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# On $(\alpha, \psi)$-$K$-Contractions in the Extended $b$-Metric Space

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**Abstract.** In this paper, we introduce a notion of $(\alpha, \psi)$-$K$-contraction in the setting of extended b-metric spaces and investigate the existence of a fixed point. The presented results generalize and unify a number of well-known fixed point theorem mainly in two distinct aspects; in the sense of the contraction conditions and in the frame of abstract spaces.

## 1. Introduction and Preliminaries

In 1993, Czerwik [16] suggested a successful and proper generalization of the metric space notion by introducing the concepts of $b$-metric space. In this paper, the author examine the basic topological properties of this new space and investigate the existence and uniqueness of certain mappings in framework of $b$-metric space. Following this famous result in the setting of $b$-metric space, a number of authors have reported several interesting results in this direction (see e.g. [2, 6–8],[11]-[14] and related references therein). Very recently, Kamran *et al.* [18] extend the b-metric space and successfully prove the analog of Banach mapping principle in this new space.

In this paper, we shall define a general contraction condition by the help of some auxiliary functions and investigate the existence and uniqueness of a fixed point for such mappings.

Throughout the manuscript, we denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the positive integers. Further, $\mathbb{R}$ represent the real numbers and $\mathbb{R}_0^+ := [0, \infty)$.

We, first, recall the notion of $b$-metric.

**Definition 1.1 (Czerwik [16]).** *Let $X$ be a nonempty set and $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions:*

($b1$) $d(x, y) = 0$ *if and only if* $x = y$.
($b2$) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.
($b3$) $d(x, y) \leq s[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$, *where* $s \geq 1$.

*The function $d$ is called a b-metric and the space $(X, d)$ is called a b-metric space, in short, bMS.*

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The immediate examples of $b$-metric are the following (see also [2, 6–8],[11]-[14].)

**Example 1.2.** *Let $X = \mathbb{R}^2$. Then, the functional $d : X \times X \to [0, \infty)$ defined by:*

$$d((x_1, y_1), (x_2, y_2)) := \begin{cases} |x_1 - x_2| + |y_1 - y_2|, & ((x_1, y_1), (x_2, y_2)) \in [0, 1] \times [0, 1] \\ |x_1 - x_2|^2 + |y_1 - y_2|^2, & ((x_1, y_1), (x_2, y_2))(1, \infty) \times (1, \infty) \\ 0, & \textit{otherwise.} \end{cases}$$

*It is a b-metric on X with coefficient $s = 2$.*

**Example 1.3.** *The space $L^p[0, 1]$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0, 1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, together with the functional*

$$d(x, y) := (\int_0^1 |x(t) - y(t)|^p dt)^{1/p}, \textit{ for each } x, y \in L^p[0, 1],$$

*is a b-metric space. Notice that $s = 2^{1/p}$.*

**Example 1.4.** *Let $X$=$\{a, b, c\}$ and $d : X \times X \to \mathbb{R}_+$ such that $d(a, b) = d(b, a) = d(a, c) = d(c, a) = b, d(b, c) = d(c, b) = \alpha \geq c, d(a, a) = d(b, b) = d(c, c) = a$. Then*

$$d(x, y) \leq \frac{\alpha}{2}\left[d(x, z) + d(z, y)\right], \textit{ for } x, y, z \in X.$$

*Then $(X, d)$ is a b-metric space. If $\alpha > c$ the ordinary triangle inequality does not hold and $(X, d)$ is not a metric space.*

**Remark 1.5.** *It is clear that for $s = 1$, the b-metric becomes a usual metric.*

In what follows, we recollect the notion of extend the $b$-metric space that is defined by Kamran *et al.* [18]

**Definition 1.6.** *[18] Let $X$ be a non empty set and $\theta : X \times X \to [1, \infty)$. A function $d_\theta : X \times X \to [0, \infty)$i is called an extended $b-$ metric if for all $x, y, z \in X$ is satisfies*

