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# On $\alpha$ - $\psi$ -Meir-Keeler contractive mappings

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## Abstract

In this paper, we introduce the notion of  $\alpha$ - $\psi$ -Meir-Keeler contractive mappings *via* a triangular  $\alpha$ -admissible mapping. We discuss the existence and uniqueness of a fixed point of such a mapping in the setting of complete metric spaces. We state a number of examples to illustrate our results.

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## 1 Introduction and preliminaries

Fixed-point theory is one of the most intriguing research fields in nonlinear analysis. The number of authors have published papers and have increased continuously in the last decades. The main reason for this involvement can be observed easily: Application potential. Fixed point theory has an application in many disciplines such as chemistry, physics, biology, computer science and many branches of mathematics. Banach contraction mapping principle or Banach fixed-point theorem is the most celebrated and pioneer result in this direction: In a complete metric space, each contraction mapping has a unique fixed point. Following Banach [1], many authors give various generalizations of this principle in various space (see e.g. [2–20]). One of the interesting results was given by Samet *et al.* [21] by defining  $\alpha$ - $\psi$ -contractive mappings *via* admissible mappings, see also [22].

In this paper, we introduce an  $\alpha$ - $\psi$ -Meir-Keeler contractive mapping in the setting of complete metric spaces *via* a triangular  $\alpha$ -admissible mapping. We prove the existence and uniqueness of a fixed point of such a mapping. We also consider a number of examples to illustrate our results.

**Definition 1** Let  $f : X \rightarrow X$  and  $\alpha : X \times X \rightarrow (-\infty, +\infty)$ . We say that  $f$  is a triangular  $\alpha$ -admissible mapping if

- (T1)  $\alpha(x, y) \geq 1$  implies  $\alpha(fx, fy) \geq 1$ ,  $x, y \in X$ ,  
(T2)  $\begin{cases} \alpha(x, z) \geq 1, \\ \alpha(z, y) \geq 1, \end{cases}$  implies  $\alpha(x, y) \geq 1$ ,  $x, y, z \in X$ .

**Example 2** Let  $X = \mathbb{R}$ ,  $fx = \sqrt[3]{x}$  and  $\alpha(x, y) = e^{x-y}$  then  $f$  is a triangular  $\alpha$ -admissible mapping. Indeed, if  $\alpha(x, y) = e^{x-y} \geq 1$  then  $x \geq y$  which implies  $fx \geq fy$ . That is,  $\alpha(fx, fy) = e^{fx-fy} \geq 1$ . Also, if  $\begin{cases} \alpha(x, z) \geq 1, \\ \alpha(z, y) \geq 1 \end{cases}$  then  $\begin{cases} x-z \geq 0, \\ z-y \geq 0. \end{cases}$  That is,  $x-y \geq 0$  and so  $\alpha(x, y) = e^{x-y} \geq 1$ .

**Example 3** Let  $X = \mathbb{R}$ ,  $fx = e^{x^7}$  and  $\alpha(x, y) = \sqrt[5]{x-y} + 1$ . Hence,  $f$  is a triangular  $\alpha$ -admissible mapping. Again, if  $\alpha(x, y) = \sqrt[5]{x-y} + 1 \geq 1$  then  $x \geq y$  which implies  $fx \geq fy$ . That is,  $\alpha(fx, fy) \geq 1$ .

Moreover, if  $\begin{cases} \alpha(x,z) \geq 1; \\ \alpha(z,y) \geq 1, \end{cases}$  then  $x - y \geq 0$ , and hence,  $\alpha(x, y) \geq 1$ .

**Example 4** Let  $X = [0, +\infty)$ ,  $fx = x^4 + \ln(x^2 + 1)$  and

$$\alpha(x, y) = \frac{x^3}{1 + x^3} - \frac{y^3}{y^3 + 1} + 1.$$

Then  $f$  is a triangular  $\alpha$ -admissible mapping. In fact, if

$$\alpha(x, y) = \frac{x^3}{1 + x^3} - \frac{y^3}{y^3 + 1} + 1 \geq 1$$

then  $x \geq y$ . Hence,  $fx \geq fy$ . That is,  $\alpha(fx, fy) \geq 1$ . Also,

$$\begin{aligned} \alpha(x, z) + \alpha(z, y) &= \frac{x^3}{1 + x^3} - \frac{z^3}{z^3 + 1} + 1 + \frac{z^3}{1 + z^3} - \frac{y^3}{y^3 + 1} + 1 \\ &= \frac{x^3}{1 + x^3} - \frac{y^3}{y^3 + 1} + 2 \leq 2 \left( \frac{x^3}{1 + x^3} - \frac{y^3}{y^3 + 1} + 1 \right) = 2\alpha(x, y). \end{aligned}$$

Thus,  $\alpha(x, z) + \alpha(z, y) \leq 2\alpha(x, y)$ . Now, if  $\begin{cases} \alpha(x,z) \geq 1; \\ \alpha(z,y) \geq 1, \end{cases}$  then  $\alpha(x, y) \geq 1$ .

