On a *q*-Extension of the Leibniz Rule via Weyl Type of *q*-Derivative Operator

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ABSTRACT. In the present paper we define a *q*-extension of the Leibniz rule for *q*-derivatives via Weyl type *q*-derivative operator. Expansions and summation formulae for the generalized basic hypergeometric functions of one and more variables are deduced as the applications of the main result.

1. Introduction

Agarwal [1], introduced the *q*-extension of the Leibniz rule for the fractional order *q*-derivative of a product of two basic functions in terms of a finite *q*-series involving fractional *q*-derivatives of the functions in the following manner:

$$(1) \qquad D_{z,q}^{\lambda} \{U(z)V(z)\} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (q^{-\lambda}; q)_n}{(q; q)_n}$$

$$D_{z,q}^{\lambda-n} \{U(zq^n)\} D_{z,q}^{n} \{V(z)\},$$

where $U(z)$ and $V(z)$ are two regular functions and the fractional *q*-differential operator $D_{z,q}^{\lambda}(.)$ of Reimann-Liouville type (see [1] and [2]) is given by

$$(2) \qquad D_{z,q}^{\lambda} \{f(z)\} = \frac{1}{\Gamma_q(-\lambda)} \int_0^z (z - tq)_{-\lambda-1} f(t) d(t; q),$$

$$(Re(\lambda) < 0; \ |q| < 1),$$

and

$$(3) \qquad (x - y)_{\nu} = x^{\nu} \prod_{n=0}^{\infty} \left[\frac{1 - (y/x)q^n}{1 - (y/x)q^{\nu+n}} \right].$$

Recently, in a series of papers [14] and [15], we have investigated certain applications of *q*-Leibniz rule given by (1) and deduced several interesting transformations

and expansions involving various basic hypergeometric functions of one and more variables including the basic analogue of Fox's H-function due to Saxena, Modi and Kalla [11]. Earlier Agarwal [1], Al-Salam and Verma [3], Denis [4] and Shukla [12] have applied the formula (1) to derive certain interesting transformations involving basic hypergeometric functions of one variable. Kim and Rim [8] investigate certain important properties of q-integrals involving q-series, and Kim [6] and [7] studied applications of q-derivative and introduced q-extensions of Euler and Genocchi numbers, and trigonometric functions.

We propose to add one more dimension to this study by introducing a q-extension of the Leibniz rule for q-derivatives by means of the Weyl type q-derivative operator. This approach will enable us for deriving the Weyl q-derivative of an integral power of z times a function $f(z)$ in terms of the Weyl type q-derivative of $f(z)$, secondly, this happens to be an important technique for deriving numerous transformations, expansions and summation formulae involving various basic hypergeometric functions of one and more variables.

Al-Salam [2] introduced the basic analogue of the Weyl fractional derivative operator as under:

$$(4) \qquad {}_zD^\mu_{\infty,q}\{f(z)\} = \frac{q^{-\mu(1+\mu)/2}}{\Gamma_q(-\mu)} \int_z^\infty (t-z)_{-\mu-1} f(tq^{1+\mu}) d(t;q),$$

where $Re(\mu) < 0$ and the basic integration cf. Gasper and Rahman [5], is defined as:

$$(5) \qquad \int_z^\infty f(t) d(t;q) = z(1-q) \sum_{k=1}^\infty q^{-k} f(zq^{-k}).$$

In view of the relation (5), the operator (4) can be expressed as:

$$(6) \qquad {}_zD^\mu_{\infty,q}\{f(z)\} = \frac{q^{\mu(1-\mu)/2} z^{-\mu}(1-q)}{\Gamma_q(-\mu)} \sum_{k=0}^\infty q^{\mu k}(1-q^{k+1})_{-\mu-1} f(zq^{\mu-k}),$$

$$(Re(\mu) < 0).$$

In particular, for $f(z) = z^{-p}$, the equation (6) reduces to

$$(7) \qquad {}_zD^\mu_{\infty,q}\{z^{-p}\} = \frac{\Gamma_q(p+\mu)}{\Gamma_q(p)} q^{-\mu p + \mu(1-\mu)/2} z^{-p-\mu}.$$

In the sequel, we shall also require the following definitions:
For real or complex a and $|q| < 1$, the q-shifted factorial is defined as:

$$(8) \qquad (a;q)_n = \begin{cases} 1 & ;\text{if} \quad n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & ;\text{if} \quad n \in N. \end{cases}$$