($d_\theta 1$) *$d_\theta(x, y) = 0$ if and only if $x = y$;*

($d_\theta 2$) *$d_\theta(x, y) = d_\theta(y, x)$;*

($d_\theta 3$) *$d_\theta(x, z) \leq \theta(x, z)\left[d_\theta(x, y) + d_\theta(y, z)\right]$.*

*The pair $(X, d_\theta)$ is called an extended $b-$metric space, in short extended-bMS.*

**Remark 1.7.** *If $\theta(x, y) = s$, for $s \geq 1$ then we obtain the definition of bMS.*

**Example 1.8.** *Let $X = \{a, b, c\} \cup \mathbb{R}_0^+$ and $d : X \times X \to [0, \infty)$ be defined by*

*Case 1. if $x, y \in \mathbb{R}_0^+$ then $d_\theta(x, y) = |x - y|^2$,*
*Case 2. if $x \in \{a, b, c\}$ and $y \in \mathbb{R}_0^+$ then $d_\theta(x, y) = 1 = d_\theta(y, x)$ and $d_\theta(x, x) = 0$ ,*
*Case 3. if $x, y \in \{a, b, c\}$*

$$d_\theta(a, b) = 1, \quad d_\theta(a, c) = \frac{1}{2} \quad \textit{and} \quad d_\theta(b, c) = 2,$$

*with $d_\theta(x, x) = 0$ and $d_\theta(x, y) = d_\theta(y, x)$.*

*Notice that d is not a metric since $d_\theta(b, c) > d_\theta(b, a) + d_\theta(a, c)$. However, it is easy to see that d is a extended b-metric space. Indeed, for the following $\theta : X \times X \to [1, \infty)$, we conclude the desired result.*

$$\theta(x, y) = \begin{cases} 2 & \text{if } x, y \in \mathbb{R}_0^+, \\ \frac{4}{3} & \text{if } x, y \in \{a, b, c\}, \\ 1 & \text{if } (x, y) \text{ or } (y, x) \in \{a, b, c\} \times \mathbb{R}_0^+. \end{cases} \tag{1}$$

**Example 1.9.** *Let $X = \{x, y, z\}$ and $\theta : X \times X \to [1, \infty)$, $\theta(x, y) = |x| + |y| + 2$. Define $d_\theta : X \times X \to [0, \infty)$ as*

$$d_\theta(x, y) = d_\theta(y, x) = 5,\ d_\theta(x, z) = d_\theta(z, x) = 3,\ d_\theta(y, z) = d_\theta(z, y) = 1,$$
$$d_\theta(x, x) = d_\theta(y, y) = d_\theta(z, z) = 0.$$

*Obviously, $(d_\theta 1)$ and $(d_\theta 2)$ hold. For $(d_\theta 3)$, we have*

$$5 = d_\theta(x, y) \leq \theta(x, y)(d_\theta(x, z) + d_\theta(z, y)) = (|x| + |y| + 2) \cdot 4,$$

$$3 = d_\theta(x, z) \leq \theta(x, z)(d_\theta(x, y) + d_\theta(y, z)) = (|x| + |z| + 2) \cdot 6,$$

$$1 = d_\theta(y, z) \leq \theta(y, z)(d_\theta(y, x) + d_\theta(x, z)) = (|y| + |z| + 2) \cdot 8,$$

*In conclusion, for any $x, y, z \in X$,*

$$d_\theta(x, z) \leq \theta(x, z)\left[d_\theta(x, y) + d_\theta(y, z)\right].$$

*Hence, $(X, d_\theta)$ is an extended $b-$metric space. Notice also that*

$$5 = d_\theta(x, y) > 4 = d_\theta(x, z) + d_\theta(z, y),$$

*thus the standard triangle inequality does not hold in this case and $(X, d)$ is not a metric space.*

In what follows that we recollect some basic concepts, for instance, convergence, notion of the Cauchy sequence, and completeness in a extended-$b$MS. For more details, see e.g. [18].