**Example 5** Let  $X = \mathbb{R}$ ,  $fx = x^3 + \sqrt[7]{x}$  and  $\alpha(x, y) = x^5 - y^5 + 1$ . Then  $f$  is a triangular  $\alpha$ -admissible mapping.

**Example 6** Let  $X = [0, +\infty)$ ,  $fx = x^2 + e^x$  and

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Hence,  $f$  is a triangular  $\alpha$ -admissible mapping.

**Lemma 7** Let  $f$  be a triangular  $\alpha$ -admissible mapping. Assume that there exists  $x_0 \in X$  such that  $\alpha(x_0, fx_0) \geq 1$ . Define sequence  $\{x_n\}$  by  $x_n = f^n x_0$ . Then

$$\alpha(x_m, x_n) \geq 1 \quad \text{for all } m, n \in \mathbb{N} \text{ with } m < n.$$

*Proof* Since there exist  $x_0 \in X$  such that  $\alpha(x_0, fx_0) \geq 1$  then from (T1), we deduce that  $\alpha(x_1, x_2) = \alpha(fx_0, f^2x_0) \geq 1$ . By continuing this process, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbb{N} \cup 0. \tag{1.1}$$

Suppose that  $m < n$ . Since  $\begin{cases} \alpha(x_m, x_{m+1}) \geq 1, \\ \alpha(x_{m+1}, x_{m+2}) \geq 1, \end{cases}$  then from (T2) we have  $\alpha(x_m, x_{m+2}) \geq 1$ .

Again, since  $\begin{cases} \alpha(x_m, x_{m+2}) \geq 1, \\ \alpha(x_{m+2}, x_{m+3}) \geq 1, \end{cases}$  then we deduce  $\alpha(x_m, x_{m+3}) \geq 1$ .

By continuing this process, we get  $\alpha(x_m, x_n) \geq 1$ . □

Denote with  $\Psi$  the family of nondecreasing functions  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  continuous in  $t = 0$  such that

- $\psi(t) = 0$  if and only if  $t = 0$ ,
- $\psi(t + s) \leq \psi(t) + \psi(s)$ .

## 2 Main results

**Definition 8** Let  $(X, d)$  be a metric space and  $\psi \in \Psi$ . Suppose that  $f : X \rightarrow X$  is a triangular  $\alpha$ -admissible mapping satisfying the following condition: for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\varepsilon \leq \psi(d(x, y)) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x, y)\psi(d(fx, fy)) < \varepsilon \tag{2.1}$$

for all  $x, y \in X$ . Then  $f$  is called an  $\alpha$ - $\psi$ -Meir-Keeler contractive mapping.

**Remark 9** Let  $f$  be an  $\alpha$ - $\psi$ -Meir-Keeler contractive mapping. Then

$$\alpha(x, y)\psi(d(fx, fy)) < \psi(d(x, y))$$

for all  $x, y \in X$  when  $x \neq y$ . Also, if  $x = y$  then  $d(fx, fy) = 0$ . i.e.,

$$\alpha(x, y)\psi(d(fx, fy)) \leq \psi(d(x, y))$$

for all  $x, y \in X$ .

**Theorem 10** Let  $(X, d)$  be a complete metric space. Suppose that  $f$  is a continuous  $\alpha$ - $\psi$ -Meir-Keeler contractive mapping and that there exists  $x_0 \in X$  such that  $\alpha(x_0, fx_0) \geq 1$ , then  $f$  has a fixed point.

*Proof* Let  $x_0 \in X$  and define a sequence  $\{x_n\}$  by  $x_n = f^n x_0$  for all  $n \in \mathbb{N}$ . If  $x_{n_0} = x_{n_0+1}$  for some  $n_0 \in \mathbb{N} \cup \{0\}$ , then obviously  $f$  has a fixed point. Hence, we suppose that

$$x_n \neq x_{n+1} \tag{2.2}$$

for all  $n \in \mathbb{N} \cup \{0\}$ . We have  $d(x_n, x_{n+1}) > 0$  for all  $n \in \mathbb{N} \cup \{0\}$ . Now define  $s_n = \psi(d(x_n, x_{n+1}))$ . By Remark 9, we deduce that for all  $n \in \mathbb{N} \cup \{0\}$ ,

$$\alpha(x_n, x_{n+1})\psi(d(x_{n+1}, x_{n+2})) = \alpha(x_n, x_{n+1})\psi(d(fx_n, fx_{n+1})) < \psi(d(x_n, x_{n+1})).$$

