# ON A QUESTION OF DAVENPORT AND LEWIS AND NEW CHARACTER SUM BOUNDS IN FINITE FIELDS

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Abstract.

Let $\chi$ be a nontrivial multiplicative character of $\mathbb{F}_{p^n}$. We obtain the following results.

(1). Let $\varepsilon > 0$ be given. If $B = \{\sum_{j=1}^n x_j\omega_j \ : x_j \in [N_j+1, N_j+H_j]\cap\mathbb{Z}, j = 1, \dots, n\}$ is a box satisfying $\prod\limits_{j=1}^n H_j > p^{(\frac{2}{5}+\varepsilon)n}$, then for $p > p(\varepsilon)$ and some absolute constant $c > 0$, we have, denoting $\chi$ a nontrivial multiplicative character

$$|\sum_{x\in B}\chi(x)| < cnp^{-\frac{\varepsilon^2}{4}}|B|$$

unless $n$ is even, $\chi$ is principal on a subfield $F_2$ of size $p^{n/2}$ and $\max_\xi |B\cap\xi F_2| > p^{-\varepsilon}|B|$.

(2). Assume $A, B \subset \mathbb{F}_p$ such that

$$|A| > p^{\frac{4}{9}+\varepsilon}, |B| > p^{\frac{4}{9}+\varepsilon}, |B+B| < K|B|.$$

Then

$$\Big|\sum_{x\in A, y\in B}\chi(x+y)\Big| < p^{-\tau}|A|\ |B|.$$

(3). Let $I \subset \mathbb{F}_p$ be an interval with $|I| = p^\beta$ and let $\mathcal{D} \subset \mathbb{F}_p$ be a $p^\beta$- spaced set with $|\mathcal{D}| = p^\sigma$. Assume $\beta > \frac{1}{4} - \frac{\sigma}{4(1-\sigma)} + \delta$. Then for a non-principal multiplicative character $\chi$

$$\Big|\sum_{x\in I, y\in\mathcal{D}}\chi(x+y)\Big| < p^{-\frac{\delta^2}{4}}|I|\ \ |\mathcal{D}|.$$

We also improve a result of Karacuba.

**Introduction.**

In this paper we obtain new character sum bounds in finite fields $\mathbb{F}_q$ with $q = p^n$, using methods from additive combinatorics related to the sum-product phenomenon. More precisely, Burgess' classical amplification argument is combined with our estimate on the 'multiplicative energy' for subsets in $\mathbb{F}_q$. (See Proposition 1 in §1.) The latter appears as a quantitative version of the sum-product theorem in finite fields (see [BKT] and [TV]) following arguments from [G], [KS1] and [KS2].

Our first results relate to the work [DL] of Davenport and Lewis. We recall their result. Let $\{\omega_1, \ldots, \omega_n\}$ be an arbitrary basis for $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$. Then elements of $\mathbb{F}_{p^n}$ have a unique representation as

$$\xi = x_1\omega_1 + \ldots + x_n\omega_n, \qquad (0 \le x_i < p). \tag{0.1}$$

We denote $B$ a box in $n$-dimensional space, defined by

$$N_j + 1 \le x_j \le N_j + H_j, \qquad (j = 1, \ldots, n) \tag{0.2}$$

where $N_j$ and $H_j$ are integers satisfying $0 \le N_j < N_j + H_j < p$, for all $j$.

**Theorem DL.** ([DL], Theorem 2) *Let $H_j = H$ for $j = 1, \ldots, n$, with*

$$H > p^{\frac{n}{2(n+1)}+\delta} \text{ for some } \delta > 0 \tag{0.3}$$

*and let $p > p_1(\delta)$. Then, with $B$ defined as above*

$$\Big|\sum_{x\in B} \chi(x)\Big| < (p^{-\delta_1} H)^n,$$

*where $\delta_1 = \delta_1(\delta) > 0$.*

For $n = 1$ (i.e. $\mathbb{F}_q = \mathbb{F}_p$) we are recovering Burgess' result ($H > p^{\frac{1}{4}+\delta}$). But as $n$ increases, the exponent in (0.3) tends to $\frac{1}{2}$. In fact, in [DL] the authors were quite aware of the shortcoming of their approach which they formulated as follows (see [DL], p130)

*'The reason for this weakening in the result lies in the fact that the parameter q used in Burgess' method has to be a rational integer and cannot (as far as we can see) be given values in $\mathbb{F}_q$'.*

In this paper we address to some extent their problem and are able to prove the following

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**Theorem 2*.** *Let $\chi$ be a nontrivial multiplicative character of $\mathbb{F}_{p^n}$, and let $\varepsilon > 0$ be given. If*

$$B = \left\{ \sum_{j=1}^{n} x_j \omega_j \ : x_j \in [N_j + 1, N_j + H_j] \cap \mathbb{Z}, j = 1, \dots, n \right\}$$