In terms of the q-gamma function, (8) can be expressed as

$$(9) \qquad (a;q)_n \;=\; \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \qquad n > 0,$$

where the q-gamma function (cf. Gasper and Rahman [5]), is given by

$$(10) \qquad \Gamma_q(a) \;=\; \frac{(q;q)_\infty}{(q^a;q)_\infty(1-q)^{a-1}},$$

$$(a \neq 0, -1, -2, \cdots).$$

The abnormal type of generalized basic hypergeometric series $_r\Phi_s(.)$ is defined as:

$$(11) \qquad {_r\Phi_s}\left[\begin{array}{c} a_1, \cdots, a_r; q; z \\ b_1, \cdots, b_s; q^\lambda \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(q, b_1, \cdots, b_s; q)_n} z^n \, q^{\lambda n(n+1)/2},$$

where $\lambda > 0$.

For $\lambda = 0$, series (11) reduces to the generalized basic hypergeometric series $_r\Phi_s(.)$ (cf. [13]), as under:

$$(12) \qquad {_r\Phi_s}\left[\begin{array}{cc} a_1, \cdots, a_r; & \\ & q, z \\ b_1, \cdots, b_s; & \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(q, b_1, \cdots, b_s; q)_n} z^n,$$

where for the convergence of the series (12), we have $|q| < 1$, for all z if $r \leq s$ and $|z| < 1$ if $r = s + 1$.

2. A q-extension of the Leibniz rule

Theorem. *Let $\Re$ be a simply connected region containing the point $z = 0$ and let ρ be the largest real number such that the domain $|z| < \rho$ is entirely contained in $\Re$. Suppose $U(z)$ and $V(z)$ are two regular basic functions. Then, for any non negative integer α*

$$(13) \qquad {_zD_{\infty,q}^\alpha}\left\{ U(z)V(z) \right\} = \sum_{r=0}^{\alpha} \frac{(-1)^r q^{r(r+1)/2}\, (q^{-\alpha};q)_r}{(q;q)_r}$$

$$_zD_{\infty,q}^{\alpha-r}\left\{ U(z) \right\} \; _zD_{\infty,q}^{r}\left\{ V(zq^{\alpha-r}) \right\}.$$

Proof. Consider $U(z) = z^{-p_1} f(z)$ and $V(z) = z^{-p_2} g(z)$, where $f(z)$ and $g(z)$ are both analytic functions within and on $\Re$, with power series representation within $|z| < \rho$ of $f(z)$ and $g(z)$ as

$$f(z) = \sum_{m=0}^{\infty} A_m \, z^m, \qquad g(z) = \sum_{n=0}^{\infty} B_n \, z^n.$$

The left-hand side, say L of the theorem (13) leads to

$$L \equiv {}_z D_{\infty,q}^{\alpha} \left\{ z^{-(p_1+p_2)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_m \, B_n \, z^{m+n} \right\}.$$

On interchanging the order of the q-derivative operator and summations, which is valid if $Re(p_1 + p_2) > -Re(\alpha) > 0$, $|z| < \rho$, and on using the result (7), the above expression reduces to

$$(14) \qquad L = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_m \, B_n \, \frac{\Gamma_q(p_1 + p_2 - m - n + \alpha)}{\Gamma_q(p_1 + p_2 - m - n)}$$

$$q^{-\alpha(p_1+p_2-m-n)+\alpha(1-\alpha)/2} \, z^{-(p_1+p_2-m-n+\alpha)}.$$

To evolute the right-hand side, of (13), we write

$$(15) \qquad R \equiv \sum_{r=0}^{\alpha} \frac{(-1)^r q^{r(r+1)/2} \, (q^{-\alpha};q)_r}{(q;q)_r} \, {}_z D_{\infty,q}^{\alpha-r} \left\{ \sum_{m=0}^{\infty} A_m \, z^{-(p_1-m)} \right\}$$

$$ {}_z D_{\infty,q}^{r} \left\{ \sum_{n=0}^{\infty} B_n \, z^{-(p_2-n)} q^{-(\alpha-r)(p_2-n)} \right\}.$$