Then by applying Lemma 7

$$\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_n, x_{n+1})).$$

Hence, the sequence  $\{s_n\}$  is decreasing in  $\mathbb{R}_+$  and so it is convergent to  $s \in \mathbb{R}_+$ . We will show that  $s = 0$ . Suppose, to the contrary, that  $s > 0$ . Hence, we have

$$0 < s < \psi(d(x_n, x_{n+1})) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

Let  $\varepsilon = s > 0$ . Then by hypothesis, there exists a  $\delta(\varepsilon) > 0$  such that (2.1) holds. On the other hand, by the definition of  $\varepsilon$ , there exists  $n_0 \in \mathbb{N}$  such that

$$\varepsilon < s_{n_0} = \psi(d(x_{n_0}, x_{n_0+1})) < \varepsilon + \delta.$$

Now by (2.1), we have

$$\begin{aligned} s_{n_0+1} &= \psi(d(x_{n_0+1}, x_{n_0+2})) \leq \alpha(x_{n_0}, x_{n_0+1})\psi(d(x_{n_0+1}, x_{n_0+2})) \\ &= \alpha(x_{n_0}, x_{n_0+1})\psi(d(fx_{n_0}, fx_{n_0+1})) < \varepsilon = s \end{aligned}$$

which is a contradiction. Hence,  $s = 0$ , that is,  $\lim_{n \rightarrow +\infty} s_n = 0$ . Now, by the continuity of  $\psi$  in  $t = 0$ , we have  $\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = 0$ . For given  $\varepsilon > 0$ , by the hypothesis, there exists a  $\delta = \delta(\varepsilon) > 0$  such that (2.1) holds. Without loss of generality, we assume  $\delta < \varepsilon$ . Since  $s = 0$ , then there exists  $N \in \mathbb{N}$  such that

$$s_{n-1} = \psi(d(x_{n-1}, x_n)) < \delta, \quad \text{for all } n \geq N. \tag{2.4}$$

We will prove that for any fixed  $k \geq N_0$ ,

$$\psi(d(x_k, x_{k+l})) \leq \varepsilon, \quad \text{for all } l \in \mathbb{N}, \tag{2.5}$$

holds. Note that (2.5), by (2.4), holds for  $l = 1$ . Suppose the condition (2.1) is satisfied for some  $m \in \mathbb{N}$ . For  $l = m + 1$ , by (2.4), we get

$$\begin{aligned} \psi(d(x_{k-1}, x_{k+m})) &\leq \psi(d(x_{k-1}, x_k) + d(x_k, x_{k+m})) \\ &\leq \psi(d(x_{k-1}, x_k)) + \psi(d(x_k, x_{k+m})) \\ &< \varepsilon + \delta. \end{aligned} \tag{2.6}$$

If  $\psi(d(x_{k-1}, x_{k+m})) \geq \varepsilon$ , then by (2.1) we get

$$\begin{aligned} \psi(d(x_k, x_{k+m+1})) &\leq \alpha(x_{k-1}, x_{k+m})\psi(d(x_k, x_{k+m+1})) \\ &= \alpha(x_k, x_{k+m+1})\psi(d(fx_{k-1}, fx_{k+m})) < \varepsilon \end{aligned}$$

and hence (2.5) holds.

If  $\psi(d(x_{k-1}, x_{k+m})) < \varepsilon$ , by Remark 9, we get

$$\psi(d(x_k, x_{k+m+1})) \leq \alpha(x_{k-1}, x_{k+m})\psi(d(x_k, x_{k+m+1})) \leq \psi(d(x_{k-1}, x_{k+m})) < \varepsilon.$$

Consequently, (2.5) holds for  $l = m + 1$ . Hence,  $\psi(d(x_k, x_{k+l})) \leq \varepsilon$  for all  $k \geq N_0$  and  $l \geq 1$ , which means

$$d(x_n, x_m) < \varepsilon, \quad \text{for all } m \geq n \geq N_0. \tag{2.7}$$

Hence  $\{x_n\}$  is a Cauchy sequence. Since  $(X, d)$  is complete, there exists  $z \in X$  such that  $x_n \rightarrow z$  as  $n \rightarrow \infty$ . Now, since,  $f$  is continuous then

$$fz = f\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} x_{n+1} = z,$$

that is,  $f$  has a fixed point. □

**Theorem 11** Let  $(X, d)$  be a complete metric space and let  $f$  be a  $\alpha$ - $\psi$ -Meir-Keeler contractive mapping. If the following conditions hold:

- (i) there exists  $x_0 \in X$  such that  $\alpha(x_0, fx_0) \geq 1$ ,
- (ii) if  $\{x_n\}$  is a sequence in  $X$  such that  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n$  and  $x_n \rightarrow x$  as  $n \rightarrow +\infty$ , then  $\alpha(x_n, x) \geq 1$  for all  $n$ .