*is a box satisfying*

$$\prod_{j=1}^{n} H_j > p^{(\frac{2}{5}+\varepsilon)n},$$

*then for $p > p(\varepsilon)$ and some absolute constant $c$*

$$\Big| \sum_{x \in B} \chi(x) \Big| < cnp^{-\frac{\varepsilon^2}{4}} |B|,$$

*unless $n$ is even and $\chi|_{F_2}$ is principal, where $F_2$ is the subfield of size $p^{n/2}$, in which case*

$$\Big| \sum_{x \in B} \chi(x) \Big| \le \max_{\xi} \big| B \cap \xi F_2 \big| + cnp^{-\frac{\varepsilon^2}{4}} |B|.$$

Hence our exponent is uniform in $n$ and supersedes [DL] for $n > 4$. The novelty of the method in this paper is to exploit the finite field combinatorics without the need to reduce the problem to a divisor issue in $\mathbb{Z}$ or in the integers of an algebraic number field $K$ (as in the papers [Bu3] and [Kar2]).

Let us emphasize that there are no further assumptions on the basis $\omega_1, \dots, \omega_n$. If one assumes $\omega_i = g^{i-1}, (1 \le i \le n)$, where $g$ satisfies a given irreducible polynomial equation (mod $p$)

$$a_0 + a_1 g + \cdots + a_{n-1} g^{n-1} + g^n = 0, \text{ with } a_i \in \mathbb{Z},$$

or more generally, if

$$\omega_i \omega_j = \sum_{k=1}^{n} c_{ijk} \omega_k, \tag{0.4}$$

with $c_{ijk}$ bounded and $p$ taken large enough, a result of the strength of Burgess' theorem was indeed obtained (see [Bu3] and [Kar2]) by reducing the problem of bounding the multiplicative energy in the finite field to counting divisors in the ring of integers

*The author is grateful to Andrew Granville for removing an additional restriction on the set $B$ from an earlier version of this theorem.

of an appropriate number field. But such reduction seems not possible in the general context considered in [DL].

Character estimates as considered above have many applications, e.g. quadratic non-residues, primitive roots, coding theory, etc. Corollary 3 in §2 is a standard consequence of Theorem 2 to the problem of primitive roots (see for instance [DL], p131).

The aim of [DL] (and in an extensive list of other works starting from Burgess' seminal paper [Bu1]) was to improve on the Polya-Vinogradov estimate (i.e. breaking the $\sqrt{q}$-barrier), when considering incomplete character sums of the form

$$\Big|\sum_{x\in A}\chi(x)\Big|, \tag{0.5}$$

where $A\subset\mathbb{F}_q$ has certain additive structure.

Note that the set $B$ considered above has a small doubling set, i.e.

$$|B+B|<c(n)|B| \tag{0.6}$$

and this is the property relevant to us in our combinatorial Proposition 1 in §1.

In the case of a prime field ($q=p$), our method provides the following generalization of Burgess' inequality.

**Theorem 4.** *Let $\mathcal{P}$ be a proper $d$-dimensional generalized arithmetic progression in $\mathbb{F}_p$ with*

$$|\mathcal{P}|>p^{2/5+\varepsilon}$$

*for some $\varepsilon>0$. If $\mathcal{X}$ is a non-principal multiplicative character of $\mathbb{F}_p$, we have*

$$\Big|\sum_{x\in\mathcal{P}}\mathcal{X}(x)\Big|<p^{-\tau}|\mathcal{P}|$$

*where $\tau=\tau(\varepsilon,d)>0$ and assuming $p>p(\varepsilon,d)$.*

See §4, where we also recall the notion of a 'proper generalized arithmetic progression'. Let us point out here that the proof of Proposition 1 below and hence Theorem 2, uses the full linear independence of the elements $\omega_1,\dots,\omega_n$ over the base field $\mathbb{F}_p$. Assuming in Theorem 2 only that $B$ is a proper generalized arithmetic progression requires us to make more restrictive assumptions on the size $|B|$.

Next, we consider the problem of estimating character sums over sumsets of the form

$$\sum_{x\in A, y\in B} \chi(x+y), \tag{0.7}$$

where $\chi$ is a non-principal multiplicative character modulo $p$ (we consider again only the prime field case for simplicity). In this situation, a well-known conjecture* predicts a nontrivial bound on (0.7) as soon as $|A|, |B| > p^\delta$, for some $\delta > 0$. (See [C] and [S] p.305.) Presently, such a result is only known (with no further assumptions) provided $|A| > p^{\frac{1}{2}+\delta}$ and $|B| > p^\delta$ for some $\delta > 0$. (See [Kar1].) The problem is open even for the case $|A| \sim p^{\frac{1}{2}} \sim |B|$. Using Proposition 1 (combined with Freiman's theorem), we prove the following result.

**Theorem 6.** *Assume $A, B \subset \mathbb{F}_p$ such that*

(a) $|A| > p^{\frac{4}{9}+\varepsilon}, |B| > p^{\frac{4}{9}+\varepsilon}$

(b) $|B+B| < K|B|$.