**Definition 1.10.** *[18] Let $(X, d_\theta)$ be an extended-bMS.*

(*i*) *A sequence $x_n$ in $X$ is said to converge to $x \in X$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_n, x) < \epsilon$, for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$.*

(*ii*) *A sequence $x_n$ in $X$ is said to be Cauchy if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_m, x_n) < \epsilon$, for all $m, n \geq N$.*

**Definition 1.11.** *An extended-bmetric space $(X, d_\theta)$ is complete if every Cauchy sequence in $X$ is convergent.*

**Lemma 1.12.** *Let $(X, d_\theta)$ be an complete extended-bMS. If $d_\theta$ is continuous, then every convergent sequence has a unique limit.*

**Theorem 1.13.** *[18] Let $(X, d_\theta)$ be an extended-bMS such that $d_\theta$ is a continuous functional. Let $T : X \to X$ satisfy:*

$$d_\theta(Tx, Ty) \leq k d_\theta(x, y) \tag{2}$$

*for all $x, y \in X$, where $k \in [0, 1)$ be such that for each $x_0 \in X$, $\lim_{n,m\to\infty} \theta(x_n, x_m) < \frac{1}{k}$, here $x_n = T^n x_0$, $n = 1, 2, ....$ Then $T$ has precisely one fixed point $u$. Moreover for each $y \in X$, $T^n y \to u$.*

For our purposes, we need to recall the following definition of $\alpha-$orbital admissible mappings given by Popescu [29]

**Definition 1.14.** *Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that $T$ is an $\alpha-$orbital admissible if for all $x, y \in X$ we have*

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1. \tag{3}$$

**Remark 1.15.** *Each $\alpha-$admissible mapping is an $\alpha-$orbital admissible mapping.(see [29]).*

Let $\Phi$ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

($\Phi_1$) $\phi$ is nondecreasing;

($\Phi_2$) $\phi(t) < t$.

## 2. Main results

We start with the definition of $(\alpha,\psi)$-$K$-contraction.

**Definition 2.1.** *Let $(X,d)$ be an extended b-metric space $\alpha : X\times X \to [0,\infty)$ and $\theta : X\times X \to [1,\infty)$. A mapping $T : X \to X$ is called $(\alpha,\psi)$-$K$-*contraction *if it satisfies*

$$\alpha(x,y)d_\theta(Tx,Ty) \le \phi(K(x,y)), \text{ for all } x,y\in X, \tag{4}$$

*where $\phi\in\Phi$ and*

$$K(x,y) = \max\{d_\theta(x,y), d_\theta(x,Tx), d_\theta(y,Ty), \frac{d_\theta(x,Tx)d_\theta(y,Ty)}{d_\theta(x,y)}, \frac{d_\theta(x,Ty)+d_\theta(y,Tx)}{2\max\{\theta(y,Tx),\theta(x,Ty)\}}\}. \tag{5}$$

The following is the first main result of this paper.

**Theorem 2.2.** *Let $(X,d)$ be a complete extended b-metric space and $T : X\to X$ be a $(\alpha,\psi)$-$K$-contraction mapping. Suppose that for each $x_0\in X$ and for each $t>0$,*

$$\limsup_{n,m\to\infty} \frac{\phi^{n+1}(t)}{\phi^n(t)}\theta(x_n,x_m) < 1$$

*where $x_n = T^n x_0$, $n\in\mathbb{N}$. Suppose also that*

(*i*) *$T$ is $\alpha$-orbital admissible,*
(*ii*) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$*
(*iii*) *$T$ is continuous.*