Then  $f$  has a fixed point.

*Proof* Following the proof of the Theorem 10, we say that  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n \in \mathbb{N} \cup \{0\}$  and that there exist  $z \in X$  such that  $x_n \rightarrow z$  as  $n \rightarrow +\infty$ . Hence, from (ii)  $\alpha(x_n, z) \geq 1$ . By Remark 9, we have

$$\begin{aligned} \psi(d(fz, z)) &\leq \psi(d(fz, fx_n) + d(fx_n, z)) \\ &\leq \psi(d(fz, fx_n)) + \psi(d(fx_n, z)) \\ &\leq \alpha(x_n, z)\psi(d(fz, fx_n)) + \psi(d(fx_n, z)) \\ &\leq \psi(d(z, x_n)) + \psi(d(x_{n+1}, z)). \end{aligned}$$

By taking limit as  $n \rightarrow +\infty$ , in the above inequality, we get  $\psi(d(fz, z)) \leq 0$ , that is,  $d(fz, z) = 0$ . Hence,  $fz = z$ . □

Next, we give some examples to validate our main result.

**Example 12** Let  $X = [0, \infty)$  and  $d(x, y) = |x - y|$  be a metric on  $X$ . Define  $f : X \rightarrow X$  by

$$fx = \begin{cases} \frac{x^2}{4}, & \text{if } x \in [0, 1], \\ 2x, & \text{if } x \in (1, \infty), \end{cases} \quad \text{and} \quad \alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ -1, & \text{otherwise} \end{cases}$$

and  $\psi(t) = \frac{1}{4}t$ . Clearly,  $(X, d)$  is a complete metric space. We show that  $f$  is a triangular  $\alpha$ -admissible mapping. Let  $x, y \in X$ , if  $\alpha(x, y) \geq 1$  then  $x, y \in [0, 1]$ . On the other hand, for all  $x, y \in [0, 1]$  we have  $fx \leq 1$  and  $fy \leq 1$ . It follows that  $\alpha(fx, fy) \geq 1$ . Also, if  $\alpha(x, z) \geq 1$  and  $\alpha(z, y) \geq 1$  then  $x, y, z \in [0, 1]$  and hence,  $\alpha(x, y) \geq 1$ . Thus, the assertion holds by the same arguments. Notice that  $\alpha(0, f0) \geq 1$ .

Now, if  $\{x_n\}$  is a sequence in  $X$  such that  $\alpha(x_n, x_{n+1}) \geq 1$  for all  $n \in \mathbb{N} \cup \{0\}$  and  $x_n \rightarrow x$  as  $n \rightarrow +\infty$ , then  $\{x_n\} \subset [0, 1]$  and hence  $x \in [0, 1]$ . This implies that  $\alpha(x_n, x) \geq 1$  for all  $n \in \mathbb{N} \cup \{0\}$ . Let  $x, y \in [0, 1]$ . Without loss of generality, take  $x \leq y$ . Then

$$\begin{aligned} \psi(d(fx, fy)) &= \frac{y^2}{16} - \frac{x^2}{16}, \\ \psi(d(x, y)) &= \frac{y}{4} - \frac{x}{4}. \end{aligned}$$

Clearly, by taking  $\delta = \varepsilon$  the condition (2.1) holds. Otherwise,  $\alpha(x, y) = -1$ . Hence, for given  $\varepsilon > 0$  we have  $\alpha(x, y)\psi(d(fx, fy)) \leq 0 < \varepsilon$ . Hence, conditions of Theorem 11 holds and  $f$  has a fixed point. But, if  $x, y \in (1, \infty)$  and

$$\varepsilon \leq d(x, y) < \varepsilon + \delta,$$

where  $\varepsilon > 0$  and  $\delta > 0$ . Then

$$d(fx, fy) = 2|x - y| = 2d(x, y) \geq d(x, y) \geq \varepsilon.$$

That is, the Meir-Keeler theorem cannot be applied for this example.