*Then*

$$\Big|\sum_{x\in A, y\in B} \chi(x+y)\Big| < p^{-\tau}|A|\,|B|,$$

*where $\tau = \tau(\varepsilon, K) > 0$, $p > p(\varepsilon, K)$ and $\chi$ is a non-principal multiplicative character of $\mathbb{F}_p$.*

Assuming $B = I$ an interval, we obtain the next estimate.

**Theorem 8.** *Let $A \subset \mathbb{F}_p$ be a subset with $|A| = p^\alpha$ and let $I \subset [1, p]$ be an arbitrary interval with $|I| = p^\beta$, where*

$$(1-\alpha)(1-\beta) < \frac{1}{2} - \delta$$

*and $\beta > \delta > 0$. Then for a non-principal multiplicative character $\chi$, we have*

$$\Big|\sum_{\substack{x\in I\\ y\in A}} \chi(x+y)\Big| < p^{-\frac{\delta^2}{13}}|A|\,|I|.$$

The following variant of Theorem 8 may be compared with Theorem 2' in [FI]. (See the discussion in §4.)

*This conjecture was partly motivated by the 'Paley-Graph conjecture' on the maximal size of a set $C \subset \mathbb{F}_p$ such that $x - y$ is a quadratic residue (mod $p$) for all $x, y \in C$.

**Theorem 9.** *Let $I \subset \mathbb{F}_p$ be an interval with $|I| = p^\beta$ and let $\mathcal{D} \subset \mathbb{F}_p$ be a $p^\beta$-spaced set modulo $p$ with $|\mathcal{D}| = p^\sigma$. Assume $\beta > \sigma$ and*

$$(1-2\beta)(1-\sigma) < \frac{1}{2} - \delta \tag{0.8}$$

*for some $\delta > 0$. Then*

$$\Big| \sum_{x \in I, y \in \mathcal{D}} \chi(x+y) \Big| < p^{-\frac{\delta^2}{17}} |I| \cdot |\mathcal{D}| \tag{0.9}$$

*for a non-principal multiplicative character $\chi$.*

Rewriting (0.8) as $\beta > \frac{1}{4} - \frac{\sigma}{4(1-\sigma)}$, we note that Theorem 9 breaks Burgess' $\frac{1}{4}$-threshold as soon as $\sigma > 0$.

The next result is a slight improvement of Karacuba's [Kar1].

**Theorem 10.** *Let $I \subset [1, p]$ be an interval with $|I| = p^\beta$ and $S \subset [1, p]$ be an arbitrary set with $|S| = p^\alpha$. Assume that $\alpha, \beta$ satisfy*

$$\varepsilon < \beta \leq \frac{1}{k} \text{ and } \Big(1 - \frac{2}{3k}\Big)\alpha + \frac{2}{3}\Big(1 + \frac{2}{k}\Big)\beta > \frac{1}{2} + \frac{1}{3k} + \varepsilon$$

*for some $\varepsilon > 0$ and $k \in \mathbb{Z}_+$. Then*

$$\sum_{y \in I} \Big| \sum_{x \in S} \chi(x+y) \Big| < p^{-\varepsilon'} |I| \ |S|$$

*for some $\varepsilon' = \varepsilon'(\varepsilon) > 0$.*

We believe that this is the first paper exploring the application of recent developments in combinatorial number theory (for which we especially refer to [TV]) to the problem of estimating (multiplicative) character sums. (Those developments have been particularly significant in the context of exponential sums with additive characters. See [BGK] and subsequent papers.) One could clearly foresee more investigations along these lines.

The paper is organized as follows. We prove Proposition 1 in §1, Theorem 2 in §2, Theorems 6 in §3, and Theorems 8, 9, 10 in §4.

**Notations.** Let * be a binary operation on some ambient set $S$ and let $A, B$ be subsets of $S$. Then

(1) $A * B := \{a * b \,:\, a \in A \text{ and } b \in B\}$.

(2) $a * B := \{a\} * B$.

(3) $AB := A * B$, if *=multiplication.

(4) $A^n := AA^{n-1}$.

Note that we use $A^n$ for both the $n$-fold product set and $n$-fold Cartesian product when there is no ambiguity.

(5) $[a, b] := \{i \in \mathbb{Z} : a \le i \le b\}$.

## §1. Multiplicative energy of a box.

Let $A, B$ be subsets of a commutative ring. Recall that the multiplicative energy of $A$ and $B$ is

$$E(A, B) = \Big|\Big\{(a_1, a_2, b_1, b_2) \in A \times A \times B \times B : a_1 b_1 = a_2 b_2\Big\}\Big|. \tag{1.1}$$

(See [TV] p.61.)

