Then  
 $a_1 = ik$ . Therefore  $\hat{\beta}_e(u) = k \delta_0(u) + \sum_{n \neq 1} a_n u^n$ . Since  $\hat{\beta}_e$   
is a \*-derivation, we deduce that  $\hat{\beta}_e(u^n) = n \hat{\beta}_e(u) u^{n-1} =$   
 $kn \delta_0(u) u^{n-1} + \sum_{m \neq 1} n a_m u^{m+n-1} = k \delta_0(u^n) + \sum_{m \neq 1} n a_m u^{m+n-1}$ .  
Hence  $\hat{\beta}_e(u^n) \lambda(g) = k \tilde{\delta}_0(u^n \lambda(g)) + \sum_{m \neq 1} n a_m u^{m+n-1} \lambda(g)$ .  
Consequently, we have that  $\delta(u^n \lambda(g)) = \delta(u^n) \lambda(g) + u^n \delta(\lambda(g))$   
 $= (k \tilde{\delta}_0 + \delta'_1)(u^n \lambda(g)) + \sum_{h \neq e} [f_h \lambda(h), u^n] \lambda(g) + \sum_{m \neq 0} u^m \tilde{\beta}_m(\lambda(g))$

$+ \sum_{m \neq 0} n a_{m+1} u^{m+m} \lambda(g)$ . Since  $\delta - k \tilde{\delta}_0 - \tilde{\delta}_1$  is a \*-derivation, so is  $\sum_{h \in e} [f_h \lambda(h), u^n] \lambda(g) + \sum_{m \neq 0} u^{m+m} \tilde{S}_m(\lambda(g)) + \sum_{m \neq 0} n a_{m+1} u^{m+m} \lambda(g)$ . Since  $\text{ad}(f_h \lambda(h))(u^n) \lambda(g) + u^n \text{ad}(f_h \lambda(h))(\lambda(g)) = \text{ad}(f_h \lambda(h))(u^n \lambda(g))$ , we deduce that  $u^n (\sum_{m \neq 0} \tilde{S}_m(\lambda(g)) - \sum_{h \in e} [f_h \lambda(h), \lambda(g)]) + \sum_{m \neq 0} n a_{m+1} u^{m+m} \lambda(g)$  is a \*-derivation. Let  $a = \sum_{m \neq 0} a_{m+1} u^m \in D$ . Conventionally put  $\sigma(\lambda(g)) = \sum_{m \neq 0} \tilde{S}_m(\lambda(g)) - \sum_{h \in e} [f_h \lambda(h), \lambda(g)]$ . Moreover, put  $\Delta(u^n \lambda(g)) = u^n \sigma(\lambda(g)) + n a u^n \lambda(g)$ . Since  $\delta_0(u^n) = i n u$ , we see  $n a u^n \lambda(g) = (-\Delta) a \tilde{\delta}_0(u^n \lambda(g))$ . Now since  $\Delta(u^n \lambda(g) u^m \lambda(h)) = \Delta(u^n \lambda(g)) u^m \lambda(h) + u^n \lambda(g) \Delta(u^m \lambda(h))$ , we can show that  $u^{m+m} (\sigma(\lambda(g)) \lambda(h) - \lambda(g) \sigma(\lambda(h))) = m(\alpha_g(a) - a) u^{m+m} \lambda(g+h)$ . Put  $h=e$  and  $m=1$ . Then we have  $u^n \sigma(\lambda(g)) = (\alpha_g(a) - a) u^n \lambda(g)$  for all  $n \in \mathbb{Z}$  and  $g \in G$ . Therefore  $\Delta(u^n \lambda(g)) = (\alpha_g(a) - a) u^n \lambda(g) + n a u^n \lambda(g) = (\alpha_g(a) + (n-1)a) u^n \lambda(g)$ . Since  $\Delta$  is a derivation, we get  $\alpha_g(a) = a$  for all  $g \in G$ . So  $a = z 1$  for some  $z \in \mathbb{C}$ . Then  $\Delta(u^n \lambda(g)) = z n u^n \lambda(g) = i z \tilde{\delta}_0(u^n \lambda(g))$ . Finally, we obtain that  $\delta(u^n \lambda(g)) = (c \tilde{\delta}_0 + \tilde{\delta}_1)(u^n \lambda(g)) + \sum_{h \in F} [f_h \lambda(h), u^n \lambda(g)]$  for some  $c \in \mathbb{R}$ . Let  $\delta_F(a \lambda(g)) = \sum_{h \in F} [f_h \lambda(h), a \lambda(g)]$  for  $a \in D(\delta_0)$  and  $g \in G$  where  $F$  is a finite set of  $G$ -elts with  $F = -F$ . Then  $\delta_F$  is a bounded \*-derivation of  $D \times_{\mathbb{R}} G$  for all  $F$  and  $\delta_F \rightarrow \tilde{\delta}_2$  pointwisely. Hence  $\tilde{\delta}_2$  is approximately bounded. This completes the proof.

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