On making use of the result (7) and (9), R can further be written as

$$(16) \qquad R = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_m \, B_n \, z^{-(p_1+p_2-m-n+\alpha)} \, q^{-\alpha(p_1+p_2-m-n)+\alpha(1-\alpha)/2}$$

$$\sum_{r=0}^{\alpha} \frac{(-1)^r \, (q^{-\alpha};q)_r \, (q^{p_1-m};q)_{\alpha-r}(q^{p_2-n};q)_r}{(q;q)_r \, (1-q)^{\alpha}} \, q^{r\alpha+r(p_1-m)+r(1-r)/2}.$$

In view of the q-identity (cf. Gasper and Rahman [5, I.10]), namely

$$(17) \qquad (a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k} \left(-\frac{q}{a} \right)^k q^{k(k-1)/2-nk},$$

relation (16) reduces to

$$(18) \qquad R = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_m \, B_n \, z^{-(p_1+p_2-m-n+\alpha)} \, q^{-\alpha(p_1+p_2-m-n)+\alpha(1-\alpha)/2}$$

$$\frac{(q^{p_1-m};q)_{\alpha}}{(1-q)^{\alpha}} \, {}_2\Phi_1 \left[\begin{array}{cc} q^{-\alpha}, q^{p_2-n}; & \\ & q, q \\ q^{1-\alpha-p_1+m}; & \end{array} \right].$$

On making use of the q-Vondermonde summation theorem, namely

$$(19) \qquad {}_2\Phi_1 \left[\begin{array}{c} q^{-n}, a; \\ \\ c; \end{array} \quad q, q \right] = \frac{(c/a; q)_n}{(c; q)_n}(a)^n,$$

the right hand side R, yields to

$$(20) \qquad R = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_m \ B_n \ z^{-(p_1+p_2-m-n+\alpha)} \ q^{-\alpha(p_1-m)+\alpha(1-\alpha)/2}$$

$$\frac{\Gamma_q(p_1 - m + \alpha) \ (q^{1-\alpha-p_1+m-p_2+n}; q)_\alpha}{\Gamma_q(p_1 - m)(q^{1-\alpha-p_1+m}; q)_\alpha},$$

which on further simplifications reduces equivalently to the value of left hand side (14). This verifies the theorem (13). □

3. Applications of the q-Leibniz rule

We shall now illustrate the applications of the q-Leibniz rule introduced by means of the equation (13) to derive a number of transformations, expansions and summation formulae involving various basic hypergeometric functions of one and more variables by assigning suitable values to the functions $U(z)$, $V(z)$, and α.

3.1. Expansion formulae

We shall establish the following expansion formulae involving the basic hypergeometric function ${}_r\Phi_s(.)$ as under:

$$(21) \qquad {}_{r+1}\Phi_{s+1} \left[\begin{array}{c} (a_r), q^{\lambda+\delta+\mu}; \ q; \rho/zq^\mu \\ (b_s), q^{\lambda+\delta} \ ; \ q^\sigma \end{array} \right] = \frac{1}{(q^{\lambda+\delta}; q)_\mu} \sum_{k=0}^{\mu} \frac{(-1)^k \ (q^{-\mu}; q)_k}{(q; q)_k}$$

$$(q^\delta; q)_k \ (q^\lambda; q)_{\mu-k} \ q^{k\lambda+k(k+1)/2} \ {}_{r+1}\Phi_{s+1} \left[\begin{array}{c} (a_r), q^{\delta+k}; \ q; \rho/zq^\mu \\ (b_s), q^\delta \ ; \ q^\sigma \end{array} \right],$$

where the symbol (a_r) denotes a sequence of r parameters $a_1, a_2, \cdots, a_r$.

$$(22) \qquad {}_{r+1}\Phi_{s+1} \left[\begin{array}{c} (a_r), q^{\lambda+\delta+\mu}; \\ \\ (b_s), q^{\lambda+\delta}; \end{array} \quad q, \rho/zq^\mu \right] = \frac{1}{(q^{\lambda+\delta}; q)_\mu} \sum_{k=0}^{\mu} \frac{(-1)^k \ (q^{-\mu}; q)_k}{(q; q)_k}$$

$$(q^\delta; q)_k \ (q^\lambda; q)_{\mu-k} \ q^{k\lambda+k(k+1)/2} \ {}_{r+1}\Phi_{s+1} \left[\begin{array}{c} (a_r), q^{\delta+k}; \\ \\ (b_s), q^\delta; \end{array} \quad q, \rho/zq^\mu \right],$$

provided that both the sides of the results (21) and (22) exists.