We will use the following (see [TV] Corollary 2.10)

**Fact 1.** $E(A, B) \le E(A, A)^{1/2} E(B, B)^{1/2}$.

**Proposition 1.** *Let $\{\omega_1, \dots, \omega_n\}$ be a basis for $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$ and let $B \subset \mathbb{F}_{p^n}$ be the box*

$$B = \left\{ \sum_{j=1}^{n} x_j \omega_j \ : x_j \in [N_j + 1, N_j + H_j], j = 1, \dots, n \right\},$$

*where $1 \le N_j < N_j + H_j < p$ for all $j$. Assume that*

$$\max_j H_j < \frac{1}{2}(\sqrt{p} - 1) \tag{1.2}$$

*Then we have*

$$E(B, B) < C^n (\log\ p)\, |B|^{11/4} \tag{1.3}$$

*for an absolute constant $C < 2^{\frac{9}{4}}$.*

The argument is an adaptation of [G] and [KS1] with the aid of a result in [KS2]. The structure of $B$ allows us to carry out the argument directly from [KS1] leading to the same statement as for the case $n = 1$.

We will use the following estimates from [KS1] (Corollaries 1.4-1.6). (See also [G].)

Let $X, B_1, \cdots, B_k$ be subsets of a commutative ring and $a, b \in X$. Then

**Fact 2.** $|B_1 + \cdots + B_k| \le \frac{|X+B_1|\cdots|X+B_k|}{|X|^{k-1}}$.

**Fact 3.** $\exists X' \subset X$ with $|X'| > \frac{1}{2}|X|$ and $|X' + B_1 + \cdots + B_k| \le 2^k \frac{|X+B_1|\cdots|X+B_k|}{|X|^{k-1}}$.

**Fact 4.** $|aX \pm bX| \le \frac{|X+X|^2}{|aX \cap bX|}$.

*Proof of Proposition 1.*

*Claim 1.* $\mathbb{F}_p \not\subset \frac{B-B}{B-B}$.

*Proof of Claim 1.* Take $t \in \mathbb{F}_p \cap \frac{B-B}{B-B}$. Then $t\Sigma x_j\omega_j = \Sigma y_j\omega_j$ for some $x_j, y_j \in [-H_j, H_j]$, where $1 \le j \le n$ and $\Sigma x_j\omega_j \neq 0$. Since $tx_j = y_j$ for all $j = 1, \dots, n$, choosing $i$ such that $x_i \neq 0$, it follows that

$$t \in \frac{[-H_i, H_i]}{[-H_i, H_i]\backslash\{0\}} \subset \frac{[-\frac{1}{2}(\sqrt{p}-1), \frac{1}{2}(\sqrt{p}-1)]}{[-\frac{1}{2}(\sqrt{p}-1), \frac{1}{2}(\sqrt{p}-1)]\backslash\{0\}}. \tag{1.4}$$

Since the set (1.4) is of size at most $\sqrt{p}(\sqrt{p}-1) < p$, it cannot contain $\mathbb{F}_p$. This proves our claim.

We may now repeat verbatim the argument in [KS1], with the additional input of the multiplicative energy.

*Claim 2. There exist $b_0 \in B$, $A_1 \subset B$ and $N \in \mathbb{Z}_+$ such that*

$$|aB \cap b_0B| \sim N \text{ for all } a \in A_1, \tag{1.5}$$

$$N\,|A_1| > \frac{E(B,B)}{|B|\log|B|} \tag{1.6}$$

*and*

$$\frac{A_1 - A_1}{A_1 - A_1} + 1 \neq \frac{A_1 - A_1}{A_1 - A_1}. \tag{1.7}$$

*Proof of Claim 2.*

From (1.1)

$$E(B,B) = \sum_{a,b\,\in B} |aB \cap bB|.$$

Therefore, there exists $b_0 \in B$ such that

$$\sum_{a\in B} |aB \cap b_0B| \ge \frac{E(B,B)}{|B|}.$$

Let $A_s$ be the level set

$$A_s = \{a \in B : 2^{s-1} \leq |aB \cap b_0 B| < 2^s\}.$$

Then for some $s_0$ with $1 \leq s_0 \leq \log_2 |B|$ we have

$$2^{s_0}\ |A_{s_0}|\ \log_2 |B| \geq \sum_{s=0}^{\log_2 |B|} 2^s |A_s| > \sum_{a \in B} |aB \cap b_0 B| \geq \frac{E(B,B)}{|B|}.$$

(1.5) and (1.6) are obtained by taking $A_1 = A_{s_0}$ and $N = 2^{s_0}$.

Next we prove (1.7) by assuming the contrary. By iterating $t$ times, we would have

$$\frac{A_1 - A_1}{A_1 - A_1} + t = \frac{A_1 - A_1}{A_1 - A_1} \quad \text{for} \quad t = 0, 1, \dots, p-1. \tag{1.8}$$

Since $0 \in \frac{A_1 - A_1}{A_1 - A_1}$, (1.8) would imply that $\mathbb{F}_p \subset \frac{A_1 - A_1}{A_1 - A_1} \subset \frac{B-B}{B-B}$, contradicting Claim 1. Hence (1.7) holds.