*Then the mappings $T$ posses a fixed point $u$, that is, $Tu=u$.*

*Proof.* By assumption, for a given $x_0\in X$, we have a constructive sequence $\{x_n\}$ that is defined by $x_n = T^n x_0$ for each $n\in\mathbb{N}$. If for some $n_0$, we have $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then $x_{n_0}$ is a fixed point of $T$. From now on, we assume that $x_n \ne x_{n+1}$ for all $n\ge 0$. Since $T$ is $\alpha-$admissible, we have

$$\alpha(x_0,x_1) = \alpha(x_0,Tx_0)\ge 1 \Rightarrow \alpha(Tx_0,Tx_1) = \alpha(x_1,x_2)\ge 1.$$

Recursively, we find that

$$\alpha(x_n,x_{n+1})\ge 1, \text{ for all } n = 0,1,\dots \tag{6}$$

On account of (6) and (4), we have

$$d_\theta(x_n,x_{n+1}) = d_\theta(Tx_{n-1},Tx_n) \le \phi(M(x_{n-1},x_n)),$$

where

$$\begin{aligned}
K(x_{n-1},x_n) &= \max\{d_\theta(x_{n-1},x_n), d_\theta(x_{n-1},Tx_{n-1}), d_\theta(x_n,Tx_n),\\
&\quad \frac{d_\theta(x_{n-1},Tx_{n-1})d_\theta(x_n,Tx_n)}{d_\theta(x_{n-1},x_n)}, \frac{d_\theta(x_n,Tx_{n-1})+d_\theta(x_{n-1},Tx_n)}{2\max\{\theta(y,Tx),\theta(x,Ty)\}}\Big\}\\
&= \max\{d_\theta(x_{n-1},x_n), d_\theta(x_{n-1},x_n), d_\theta(x_n,x_{n+1}),\\
&\quad \frac{d_\theta(x_{n-1},x_n)d_\theta(x_n,x_{n+1})}{d_\theta(x_{n-1},x_n)}, \frac{d_\theta(x_{n-1},x_{n+1})}{2\max\{\theta(x_n,x_n),\theta(x_{n-1},x_{n+1})\}}\Big\}\\
&\le \max\Big\{d_\theta(x_{n-1},x_n), d_\theta(x_n,x_{n+1}), \frac{d_\theta(x_n,x_{n+1})+d_\theta(x_{n-1},x_n)}{2}\Big\}\\
&= \max\{d_\theta(x_{n-1},x_n), d_\theta(x_n,x_{n+1})\}.
\end{aligned}$$

If for some $n$, we have $K(x_{n-1}, x_n) = \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1}) = d_\theta(x_n, x_{n+1})\}$, then

$$0 < d_\theta(x_n, x_{n+1}) \leq \phi(d_\theta(x_n, x_{n+1})) < d_\theta(x_n, x_{n+1}),$$

a contradiction. Accordingly, we conclude, for all $n \geq 1$, that

$$K(x_{n-1}, x_n) = \max\{d_\theta(x_{n-1}, x_n), d_\theta(x_n, x_{n+1}) = d_\theta(x_{n-1}, x_n).$$

We deduce that

$$0 < d_\theta(x_n, x_{n+1}) \leq \phi(d_\theta(x_{n-1}, x_n)) < d_\theta(x_{n-1}, x_n), \quad \forall\ n \geq 1. \tag{7}$$

We deduce

$$0 < d_\theta(x_n, x_{n+1}) \leq \phi^n(d_\theta(x_0, x_1)), \quad \forall\ n \geq 0. \tag{8}$$