Proof of (21). To prove the result (21), we choose $U(z) = z^{-\lambda}$ and $V(z) = z^{-\delta} \, {}_r\Phi_s \left[\begin{array}{l} (a_r); \; q; \rho/z \\ (b_s); \; q^\sigma \end{array} \right]$, in the q-Leibniz rule (13), to obtain

$$(23) \qquad {}_zD^\mu_{\infty,q} \left\{ z^{-(\lambda+\delta)} \, {}_r\Phi_s \left[\begin{array}{l} (a_r); \; q; \rho/z \\ (b_s); \; q^\sigma \end{array} \right] \right\} = \sum_{k=0}^\mu \frac{(-1)^k q^{k(k+1)/2} \, (q^{-\mu}; q)_k}{(q; q)_k}$$

$$ {}_zD^{\mu-k}_{\infty,q} \left\{ z^{-\lambda} \right\} \; {}_zD^k_{\infty,q} \left\{ (zq^{\mu-k})^{-\delta} \, {}_r\Phi_s \left[\begin{array}{l} (a_r); \; q; \rho/zq^{\mu-k} \\ (b_s); \; q^\sigma \end{array} \right] \right\}.$$

In view of the definition (11), the left hand side, say L of (23) becomes

$$L \equiv \sum_{n=0}^\infty \frac{(a_1, \cdots, a_r; q)_n \, (\rho)^n}{(q, b_1, \cdots, b_s; q)_n} \, q^{\sigma n(n+1)/2} \; {}_zD^\mu_{\infty,q} \left\{ z^{-(\lambda+\delta+n)} \right\}.$$

On making use of formula (7), we obtain following Weyl type q-derivative formula for basic hypergeometric functions after certain simplifications

$$(24) \qquad L = \frac{\Gamma_q(\lambda+\delta+\mu)}{\Gamma_q(\lambda+\delta)} \, z^{-\lambda-\delta-\mu} q^{-\mu(\lambda+\delta)+\mu(1-\mu)/2}$$

$$ {}_{r+1}\Phi_{s+1} \left[\begin{array}{l} (a_r), q^{\lambda+\delta+\mu}; \; q; \rho/zq^\mu \\ (b_s), q^{\lambda+\delta} \quad ; q^\sigma \end{array} \right].$$

Further, if we put $\lambda = 0$, replace μ by k and then z by $zq^{\mu-k}$ in equation (23), we obtain yet another Weyl type q-derivative formula for ${}_r\Phi_s(.)$

$$(25) \qquad {}_zD^k_{\infty,q} \left\{ (zq^{\mu-k})^{-\delta} \, {}_r\Phi_s \left[\begin{array}{l} (a_r); \; q; \rho/zq^{\mu-k} \\ (b_s); \; q^\sigma \end{array} \right] \right\} = \frac{\Gamma_q(\delta+k)}{\Gamma_q(\delta)} \, z^{-\delta-k}$$

$$ q^{-\mu\delta+k(1+k)/2-\mu k} \, {}_{r+1}\Phi_{s+1} \left[\begin{array}{l} (a_r), q^{\delta+k}; \; q; \rho/zq^\mu \\ (b_s), q^\delta \; ; q^\sigma \end{array} \right].$$

Also we have from formula (7),

$$(26) \qquad {}_zD^{\mu-k}_{\infty,q} \left\{ z^{-\lambda} \right\} = \frac{\Gamma_q(\lambda+\mu-k)}{\Gamma_q(\lambda)} q^{-(\mu-k)\lambda+(\mu-k)(1-\mu+k)/2} \, z^{-\lambda-\mu+k}.$$

On substituting the values of the various expressions involved in the equation (23), from equations (24), (25) and (26), we arrive at the main result (21). The proof of the result (22) follows similarly when $\sigma = 0$.

3.2. Summation formulae

In this section, we shall establish summation formula for the basic Lauricella function $\Phi_D^{(n)}(.)$ as the application of the newly defined q-Leibniz rule (13). The main summation formula, runs as:

(27) $\qquad \Phi_D^{(m)} \left[q^{-n}, q^{b_1}, \cdots, q^{b_m}; q^c; q; q^{1+b_2+\cdots+b_m}, q^{1+b_3+\cdots+b_m}, \cdots, q^{1+b_m}, q \right]$

$$= \frac{(q^{c-b_1-b_2-\cdots-b_m}; q)_n}{(q^c; q)_n} q^{(b_1+b_2+\cdots+b_m)n}.$$