**Example 13** Let  $X = [0, \infty)$  and  $d(x, y) = |x - y|$  be a metric on  $X$ . Define  $f : X \rightarrow X$  by

$$fx = \begin{cases} \frac{1}{2}(1 - x), & \text{if } x \in [0, 1], \\ \frac{\sqrt{4-x^2}}{2+\sin(x^2)}, & \text{if } x \in (1, 2], \\ \frac{\sqrt[3]{x-2}}{x^3+1}, & \text{if } x \in (2, \infty) \end{cases}$$

and  $\psi(t) = \frac{1}{2}t$ ,

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y, z \in [0, 1], \\ -1, & \text{otherwise.} \end{cases}$$

Let  $x, y \in [0, 1]$ . Without loss of generality, take  $x \leq y$ . Then

$$\begin{aligned} \psi(d(fx, fy)) &= \frac{1}{4}(y - x), \\ \psi(d(x, y)) &= \frac{1}{2}(y - x). \end{aligned}$$

Clearly, by taking  $\delta = \varepsilon$  the condition (2.1) holds. Otherwise,  $\alpha(x, y) = -1$ . Hence, for given  $\varepsilon > 0$  we have  $\alpha(x, y)\psi(d(fx, fy)) \leq 0 < \varepsilon$ . Hence, conditions of Theorem 11 hold and  $f$  has a fixed point.

Denote with  $\Psi_{st}$  the family of strictly nondecreasing functions  $\psi_{st} : [0, +\infty) \rightarrow [0, +\infty)$  continuous in  $t = 0$  such that

- $\psi_{st}(t) = 0$  if and only if  $t = 0$ ,
- $\psi_{st}(t + s) \leq \psi_{st}(t) + \psi_{st}(s)$ .

**Definition 14** Let  $(X, d)$  be a metric space and  $\psi_{st} \in \Psi_{st}$ . Suppose that  $f : X \rightarrow X$  is a triangular  $\alpha$ -admissible mapping satisfying the following condition: for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\varepsilon \leq \psi_{st}(M(x, y)) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x, y)\psi_{st}(d(fx, fy)) < \varepsilon \tag{2.8}$$

for all  $x, y \in X$  where

$$M(x, y) = \max \left\{ d(x, y), d(fx, x), d(fy, y), \frac{1}{2}[d(fx, y) + d(x, fy)] \right\}.$$

Then  $f$  is called generalized an  $\alpha$ - $\psi_{st}$ -Meir-Keeler contractive mapping.

**Remark 15** Let  $f$  be a generalized  $\alpha$ - $\psi_{st}$ -Meir-Keeler contractive mapping. Then

$$\alpha(x, y)\psi_{st}(d(fx, fy)) < \psi_{st}(M(x, y))$$

for all  $x, y \in X$  when  $M(x, y) > 0$ . Also, if  $M(x, y) = 0$  then  $x = y$  which implies  $\psi(d(fx, fy)) = 0$ , i.e.,

$$\alpha(x, y)\psi_{st}(d(fx, fy)) \leq \psi_{st}(M(x, y))$$

for all  $x, y \in X$ .

**Proposition 16** Let  $(X, d)$  be a metric space and  $f : X \rightarrow X$  a generalized  $\alpha$ - $\psi_{st}$ -Meir-Keeler contractive mapping, if there exists  $x_0 \in X$  such that  $\alpha(x_0, fx_0) \geq 1$ . Then  $\lim_{n \rightarrow \infty} d(f^{n+1}x_0, f^n x_0) = 0$ .

*Proof* Define a sequence  $\{x_n\}$  by  $x_n = f^n x_0$  for all  $n \in \mathbb{N}$ . If  $x_{n_0} = x_{n_0+1}$  for some  $n_0 \in \mathbb{N} \cup \{0\}$ , then obviously  $f$  has a fixed point. Hence, we suppose that

$$x_n \neq x_{n+1} \tag{2.9}$$

for all  $n \in \mathbb{N} \cup \{0\}$ . Then we have  $M(x_{n+1}, x_n) > 0$  for every  $n \geq 0$ . Then by Lemma 7 and Remark 15, we have

$$\begin{aligned} \psi_{st}(d(x_{n+1}, x_{n+2})) &\leq \alpha(x_n, x_{n+1})\psi_{st}(d(x_{n+1}, x_{n+2})) \\ &= \alpha(x_n, x_{n+1})\psi_{st}(d(fx_n, fx_{n+1})) < \psi_{st}(M(x_n, x_{n+1})) \\ &= \psi_{st}\left(\max\left\{d(x_n, x_{n+1}), d(fx_n, x_n), d(fx_{n+1}, x_{n+1}), \right. \right. \\ &\quad \left. \left. \frac{1}{2}[d(fx_n, x_{n+1}) + d(x_n, fx_{n+1})]\right\}\right) \\ &\leq \psi_{st}(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}). \end{aligned}$$