Take $c_1, c_2, d_1, d_2 \in A_1, d_1 \neq d_2$, such that

$$\xi = \frac{c_1 - c_2}{d_1 - d_2} + 1 \not\subset \frac{A_1 - A_1}{A_1 - A_1}.$$

It follows that for any subset $A' \subset A_1$, we have

$$\begin{aligned} |A'|^2 &= |A' + \xi A'| = |(d_1 - d_2)A' + (d_1 - d_2)A' + (c_1 - c_2)A'| \\ &\leq |(d_1 - d_2)A' + (d_1 - d_2)A_1 + (c_1 - c_2)A_1|. \end{aligned} \tag{1.9}$$

In Fact 3, we take $X = (d_1 - d_2)A_1$, $B_1 = (d_1 - d_2)A_1$ and $B_2 = (c_1 - c_2)A_1$. Then there exists $A' \subset A_1$ with $|A'| = \frac{1}{2}|A_1|$ and by (1.9)

$$\begin{aligned} |A'|^2 &\leq |(d_1 - d_2)A' + (d_1 - d_2)A_1 + (c_1 - c_2)A_1| \\ &\leq \frac{2^2}{|A_1|}|A_1 + A_1|\ |\ (d_1 - d_2)A_1 + (c_1 - c_2)A_1|. \end{aligned} \tag{1.10}$$

Since $|A_1 + A_1| \leq |B + B| \leq 2^n |B|$,

$$\begin{aligned} 2^{-2}|A_1|^3 &\leq 2^{n+2}|B|\ |\ (d_1 - d_2)A_1 + (c_1 - c_2)A_1| \\ &\leq 2^{n+2}|B|\ |\ c_1 B - c_2 B + d_1 B - d_2 B|. \end{aligned} \tag{1.11}$$

Facts 2, 4 and (1.5) imply

$$2^{-2}|A_1|^3 \leq 2^{n+2}|B|\frac{|B+B|^8}{N^4\ |B|^3}. \tag{1.12}$$

Thus

$$N^4|A_1|^3 \leq 2^{9n+4}|B|^6 \tag{1.13}$$

and recalling (1.6)

$$E(B,B)^4 \leq (\log|B|)^4|B|^5N^4|A_1|^3 < 2^{9n+4}(\log\ p)^4|B|^{11}$$

implying (1.3). $\square$

## §2. Burgess' method and the proof of Theorem 2.

The goal of this section is to prove the theorem below.

**Theorem 2.** *Let $\chi$ be a non-principal multiplicative character of $\mathbb{F}_{p^n}$. Given $\varepsilon > 0$, there is $\tau > \frac{\varepsilon^2}{4}$ such that if*

$$B = \left\{ \sum_{j=1}^n x_j\omega_j \ : x_j \in [N_j+1, N_j+H_j] \cap \mathbb{Z}, j=1,\dots,n \right\}$$

*is a box satisfying*

$$\prod_{j=1}^n H_j > p^{(\frac{2}{5}+\varepsilon)n},$$

*then for $p > p(\varepsilon)$ and some absolute constant $c$*

$$\Big|\sum_{x\in B}\chi(x)\Big| < cnp^{-\tau}|B|,$$

*unless $n$ is even and $\chi|_{F_2}$ is principal, where $F_2$ is the subfield of size $p^{n/2}$, in which case*

$$\Big|\sum_{x\in B}\chi(x)\Big| \leq \max_{\xi}\big|B\cap\xi F_2\big| + cnp^{-\tau}|B|.$$

First we will prove a special case of Theorem 2, assuming some further restriction on the box $B$.

**Theorem 2'.** *Let $\chi$ be a non-principal multiplicative character of $\mathbb{F}_{p^n}$. Given $\varepsilon > 0$, there is $\tau > \frac{\varepsilon^2}{4}$ such that if*

$$B = \Big\{ \sum_{j=1}^{n} x_j \omega_j \ : x_j \in [N_j + 1, N_j + H_j], j = 1, \dots, n \Big\}$$

*is a box satisfying*

$$\prod_{j=1}^{n} H_j > p^{(\frac{2}{5}+\varepsilon)n}$$

*and also*

$$H_j < \frac{1}{2}(\sqrt{p} - 1) \text{ for all } j, \tag{2.1}$$

*then for $p > p(\varepsilon)$*

$$\Big| \sum_{x \in B} \chi(x) \Big| < cnp^{-\tau} |B|. \tag{2.2}$$

We will need the following version of Weil's bound on exponential sums. (See Theorem 11.23 in [IK])

**Theorem W.** *Let $\chi$ be a non-principal multiplicative character of $\mathbb{F}_{p^n}$ of order $d > 1$. Suppose $f \in \mathbb{F}_{p^n}[x]$ has $m$ distinct roots and $f$ is not a $d$-th power. Then for $n \geq 1$ we have*

$$\Big| \sum_{x \in \mathbb{F}_{p^n}} \chi(f(x)) \Big| \leq (m-1) p^{\frac{n}{2}}.$$