Proof. To prove the summation formula (27), we take $U(z) = z^{c+n-1}$, $V(z) = z^{-(b_1+\cdots+b_m)}$ and $\alpha = n$ in the q-Leibniz rule (13), we obtain

(28) $\qquad {}_z D_{\infty,q}^n \left\{ z^{c+n-b_1-\cdots-b_m-1} \right\} = \sum_{r=0}^{n} \frac{(-1)^r q^{r(r+1)/2} (q^{-n}; q)_r}{(q; q)_r}$

$$ {}_z D_{\infty,q}^{n-r} \left\{ z^{c+n-1} \right\} {}_z D_{\infty,q}^r \left\{ (zq^{n-r})^{-(b_1+\cdots+b_m)} \right\}.$$

On using the Weyl type q-derivative formula (7) for various expressions involving in the equation (28), we obtain

$$\frac{\Gamma_q(1+b_1+\cdots+b_m-c)\,\Gamma_q(1-c-n)}{\Gamma_q(1+b_1+\cdots+b_m-c-n)\,\Gamma_q(1-c)} = \sum_{r=0}^{n} \frac{(-1)^r\,(q^{-n}; q)_r}{(q; q)_r}$$

$$q^{r(1-c)+r(1-r)/2}\,\frac{\Gamma_q(1-c-r)\,\Gamma_q(1+b_1+\cdots+b_m+r)}{\Gamma_q(1-c)\,\Gamma_q(1+b_1+\cdots+b_m)},$$

which on further simplifications, yields to an interesting summation formula involving a ${}_2\Phi_1(.)$ series, namely

(29) $\qquad \dfrac{(q^{c-b_1-b_2-\cdots-b_m}; q)_n}{(q^c; q)_n} q^{(b_1+b_2+\cdots+b_m)n} = {}_2\Phi_1 \left[\begin{array}{cc} q^{-n}, q^{b_1+\cdots+b_m}; & \\ & q, q \\ q^c; & \end{array} \right].$

By making use of the q-multinomial theorem cf. Gasper and Rahmann ([5], p. 21), namely

(30) $\qquad (a_1 a_2 \cdots a_{m+1}; q)_n$

$$= \sum_{\substack{k_1, k_2, \cdots, k_m \geq 0, \\ n \geq k_1+k_2+\cdots+k_m}} \frac{(q; q)_n\, a_2^{k_1} a_3^{k_1+k_2} \cdots a_{m+1}^{k_1+k_2+\cdots+k_m}}{(q; q)_{k_1} (q; q)_{k_2} \cdots (q; q)_{k_m} (q; q)_{n-(k_1+k_2+\cdots+k_m)}}$$

$$(a_1; q)_{k_1} (a_2; q)_{k_2} \cdots (a_m; q)_{k_m} (a_{m+1}; q)_{n-(k_1+k_2+\cdots+k_m)},$$

where $m = 1, 2, \cdots$, $n = 0, 1, \cdots$, the equation (30) reduces to

$$(31) \quad \frac{(q^{c-b_1-b_2-\cdots-b_m}; q)_n}{(q^c; q)_n} q^{(b_1+b_2+\cdots+b_m)n} = \sum_{n \geq k_1+\cdots+k_{m-1}} \frac{(q^{-n}; q)_n \; q^n}{(q; q)_n (q^c; q)_n}$$

$$\sum_{k_1,\cdots,k_{m-1} \geq 0} \frac{(q; q)_n \; (q^{b_2})^{k_1} \; (q^{b_3})^{k_1+k_2} \cdots (q^{b_m})^{k_1+\cdots+k_{m-1}}}{(q; q)_{k_1} \cdots (q; q)_{k_{m-1}} (q; q)_{n-(k_1+\cdots+k_{m-1})}}$$

$$(q^{b_1}; q)_{k_1} \cdots (q^{b_{m-1}}; q)_{k_{m-1}} (q^{b_m}; q)_{n-(k_1+\cdots+k_{m-1})}.$$

In view of the definition of basic Lauricella function $\Phi_D^{(n)}(.)$, namely

$$(32) \qquad \Phi_D^{(n)} [a, b_1, \cdots, b_n; c; q; x_1, \cdots, x_n]$$

$$= \sum_{m_1,\cdots,m_n \geq 0} \frac{(a; q)_{m_1+\cdots+m_n}}{(c; q)_{m_1+\cdots+m_n}} \prod_{j=1}^{n} \left\{ \frac{(b_j; q)_{m_j} \; x_j^{m_j}}{(q; q)_{m_j}} \right\},$$

where for convergence $|x_1| < 1, \cdots, |x_n| < 1, |q| < 1$, the equation (31) yields to the summation formula (27).