Therefore, there exists $L \geq 0$ such that

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = L.$$

Letting $n \to \infty$ in (7), we get

$$L \leq \phi(L),$$

which holds unless $l = 0$. Thus

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0. \tag{9}$$

We claim that $\{x_n\}$ is a Cauchy sequence. By using the modified triangle inequality $(b3)$ together with (7) and (8), we find that

$$\begin{aligned}
d_\theta(x_n, x_{n+k}) &\leq \theta(x_n, x_{n+k})[d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_{n+k})] \\
&\leq \theta(x_n, x_{n+k})d_\theta(x_n, x_{n+1}) + \theta(x_n, x_{n+k})\theta(x_{n+1}, x_{n+k})d_\theta(x_{n+1}, x_{n+2}) \\
&\quad + ... + \theta(x_n, x_{n+k})\theta(x_{n+1}, x_{n+k})...\theta(x_{n+k-1}, x_{n+k})d_\theta(x_{n+k-1}, x_{n+k}) \\
&\leq \theta(x_n, x_{n+k})\phi^n(d_\theta(x_0, x_1)) + \theta(x_n, x_{n+k})\theta(x_{n+1}, x_{n+k})\phi^{n+1}(d_\theta(x_0, x_1)) \\
&\quad + ... + \theta(x_n, x_{n+k})\theta(x_{n+1}, x_{n+k})...\theta(x_{n+k-1}, x_{n+k})\phi^{n+k-1}(d_\theta(x_0, x_1)) \\
&\leq \theta(x_1, x_{n+k})\theta(x_2, x_{n+k})...\theta(x_n, x_{n+k})\phi^n(d_\theta(x_0, x_1)) \\
&\quad + \theta(x_1, x_{n+k})\theta(x_2, x_{n+k})...\theta(x_n, x_{n+k})\theta(x_{n+1}, x_{n+k})\phi^{n+1}(d_\theta(x_0, x_1)) \\
&\quad + ... + \cdots \\
&\quad + \theta(x_1, x_{n+k})\theta(x_2, x_{n+k})...\theta(x_{n+k-1}, x_{n+k})\phi^{n+k-1}(d_\theta(x_0, x_1)) \\
&= \sum_{j=n}^{n+k-1} \phi^j(d_\theta(x_0, x_1)) \prod_{i=1}^{j} \theta(x_i, x_{n+k}).
\end{aligned}$$

We deduce that

$$d_\theta(x_n, x_{n+k}) \leq S_{n+k-1} - S_n, \tag{10}$$

for the series

$$\sum_{j=1}^{\infty} \phi^j(d_\theta(x_0, x_1)) \prod_{i=1}^{j} \theta(x_i, x_{n+k}).$$

Put $a_n = \phi^n(d_\theta(x_0,x_1)) \prod_{i=1}^{n} \theta(x_i,x_{n+k})$. We have

$$\frac{a_{n+1}}{a_n} = \frac{\phi^{n+1}(d_\theta(x_0,x_1))}{\phi^j(d_\theta(x_0,x_1))}\theta(x_{n+1},x_{n+k}).$$

In view of the assumption,

$$\limsup_{n\to\infty} \frac{\phi^{n+1}(t)}{\phi^j(t)}\theta(x_{n+1},x_{n+k}) < 1,$$

the above series converges by ratio test. Consequently, in view of (10), we get

$$\lim_{n,m\to\infty} d_\theta(x_n,x_{n+k}) = 0, \tag{11}$$

that is, $\{x_n\}$ is a Cauchy sequence. Since $(X,d)$ is a complete extended $b$-metric space, there exists $z \in X$ such that

$$\lim_{n\to\infty} d_\theta(x_n,z) = 0. \tag{12}$$

Since the mapping $T$ and the extended $b$-metric are continuous, we derive that

$$\lim_{n\to\infty} d_\theta(Tx_n,Tz) = 0 = \lim_{n\to\infty} d_\theta(x_{n+1},Tz) = d_\theta(z,Tz). \tag{13}$$

Hence, we conclude that $Tz = z$. □

In what follows, we refine the definition of $(\alpha,\psi)$-$K$-contraction as $(\alpha,\psi)$-$M$-contraction to remove the heavy condition, continuity, on the given self-mapping.