*Proof of Theorem 2'.*

By breaking up $B$ in smaller boxes, we may assume

$$\prod_{j=1}^{n} H_j \sim p^{(\frac{2}{5}+\varepsilon)n}. \tag{2.3}$$

Let $\delta > 0$ be specified later. Let

$$I = [1, p^{\delta}] \tag{2.4}$$

and

$$B_0 = \Big\{ \sum_{j=1}^{n} x_j \omega_j \ : x_j \in [0, p^{-2\delta} H_j], j = 1, \dots, n \Big\}. \tag{2.5}$$

Since $B_0 I \subset \left\{ \sum_{j=1}^n x_j\omega_j \ : x_j \in [0, p^{-\delta} H_j], j = 1, \dots, n\right\}$, clearly

$$\Big|\sum_{x\in B}\chi(x) - \sum_{x\in B}\chi(x+yz)\Big| < |B\backslash(B+yz)| + |(B+yz)\backslash B| < 2np^{-\delta}|B|$$

for $y \in B_0, z \in I$. Hence

$$\sum_{x\in B}\chi(x) = \frac{1}{|B_0|\,|I|}\sum_{x\in B, y\in B_0, z\in I}\chi(x+yz) + O\big(np^{-\delta}|B|\big). \tag{2.6}$$

*Estimate following Burgess' method

$$\begin{aligned}\Big|\sum_{x\in B, y\in B_0, z\in I}\chi(x+yz)\Big| &\le \sum_{x\in B, y\in B_0}\big|\sum_{z\in I}\chi(x+yz)\big|\\ &= \sum_{x\in B, y\in B_0}\big|\sum_{z\in I}\chi(xy^{-1}+z)\big|\\ &= \sum_{u\in\mathbb{F}_{p^n}} w(u)\,\big|\sum_{z\in I}\chi(u+z)\big|,\end{aligned} \tag{2.7}$$

where

$$\omega(u) = \Big|\Big\{(x,y)\in B\times B_0 : \frac{x}{y} = u\Big\}\Big|. \tag{2.8}$$

Next, observe that

$$\begin{aligned}\sum_{u\in\mathbb{F}_{p^n}}\omega(u)^2 &= |\{(x_1,x_2,y_1,y_2)\in B\times B\times B_0\times B_0 : x_1y_2 = x_2y_1\}|\\ &= \sum_\nu \big|\{(x_1,x_2) : \frac{x_1}{x_2} = \nu\}\big|\ \big|\{(y_1,y_2) : \frac{y_1}{y_2} = \nu\}\big|\\ &\le E(B,B)^{\frac12}E(B_0,B_0)^{\frac12}\\ &< 2^{\frac94 n+1}(\log\ p)|B|^{\frac{11}{8}}|B_0|^{\frac{11}{8}}\\ &< 2^{\frac94 n+1}(\log\ p)\Big(|B|\Big)^{\frac{11}{4}}p^{-\frac{11}{4}n\delta},\end{aligned} \tag{2.9}$$

by the Cauchy-Schwarz inequality, Proposition 1 and (2.5). Compared with Burgess' argument (where $\omega(u) < p^{o(1)}$), obtaining good bounds on $\sum_u \omega(u)^2$ in our setting is considerably harder and (2.9), based on Proposition 1 is the main new ingredient.

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*This initial step of translation by a product is by now standard and was first used in [Kar2] in the context of character sums.

Let $r$ be the nearest integer to $\frac{n}{\varepsilon}$. Hence

$$\left|r - \frac{n}{\varepsilon}\right| \leq \frac{1}{2}. \tag{2.10}$$

By Hölder's inequality, (2.7) is bounded by

$$\Big(\sum_{u\in\mathbb{F}_{p^n}} \omega(u)^{\frac{2r}{2r-1}}\Big)^{1-\frac{1}{2r}}\Big(\sum_{u\in\mathbb{F}_{p^n}} \big|\sum_{z\in I}\chi(u+z)\big|^{2r}\Big)^{\frac{1}{2r}}. \tag{2.11}$$

Since $\sum_u \omega(u) = |B_0|\cdot|B|$ and (2.9) holds, we have

$$\begin{aligned}\Big(\sum_u \omega(u)^{\frac{2r}{2r-1}}\Big)^{1-\frac{1}{2r}} \leq & \Big[\sum \omega(u)\Big]^{1-\frac{1}{r}}\Big[\sum\omega(u)^2\Big]^{\frac{1}{2r}}\\ < & 2^{(\frac{9}{4}n+1)\frac{1}{2r}}\Big(|B_0|\cdot|B|\Big)^{1-\frac{1}{r}}\Big(|B|\Big)^{\frac{11}{8r}}(\log\ p)\, p^{-\frac{11}{8}\frac{n}{r}\delta}.\end{aligned} \tag{2.12}$$

The first inequality follows from the following fact, which is proved by using Hölder's inequality with $\frac{2r-2}{2r-1}+\frac{1}{2r-1}=1$.