If we put $m = 1$ in the summation formula (27), we get the following well known summation formula for basic hypergeometric functions $_2\Phi_1(.)$ as under:

$$(33) \qquad _2\Phi_1 \left[\begin{array}{cc} q^{-n}, q^{b_1}; & \\ & q, q \\ q^c; & \end{array} \right] = \frac{(q^{c-b_1}; q)_n}{(q^c; q)_n} q^{b_1 n},$$

which is well-known summation theorem (q-Vondermonde summation theorem) (see [5]).

Again, on putting $m = 2$, equation (27) reduces to terminating summation formulae for basic Appell function $\Phi^{(1)}(.)$ as under:

$$(34) \qquad \Phi^{(1)} \left[q^{-n}, q^{b_1}, q^{b_2}; q^c; q; q^{1+b_2}, q \right] = \frac{(q^{c-b_1-b_2}; q)_n}{(q^c; q)_n} q^{(b_1+b_2)n},$$

where the basic Appell function $\Phi^{(1)}(.)$ defined as:

$$(35) \qquad \Phi^{(1)} [a, b, b'; c; q; x, y] = \sum_{m,n \geq 0} \frac{(a; q)_{m+n}(b; q)_m(b'; q)_n}{(c; q)_{m+n}(q; q)_m(q; q)_n} x^m y^m,$$

$$(|x| < 1, \;\; |y| < 1; \;\; |q| < 1).$$

Further, it is interesting to observe that if we put $b_1 = \beta - \delta$, $b_2 = \alpha - \beta$, $c = \alpha + 1 - n - \gamma$ in result (34), we obtain a known summation formula due to Kumar, Saxena and Srivastava [9], namely

$$(36) \qquad \Phi^{(1)} \left[q^{-n}, q^{\beta-\delta}, q^{\alpha-\beta}; q^{\alpha+1-n-\gamma}; q; q^{1+\alpha-\beta}, q \right] = \frac{(q^{\gamma-\delta}; q)_n}{(q^{\gamma-\alpha}; q)_n}.$$

On putting $m = 3$, the formula (27) reduces to summation formula for basic hypergeometric function of three variables, namely

$$(37) \quad \Phi_D^{(3)} \left[q^{-n}, q^{b_1}, q^{b_2}, q^{b_3}; q^c; q; q^{1+b_2+b_3}, q^{1+b_3}, q \right] = \frac{(q^{c-b_1-b_2-b_3}; q)_n}{(q^c; q)_n} q^{(b_1+b_2+b_3)n},$$

where the basic hypergeometric function $\Phi_D^{(3)}(.)$ defined as:

$$(38) \quad \Phi_D^{(3)} [a, b, b', b''; c; q; x, y, z] = \sum_{m,n,p \geq 0} \frac{(a;q)_{m+n+p}(b;q)_m(b';q)_n(b'';q)_p}{(c;q)_{m+n+p}(q;q)_m(q;q)_n(q;q)_p} x^m y^m z^p.$$

Next, in view of the limit formula

$$(39) \quad \lim_{q \to 1^-} \Gamma_q(a) = \Gamma(a) \quad and \quad \lim_{q \to 1^-} \frac{(q^a; q)_n}{(1-q)^n} = (a)_n ,$$

where

$$(40) \quad (a)_n = a(a+1)\cdots(a+n-1).$$

one can deduce that the result (13) is the q-extension of the known Leibniz rule (via Weyl derivative) due to Miller and Ross [10].

Similarly if we take $q \to 1^-$ and using limit formula (39), the result (27) reduces to summation formula for the ordinary Lauricella function $F_D^{(n)}(.)$, namely

$$(41) \quad F_D^{(m)} [-n, b_1, \cdots, b_m; c; 1, 1, \cdots, 1] = \frac{(c - b_1 - b_2 - \cdots - b_m)_n}{(c)_n}.$$

As concluding remark, one can be observed that, the q-Leibniz rule investigated in the present communication, is an elegant technique for deriving numerous transformations, expansions and summation formulae involving various basic hypergeometric functions of one and more variables. The results thus derived in this paper are general in character and likely to find certain applications in the theory of basic hypergeometric functions.

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