Now, since  $\psi_{st}$  is strictly nondecreasing then, we get

$$d(x_{n+2}, x_{n+1}) < \max\{d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1})\}.$$

Hence, the case where

$$\max\{d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1})\} = d(x_{n+2}, x_{n+1})$$

is not possible. Therefore, we deduce that

$$d(x_{n+2}, x_{n+1}) < d(x_{n+1}, x_n) \tag{2.10}$$

for all  $n$ . That is,  $\{d(x_{n+1}, x_n)\}_{n=0}^{\infty}$  is a decreasing sequence in  $\mathbb{R}_+$  and it converges to  $\varepsilon \in \mathbb{R}_+$ , that is,

$$\lim_{n \rightarrow \infty} \psi_{st}(d(x_{n+1}, x_n)) = \lim_{n \rightarrow \infty} \psi_{st}(M(x_{n+1}, x_n)) = \psi_{st}(\varepsilon). \tag{2.11}$$

Notice that  $\varepsilon = \inf\{p(x_n, x_{n+1}) : n \in \mathbb{N}\}$ . Let us prove that  $\varepsilon = 0$ . Suppose, to the contrary, that  $\varepsilon > 0$ . Then  $\psi(\varepsilon) > 0$ . Regarding (2.11) together with the assumption that  $f$  is a generalized  $\alpha$ - $\psi_{st}$ -Meir-Keeler contractive mapping, for  $\psi_{st}(\varepsilon)$ , there exists  $\delta > 0$  and a natural number  $m$  such that

$$\psi_{st}(\varepsilon) \leq \psi_{st}(M(x_m, x_{m+1})) < \psi_{st}(\varepsilon) + \delta$$

implies that

$$\begin{aligned} \psi_{st}(d(x_{m+1}, x_{m+2})) &\leq \alpha(x_m, x_{m+1})\psi_{st}(d(x_{m+1}, x_{m+2})) \\ &= \alpha(x_m, x_{m+1})\psi_{st}(d(fx_m, fx_{m+1})) < \psi_{st}(\varepsilon). \end{aligned}$$

Now, since  $\psi_{st}$  is strictly nondecreasing then we get

$$d(x_{m+2}, x_{m+1}) < \varepsilon$$

which is a contradiction since  $\varepsilon = \inf\{p(x_n, x_{n+1}) : n \in \mathbb{N}\}$ . Then  $\varepsilon = 0$  and so  $\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0$ .  $\square$

**Theorem 17** *Let  $(X, d)$  be a complete metric space and  $f : X \rightarrow X$  a orbitally continuous generalized  $\alpha$ - $\psi_{st}$ -Meir-Keeler contractive mapping, if there exist  $x_0 \in X$  such that  $\alpha(x_0, fx_0) \geq 1$ . Then,  $f$  has a fixed point.*

*Proof* Define  $x_{n+1} = f^{n+1}x_0$  for all  $n \geq 0$ . We want to prove that  $\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0$ . If this is not, then there exist  $\varepsilon > 0$  and a subsequence  $\{x_{n(i)}\}$  of  $\{x_n\}$  such that

$$d(x_{n(i)}, x_{n(i+1)}) > 2\varepsilon. \tag{2.12}$$

For this  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $\varepsilon \leq \psi_{st}(M(x, y)) < \varepsilon + \delta$  implies that  $\alpha(x, y)\psi_{st}(d(fx, fy)) < \varepsilon$ . Put  $r = \min\{\varepsilon, \delta\}$  and  $s_n = d(x_n, x_{n+1})$  for all  $n \geq 1$ . From Proposition 16, there exists  $n_0$  such that

$$s_n = d(x_n, x_{n+1}) < \frac{r}{4} \tag{2.13}$$

for all  $n \geq n_0$ . Let  $n(i) > n_0$ . We get  $n(i) \leq n(i+1) - 1$ . If  $d(x_{n(i)}, x_{n(i+1)-1}) \leq \varepsilon + \frac{r}{2}$ , then

$$\begin{aligned} d(x_{n(i)}, x_{n(i+1)}) &\leq d(x_{n(i)}, x_{n(i+1)-1}) + d(x_{n(i+1)-1}, x_{n(i+1)}) \\ &\leq d(x_{n(i)}, x_{n(i+1)-1}) + d(x_{n(i+1)-1}, x_{n(i+1)}) \\ &< \varepsilon + \frac{r}{2} + s_{n(i+1)-1} < \varepsilon + \frac{3r}{4} < 2\varepsilon \end{aligned}$$