**Fact 5.** $(\sum_u f(u)^{\frac{2r}{2r-1}})^{1-\frac{1}{2r}} \leq [\sum f(u)]^{1-\frac{1}{r}}[\sum f(u)^2]^{\frac{1}{2r}}$.

*Proof.* Write $f(u)^{\frac{2r}{2r-1}} = f(u)^{\frac{2r-2}{2r-1}} f(u)^{\frac{2}{2r-1}}$. $\square$

Next, we bound the second factor of (2.11).

Let

$$q = p^n.$$

Write

$$\sum_{u\in\mathbb{F}_{p^n}} |\sum_{z\in I}\chi(u+z)|^{2r} \leq \sum_{z_1,\dots,z_{2r}\in I} |\sum_{u\in\mathbb{F}_q}\chi((u+z_1)\dots(u+z_r)(u+z_{r+1})^{q-2}\dots(u+z_{2r})^{q-2})|. \tag{2.13}$$

For $z_1,\dots,z_{2r}\in I$ such that at least one of the elements is not repeated twice, the polynomial $f_{z_1,\dots,z_{2r}}(x) = (x+z_1)\dots(x+z_r)(x+z_{r+1})^{q-2}\dots(x+z_{2r})^{q-2}$ clearly cannot be a $d$-th power. Since $f_{z_1,\dots,z_{2r}}(x)$ has no more that $2r$ many distinct roots, Theorem W gives

$$\Big|\sum_{u\in\mathbb{F}_q}\chi((u+z_1)\dots(u+z_r)(u+z_{r+1})^{q-2}\dots(u+z_{2r})^{q-2})\Big| < 2rp^{\frac{n}{2}}. \tag{2.14}$$

For those $z_1, \dots, z_{2r} \in I$ such that every root of $f_{z_1,\dots,z_{2r}}(x)$ appears at least twice, we bound $\sum |\sum_{u\in\mathbb{F}_q} \chi(f_{z_1,\dots,z_{2r}}(u))|$ by $|\mathbb{F}_q|$ times the number of such $z_1, \dots, z_{2r}$. Since there are at most $r$ roots in $I$ and for each $z_1, \dots, z_{2r}$ there are at most $r$ choices, we obtain a bound $|I|^r r^{2r} p^n$.

Therefore

$$\sum_{u\in\mathbb{F}_{p^n}} \Big|\sum_{z\in I}\chi(u+z)\Big|^{2r} < |I|^r r^{2r} p^n + 2r|I|^{2r} p^{\frac{n}{2}} \tag{2.15}$$

and

$$\Big(\sum_{u\in\mathbb{F}_{p^n}} \Big|\sum_{z\in I}\chi(u+z)\Big|^{2r}\Big)^{\frac{1}{2r}} \le r|I|^{\frac{1}{2}} p^{\frac{n}{2r}} + 2|I|p^{\frac{n}{4r}}. \tag{2.16}$$

Putting (2.7), (2.11), (2.12) and (2.16) together, we have

$$\begin{aligned}
&\frac{1}{|B_0|\,|I|} \sum_{x\in B, y\in B_0, z\in I} \chi(x+yz) \\
<&4^{\frac{n}{r}}(\log\ p)\Big(|B_0|\,|B|\Big)^{-\frac{1}{r}}\Big(|B|\Big)^{1+\frac{11}{8r}} p^{-\frac{11}{8}\frac{n}{r}\delta}\Big(r|I|^{-\frac{1}{2}}p^{\frac{n}{2r}} + 2p^{\frac{n}{4r}}\Big) \\
<&4^{\frac{n}{r}}(\log\ p)\, p^{\frac{1}{r}2n\delta - \frac{11}{8}\frac{n}{r}\delta}\Big(|B|\Big)^{1-\frac{5}{8r}}\Big(rp^{\frac{-\delta}{2}}p^{\frac{n}{2r}} + 2p^{\frac{n}{4r}}\Big) \\
<&4^{\frac{n}{r}}(\log\ p)\, 2rp^{\frac{n}{4r}+2\delta\frac{n}{r}-\frac{5}{8r}(\frac{2}{5}+\varepsilon)n}|B| \\
<&2\cdot 4^{\frac{n}{r}}(\log p)\, r|B|p^{-\frac{5}{8}\frac{n}{r}(\varepsilon-\delta)}.
\end{aligned} \tag{2.17}$$

The second to the last inequality holds because of (2.3) and assuming $\delta \ge n/2r$.