**Definition 2.3.** *Let $(X,d)$ be an extended b-metric space $\alpha : X\times X \to [0,\infty)$ and $\theta : X\times X \to [1,\infty)$. A mapping $T : X \to X$ is called $(\alpha,\psi)$-$M$-*contraction *if it satisfies*

$$\alpha(x,y)d_\theta(Tx,Ty) \le \phi(M(x,y)), \text{ for all } x,y \in X, \tag{14}$$

*where $\phi \in \Phi$ and*

$$M(x,y) = \max\{d_\theta(x,y), \frac{d_\theta(x,Tx)+d_\theta(y,Ty)}{2}, \frac{d_\theta(x,Ty)+d_\theta(y,Tx)}{2\max\{\theta(y,Tx),\theta(x,Ty)\}}\}. \tag{15}$$

This is the second main result in which the continuity of the mapping is removed.

**Theorem 2.4.** *Let $(X,d)$ be a complete extended b-metric space and $T : X \to X$ be a $(\alpha,\psi)$-$M$-contraction mapping. Suppose that for each $x_0 \in X$ and for each $t > 0$,*

$$\limsup_{n,m\to\infty} \frac{\phi^{n+1}(t)}{\phi^n(t)}\theta(x_n,x_m) < 1$$

*where $x_n = T^n x_0$, $n \in \mathbb{N}$. Suppose also that*

(*i*) *$T$ is $\alpha$-orbital admissible,*
(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$*
(*iii*) *if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n,x_{n+1}) \ge 1$ for all $n$ and $x_n \to x \in X$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x) \ge 1$ for all $k$.*

*Then the mappings $T$ posses a fixed point $u$, that is, $Tu = u$.*

*Proof.* Following the proof of Theorem 2.2, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges for some $u \in X$. From (6) and condition (iii), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, u) \geq 1$ for all $k$. Applying (14), for all $k$, we get that

$$d_\theta(x_{n(k)+1}, Tu) = d_\theta(Tx_{n(k)}, Tu) \leq \alpha(x_{n(k)}, u)d_\theta(Tx_{n(k)}, Tu) \leq \phi(M(x_{n(k)}, u)). \tag{16}$$

On the other hand, we have

$$M(x_{n(k)}, u) = \max\left\{d_\theta(x_{n(k)}, u), \frac{d_\theta(x_{n(k)}, x_{n(k)+1}) + d_\theta(u, Tu)}{2}, \frac{d_\theta(x_{n(k)}, Tu) + d_\theta(u, x_{n(k)+1})}{2\max\{\theta(y, Tx), \theta(x, Ty)\}}\right\}.$$

Letting $k \to \infty$ in the above equality, we get that

$$\lim_{k\to\infty} M(x_{n(k)}, u) = \frac{d_\theta(u, Tu)}{2}. \tag{17}$$

Suppose that $d_\theta(u, Tu) > 0$. From (17), for $k$ large enough, we have $M(x_{n(k)}, u) > 0$, which implies that $\phi(M(x_{n(k)}, u)) < M(x_{n(k)}, u)$. Thus, from (16), we have

$$d_\theta(x_{n(k)+1}, Tu) < M(x_{n(k)}, u).$$

Letting $k \to \infty$ in the above inequality, using (17), we obtain that

$$d_\theta(u, Tu) \leq \frac{d_\theta(u, Tu)}{2},$$

which is a contradiction. Thus we have $d_\theta(u, Tu) = 0$, that is, $u = Tu$. □

For the uniqueness of a fixed point of a $(\alpha, \psi)$-$K$-contractive mapping (respectively, $(\alpha, \psi)$-$M$-contractive mapping), we shall suggest the following hypothesis.

(U) For all $x, y \in \mathrm{Fix}(T)$, either $\alpha(x, y) \geq 1$ or $\alpha(y, x) \geq 1$.

Here, $\mathrm{Fix}(T)$ denotes the set of fixed points of $T$.

