

On a problem of Smirnov

By PER ENFLO

From the theorem that every separable metric space is isometric with a subset of $C(0, 1)$ and the theorem that all separable Banach spaces are homeomorphic it follows that every separable metric space is homeomorphic with a subset of $L_2(0, 1)$. In this paper we shall construct a countable metric space which is not uniformly homeomorphic with any subset of $L_2(0, 1)$. This gives a negative answer to a question asked by Smirnov. This question is the theme in [3] where among other things it is proved that Euclidean n -space is uniformly homeomorphic with a bounded subset of $L_2(0, 1)$. The question is also treated in [2] where a result in the negative direction is obtained and in [1] where a stronger result is obtained.

1. A geometric property of $L_2(0, 1)$

We shall say that a set of $2n+2$ points in a metric space is a double n -simplex if the points are written $a_1, a_2, \dots, a_{n+1}, b_1, b_2, \dots, b_{n+1}$. We shall call a pair of points (a_i, a_j) or (b_i, b_j) $i \neq j$ an edge and a pair of points (a_i, b_k) a connecting line. We shall say that a metric space M has generalised roundness p , if p is the supremum of the q 's with the property: for every $n \geq 1$ and every double n -simplex in M , $\Sigma c_\alpha^q \geq \Sigma s_\beta^q$ where c_α runs through the lengths of all connecting lines and s_β runs through the lengths of all edges. In [1] we defined roundness to be the supremum of the q 's for which the inequality holds for double 1-simplexes. It is obvious that the generalised roundness is not larger than the roundness. In [1] it was proved that $L_p(0, 1)$ $1 \leq p \leq 2$ has roundness p . If a metric space has the property that some pair of points (a_1, a_2) has a metric middle point m then its roundness and thus its generalised roundness is not larger than 2. We see this by choosing $b_1 = b_2 = m$.

Since in every double $(n-1)$ -simplex there are n^2 connecting lines and $n(n-1)$ edges the generalised roundness of a metric space is ≥ 0 . If in a double $(n-1)$ -simplex we put the lengths of all connecting lines $= \frac{1}{2}$ and the lengths of all edges $= 1$ then it is easy to see that we get a metric space with generalised roundness $= 2 \log(1 - 1/n)$ which tends to 0 as $n \rightarrow \infty$. If in a double $(n-1)$ -simplex we put instead the lengths of all connecting lines $= (1 - 1/n)^{1/q}$, $(1 - 1/n)^{1/q} \geq \frac{1}{2}$, $q > 0$, it is easy to see that we get a metric space with generalised roundness q .

Theorem 1.1. $L_2(0, 1)$ has generalised roundness 2.

Proof. Since $L_2(0, 1)$ has roundness 2, the generalised roundness is not larger than 2. Thus it is enough to prove $\Sigma c_\alpha^2 \geq \Sigma s_\beta^2$ for all double n -simplexes in $L_2(0, 1)$. This inequality is for a double $(n-1)$ -simplex equivalent with the inequality

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$$\int_0^1 \left(\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} (a_i - b_j)^2 - \sum_{1 \leq i < j \leq n} (a_i - a_j)^2 - \sum_{1 \leq i < j \leq n} (b_i - b_j)^2 \right) dx \geq 0$$

which holds since the integrand is equal to

$$\left(\sum_{1 \leq i \leq n} a_i - \sum_{1 \leq i \leq n} b_i \right)^2.$$

Remark. The identity above gives $\sum c_\alpha^2 = \sum s_\beta^2$ when the two simplexes have the same centre of gravity. For double 1-simplexes this is the parallelogram theorem.

2. Universal uniform embedding spaces

We shall say that a metric space M is a universal uniform embedding space if every separable metric space is uniformly homeomorphic with a subset of M . $C(0, 1)$ is a universal uniform embedding space since every separable metric space is isometric with a subset of $C(0, 1)$. We now prove

Theorem 2.1. *Every universal uniform embedding space has generalised roundness 0.*

Proof. We prove that a metric space with generalised roundness $p > 0$ is not a universal uniform embedding space.