which contradicts the assumption (2.12). Therefore, there are values of  $k$  such that  $n(i) \leq k \leq n(i+1)$  and  $d(x_{n(i)}, x_k) > \varepsilon + \frac{r}{2}$ . Now if  $d(x_{n(i)}, x_{n(i+1)}) \geq \varepsilon + \frac{r}{2}$ , then

$$s_{n(i)} = d(x_{n(i)}, x_{n(i+1)}) \geq \varepsilon + \frac{r}{2} > r + \frac{r}{2} > \frac{r}{4}$$



which is a contradiction with (2.13). Hence, there are values of  $k$  with  $n(i) \leq k \leq n(i + 1)$  such that  $d(x_{n(i)}, x_k) < \varepsilon + \frac{r}{2}$ . Choose smallest integer  $k$  with  $k \geq n(i)$  such that  $d(x_{n(i)}, x_k) \geq \varepsilon + \frac{r}{2}$ . Thus,  $d(x_{n(i)}, x_{k-1}) < \varepsilon + \frac{r}{2}$  and so

$$\begin{aligned} d(x_{n(i)}, x_k) &\leq d(x_{n(i)}, x_{k-1}) + d(x_{k-1}, x_k) \\ &\leq d(x_{n(i)}, x_{k-1}) + d(x_{k-1}, x_k) < \varepsilon + \frac{r}{2} + \frac{r}{4} = \varepsilon + \frac{3r}{4}. \end{aligned}$$

Now, we can choose a natural number  $k$  satisfying  $n(i) \leq k \leq n(i + 1)$  such that

$$\varepsilon + \frac{r}{2} \leq d(x_{n(i)}, x_k) < \varepsilon + \frac{3r}{4}. \tag{2.14}$$

Therefore, we obtain

$$d(x_{n(i)}, x_k) < \varepsilon + \frac{3r}{4} < \varepsilon + r, \tag{2.15}$$

$$d(x_{n(i)}, x_{n(i)+1}) = d_{n(i)} < \frac{r}{4} < \varepsilon + r, \tag{2.16}$$

and

$$d(x_k, x_{k+1}) = d_k < \frac{r}{4} < \varepsilon + r. \tag{2.17}$$

Thus, we have

$$\begin{aligned} &\frac{1}{2} [d(x_{n(i)}, x_{k+1}) + d(x_{n(i)+1}, x_k)] \\ &\leq \frac{1}{2} [d(x_{n(i)}, x_k) + d(x_k, x_{k+1}) + d(x_{n(i)+1}, x_{n(i)}) + d(x_{n(i)}, x_k)] \\ &\leq \frac{1}{2} [d(x_{n(i)}, x_k) + d(x_k, x_{k+1}) + d(x_{n(i)+1}, x_{n(i)}) + d(x_{n(i)}, x_k)] \\ &= d(x_{n(i)}, x_k) + \frac{1}{2} [s_k + s_{n(i)}] \\ &< \varepsilon + \frac{3r}{4} + \frac{1}{2} \left[ \frac{r}{4} + \frac{r}{4} \right] = \varepsilon + r. \end{aligned} \tag{2.18}$$

Now, the inequalities (2.15)-(2.18) imply that  $M(x_{n(i)}, x_k) < \varepsilon + r \leq \varepsilon + \delta$  and so  $\psi_{st}(M(x_{n(i)}, x_k)) < \psi_{st}(\varepsilon + \delta) \leq \psi_{st}(\varepsilon) + \psi_{st}(\delta)$  the fact that  $f$  is a generalized  $\alpha$ - $\psi_{st}$ -Meir-Keeler contractive mapping yields that,

$$d(x_{n(i)+1}, x_{k+1}) \leq \alpha(x_{n(i)}, x_k) \psi_{st}(d(x_{n(i)+1}, x_{k+1})) < \psi_{st}(\varepsilon).$$

Then  $d(x_{n(i)+1}, x_{k+1}) < \varepsilon$ . We deduce,

$$\begin{aligned} d(f^{n(i)} x_0, f^k x_0) &\leq d(f^{n(i)} x_0, f^{n(i)+1} x_0) + d(f^{n(i)+1} x_0, f^k x_0) \\ &\leq d(f^{n(i)} x_0, f^{n(i)+1} x_0) + d(f^{n(i)+1} x_0, f^k x_0) \\ &\leq d(f^{n(i)} x_0, f^{n(i)+1} x_0) + d(f^{n(i)+1} x_0, f^{k+1} x_0) + d(f^{k+1} x_0, f^k x_0). \end{aligned}$$