**Theorem 2.5.** *Adding condition (U) to hypotheses of Theorem 2.2 (respectively, Theorem 2.4), we obtain uniqueness of the fixed point of T.*

*Proof.* Suppose, on the contrary, that $u$ and $v$ are two distinct fixed points of $T$. Then we have $K(u, v) = d(u, v)$ (respectively, $M(u, v) = d(u, v)$). On account of the hypothesis of $(U)$, we employ the contraction condition (14)

$$\begin{aligned}
\phi(d(u, v)) &= \phi(d(Tu, Tv)) \\
&\leq \alpha(u, v)\phi(d(Tu, Tv)) \\
&\leq \phi(K(u, v)) \\
&< \phi(d(u, v)),
\end{aligned}$$

which is a contradiction. Hence, we conclude that the obtained fixed points are unique in Theorem 2.2 and Theorem 2.4. □

**Definition 2.6.** *Let $(X, d)$ be an extended b-metric space $\alpha : X \times X \to [0, \infty)$ and $\theta : X \times X \to [1, \infty)$. A mapping $T : X \to X$ is called $\alpha$-*contraction *if it satisfies*

$$\alpha(x, y)d_\theta(Tx, Ty) \leq \phi(d_\theta(x, y)), \text{ for all } x, y \in X, \tag{18}$$

*where $\phi \in \Phi$.*

**Corollary 2.7.** *Let $(X,d)$ be a complete extended b-metric space and $T : X \to X$ be a $\alpha$-contraction mapping. Suppose that for each $x_0 \in X$ and for each $t > 0$,*

$$\limsup_{n,m\to\infty} \frac{\phi^{n+1}(t)}{\phi^n(t)}\theta(x_n, x_m) < 1$$

*where $x_n = T^n x_0$, $n \in \mathbb{N}$. Suppose also that*

($i$) *T is $\alpha$-orbital admissible,*
($ii$) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*
($iii$) *T is continuous.*

*or*

($iii$)* *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.*

*Then the mappings T posses a fixed point u, that is, $Tu = u$. If, additionally, we assume the condition ($U$), then u is the unique fixed point of T.*

**Definition 2.8.** *Let $(X,d)$ be an extended b-metric space $\alpha : X \times X \to [0,\infty)$ and $\theta : X \times X \to [1,\infty)$. A mapping $T : X \to X$ is called* $\alpha$-Jaggi-type contraction *if it satisfies*

$$\alpha(x,y)d_\theta(Tx,Ty) \leq \phi(\frac{d_\theta(x,Tx)d_\theta(y,Ty)}{d_\theta(x,y)}), \text{ for all } x,y \in X, \tag{19}$$

*where $\phi \in \Phi$.*

**Corollary 2.9.** *Let $(X,d)$ be a complete extended b-metric space and $T : X \to X$ be a $\alpha$-Jaggi-type contraction mapping. Suppose that for each $x_0 \in X$ and for each $t > 0$,*

$$\limsup_{n,m\to\infty} \frac{\phi^{n+1}(t)}{\phi^n(t)}\theta(x_n, x_m) < 1$$

*where $x_n = T^n x_0$, $n \in \mathbb{N}$. Suppose also that*

($i$) *T is $\alpha$-orbital admissible,*
($ii$) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$*
($iii$) *T is continuous.*

*Then the mappings T posses a fixed point u, that is, $Tu = u$. If, additionally, we assume the condition ($U$), then u is the unique fixed point of T.*

## 3. Conclusion

One can easily drive several consequences from the presented main results in this paper in different aspects. For example, letting $\theta(x,y) = s \geq 1$ yields the corresponding fixed point results in the context of $b$-metric space. Moreover, the standard versions of the given results are follows when we take $\theta(x,y) = 1$. As in Corollary 2.7 and Corollary 2.9, we can derive more results by replacing $K(x,y)$ with a proper one. On the other hand, as in [19], by assign $\alpha(x,y)$ in a proper way, we can conclude results in the frame of "partially ordered spaces" and for "cyclic contraction."

## Competing interests

The authors declare that they have no competing interests.

## Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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