Consider the metric space $M = \{\exp((2\pi ki)/2^{n+1}), k=0, 1, 2, \dots, (2^{n+1}-1)\}$ where n is an even number. Take the product M_n of n^n such spaces and define a metric in M_n by letting the distance between two points in M_n be the largest of the distances in the coordinate spaces. We shall say that a pair of points (a, b) in M_n is an m -segment if the coordinates of a and b are different in exactly n^m coordinate spaces, and the difference between the coordinates in each of these spaces is $\exp((2\pi ki)/2^n) \times (\exp(\pi i/2^m) - 1)$ where k may depend on the coordinate space.

We shall consider double $(n-1)$ -simplexes in M_n where every connecting line is an $(m+1)$ -segment and every edge is an m -segment. In such a double $(n-1)$ -simplex every connecting line has length $|\exp(\pi i/2^{m+1}) - 1|$ and every edge has length $|\exp(\pi i/2^m) - 1|$. If there exists such a double $(n-1)$ -simplex then, by symmetry, it follows that for fixed m , all m -segments in M_n are edges in the same number N_1 of double $(n-1)$ -simplexes of that type and all $(m+1)$ -segments in M_n are connecting lines in the same number N_2 of double $(n-1)$ -simplexes of that type. For if s_1 and s_2 are two m -segments then there is an isometry of M_n onto itself by which s_2 is the image of s_1 and the image of each k -segment is a k -segment. We now prove the existence of such a double $(n-1)$ -simplex, $1 \leq m \leq n-1$.

We consider an ordering of the coordinate spaces and divide the first n^{m+1} coordinate spaces into $2n$ groups with $n^m/2$ coordinate spaces in each. We let the coordinates of a point in one of the simplexes be $\exp(\pi i/2^m)$ in all coordinate spaces of one of the first n groups and be 1 in the remaining coordinate spaces of the first n groups. The coordinates are $\exp(\pi i/2^{m+1})$ in all the remaining coordinate spaces of M_n . We let the coordinates of a point in the other simplex be $\exp(\pi i/2^m)$ in all coordinate spaces of one of the last n groups and be 1 in the remaining coordinate spaces of the last n groups. The coordinates are $\exp(\pi i/2^{m+1})$ in all the remaining coordinate spaces of M_n . This double $(n-1)$ -simplex has the required properties.

If M_n is embedded in a metric space B with generalised roundness $p > 0$ then we get $\sum d_{\alpha, m+1}^p \geq \sum d_{\alpha, m}^p$ for every double $(n-1)$ -simplex of the type described above,

where $d_{\alpha,k}$ runs through the lengths of the images of k -segments. We add all these inequalities for a fixed m . Then we get an inequality where all lengths of images of m -segments appear N_1 times on the right side and all lengths of images of $(m+1)$ -segments appear N_2 times on the left side. Since there are n^2 connecting lines and $n(n-1)$ edges in each double $(n-1)$ -simplex we get $S(d_{\alpha,m+1}^p) \geq [(n-1)/n] S(d_{\alpha,m}^p)$ where $S(d_{\alpha,k}^p)$ is the arithmetic mean of the p th powers of the lengths of the images of the k -segments. If we apply this last inequality $n-1$ times we get $S(d_{\alpha,n}^p) \geq [n-1/n]^{n-1} S(d_{\alpha,1}^p)$. From this we get $\sup d_{\alpha,n}^p \geq 1/e \inf d_{\alpha,1}^p$ and $\sup d_{\alpha,n} \geq (1/e)^{1/p}$ ($\inf d_{\alpha,1}$).

Now we take the union of a countable family of sets M_n where we let n tend to infinity. We put the distance between two points in different M_n 's = 2. If this metric space were uniformly homeomorphic with some subset of B and T were a uniform homeomorphism then $\inf d_{\alpha,1} \geq \varepsilon$ for some $\varepsilon > 0$ and all spaces M_n . But then $\sup d_{\alpha,n} \geq (1/e)^{1/p} \cdot \varepsilon$ for all n and this contradicts that T is uniformly continuous. The theorem is proved.

Remark. By a modification of the last construction of the proof we can get a metric space in which no non-void open set is uniformly homeomorphic with a subset of B .

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