Hence, from this with (2.14), (2.16) and (2.17), we obtain

$$\begin{aligned} d(x_{n(i)+1}, x_{k+1}) &\geq d(x_{n(i)}, x_k) - d(x_{n(i)}, x_{n(i)+1}) - d(x_k, x_{k+1}) \\ &> \varepsilon + \frac{r}{2} - \frac{r}{4} - \frac{r}{4} = \varepsilon, \end{aligned}$$

which is a contradiction. We obtained that  $\lim_{m,n \rightarrow \infty} d(x_n, x_m) = 0$  and so  $\{x_n = f^n x_0\}$  is a Cauchy sequence. Since,  $X$  is complete, then there exists  $z \in X$  such that  $f^n x_0 \rightarrow z$  as  $n \rightarrow \infty$ . Now, since  $f$  is orbitally continuous, then  $z = fz$ .  $\square$

**Example 18** Let  $X = [0, \infty)$  and  $d(x, y) = |x - y|$  be a metric on  $X$ . Define  $f : X \rightarrow X$  by

$$fx = \begin{cases} \frac{x^2}{8}, & \text{if } x \in [0, 1], \\ \frac{1}{2} + \ln x, & \text{if } x \in (1, \infty) \end{cases}$$

and  $\psi_{st}(t) = \frac{1}{2}t$ ,

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Clearly,  $f$  is a triangular  $\alpha$ -admissible mapping and orbitally continuous. Let  $x, y \in [0, 1]$ . Without loss of generality, take  $x \leq y$ . Then

$$\begin{aligned} \psi_{st}(d(fx, fy)) &= \frac{y^2}{16} - \frac{x^2}{16}, \\ \psi_{st}(M(x, y)) &= \psi_{st}\left(\max\left\{y - x, x - \frac{x^2}{8}, y - \frac{y^2}{8}\right\}\right) = \max\left\{\frac{y}{2} - \frac{x}{2}, \frac{x}{2} - \frac{x^2}{16}, \frac{y}{2} - \frac{y^2}{16}\right\}. \end{aligned}$$

Clearly, by taking  $\delta = 3\varepsilon$  the condition (2.8) holds. Otherwise,  $\alpha(x, y) = 0$ . Hence, for given  $\varepsilon > 0$ , we have  $0 = \alpha(x, y)\psi_{st}(d(fx, fy)) < \varepsilon$ . Hence, condition of Theorem 17 is held and  $f$  has a fixed point.

**Theorem 19** Assume that all the hypotheses of Theorem 10 (11 and 17) hold. Adding the following conditions:

(iii) for all  $x \neq y \in X$  there exists  $v \in X$  such that  $\alpha(x, v) \geq 1$  and  $\alpha(v, y) \geq 1$ , we obtain the uniqueness of the fixed point of  $f$ .

*Proof* Suppose that  $z$  and  $z^*$  are two fixed points of  $f$  such that  $z \neq z^*$ . Then  $\alpha(z, v) \geq 1$  and  $\alpha(v, z^*) \geq 1$ . Hence, from (T2), we have  $\alpha(z, z^*) \geq 1$ . Now, by Remark 9, we get

$$d(z, z^*) = d(fz, fz^*) \leq \alpha(z, z^*)d(fz, fz^*) < d(z, z^*)$$

which is a contradiction and so  $z = z^*$ . Similarly, for Theorem 17, we can observe that  $f$  has a unique fixed point.  $\square$

We can obtain the following corollaries intermediately.

**Corollary 20** Let  $(X, d)$  be a complete metric space and  $f : X \rightarrow X$  is self-mapping. Suppose that for each  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\varepsilon \leq \psi(d(x, y)) < \varepsilon + \delta \quad \text{implies} \quad \psi(d(fx, fy)) < \frac{\varepsilon}{L},$$

where  $\psi \in \Psi$  and  $L \geq 1$ . Then  $f$  has a unique fixed points.

**Corollary 21** Let  $(X, d)$  be a complete metric space and  $f : X \rightarrow X$  a orbitally continuous self-mapping. Suppose that for each  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\varepsilon \leq \psi_{st}(M(x, y)) < \varepsilon + \delta \quad \text{implies} \quad \psi_{st}(d(fx, fy)) < \frac{\varepsilon}{L},$$

where  $\psi_{st} \in \Psi_{st}$ ,  $L \geq 1$  and

$$M(x, y) = \max \left\{ d(x, y), d(fx, x), d(fy, y), \frac{1}{2} [d(fx, y) + d(x, fy)] \right\}.$$

Then  $f$  has unique fixed points.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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