Let

$$\delta = \frac{n}{2r}. \tag{2.18}$$

To bound the exponent $\frac{5}{8}\frac{n}{r}(\varepsilon-\delta) = \frac{5}{16}\varepsilon^2\frac{n}{r\varepsilon}(2-\frac{n}{r\varepsilon})$, we let

$$\theta = \frac{n}{\varepsilon r} - 1. \tag{2.19}$$

Then by (2.10),

$$|\theta| < \frac{1}{2r} < \frac{\varepsilon}{2n-\varepsilon} < \frac{3}{(10n-3)} \le \frac{3}{7} \tag{2.20}$$

and

$$\frac{5}{8}\frac{n}{r}(\varepsilon-\delta) = \frac{5}{16}\varepsilon^2(1+\theta)(1-\theta) > \frac{25}{98}\varepsilon^2. \tag{2.21}$$

Returning to (2.6), we have

$$\Big|\sum_{x\in B}\chi(x)\Big| < cn\varepsilon^{-1}(\log\ p)p^{-\frac{25}{98}\varepsilon^2}|B| < np^{-\frac{\varepsilon^2}{4}}|B| \tag{2.22}$$

and thus proves Theorem 2'. $\square$

Our next aim is to remove the additional hypothesis (2.1) on the shape of $B$. We proceed in several steps and rely essentially on a further key ingredient provided by the following estimate. (See [PS].)

**Proposition ♣*.** *Let $\chi$ be a non-principal multiplicative character of $\mathbb{F}_q$ and let $g \in \mathbb{F}_q$ be a generating element, i.e. $\mathbb{F}_q = \mathbb{F}_p(g)$. For any integral interval $I \subset [1, p]$,*

$$\big|\sum_{t\in I}\chi(g+t)\big| \leq cn\sqrt{p}\ \log p \tag{2.23}$$

Note that (2.23) is nontrivial as soon as $|I| \gg \sqrt{p}\ \log p$.

First we make the following observation (extending slightly the range of the applicability of Theorem 2').

Let $H_1 \geq H_2 \geq \cdots \geq H_n$. If $H_1 \leq p^{\frac{1}{2}+\frac{\varepsilon}{2}}$, we may clearly write $B$ as a disjoint union of boxes $B_\alpha \subset B$ satisfying the first condition in (2.1) and $|B_\alpha| > (\frac{1}{2}p^{-\frac{\varepsilon}{2}})^n|B| > 2^{-n}p^{(\frac{2}{5}+\frac{\varepsilon}{2})n}$. Since (2.1) holds for each $B_\alpha$, we have

$$\Big|\sum_{x\in B_\alpha}\chi(x)\Big| < cnp^{-\tau}|B_\alpha|.$$

Hence

$$\Big|\sum_{x\in B}\chi(x)\Big| < cnp^{-\tau}|B|.$$

Therefore we may assume that $H_1 > p^{\frac{1}{2}+\frac{\varepsilon}{2}}$.

*Proof of Theorem 2.*

*Case 1. $n$ is odd.*

We denote $I_i = [N_i + 1, N_i + H_i]$ and estimate using (2.23)

$$\Big|\sum_{x\in B}\chi(x)\Big| = \Bigg|\sum_{\substack{x_i\in I_i\\ 2\leq i\leq n}}\sum_{x_1\in I_1}\chi\Big(x_1 + x_2\frac{\omega_2}{\omega_1} + \cdots + x_n\frac{\omega_n}{\omega_1}\Big)\Bigg| \leq cnp^{\frac{1}{2}}\log p\frac{|B|}{H_1} + (*), \tag{2.24}$$

*This was originally communicated to the author by Nick Katz as an extension of his work [K].

where

$$(*) = \Big| \sum_{x_1 \in I_1} \sum_{(x_2, \dots, x_n) \in D} \chi\Big(x_1 + x_2 \frac{\omega_2}{\omega_1} + \cdots + x_n \frac{\omega_n}{\omega_1}\Big)\Big| \tag{2.25}$$

and

$$D = \Big\{(x_2, \dots, x_n) \in I_2 \times \cdots \times I_n : \mathbb{F}_p\Big(x_2 \frac{\omega_2}{\omega_1} + \cdots + x_n \frac{\omega_n}{\omega_1}\Big) \neq \mathbb{F}_q\Big\}.$$

In particular,

$$(*) \leq p\,|D| \leq p \sum_G \Big| G \bigcap \operatorname{Span}_{\mathbb{F}_p}\Big(\frac{\omega_2}{\omega_1}, \dots, \frac{\omega_n}{\omega_1}\Big)\Big|,$$

where $G$ runs over nontrivial subfields of $\mathbb{F}_q$. Since $q = p^n$ and $n$ is odd, obviously $[\mathbb{F}_q : G] \geq 3$. Hence $[G : \mathbb{F}_p] \leq \frac{n}{3}$. Furthermore, since $\{\omega_1, \dots, \omega_n\}$ is a basis of $\mathbb{F}_q$ over $\mathbb{F}_p$, $1 \notin \operatorname{Span}_{\mathbb{F}_p}(\frac{\omega_2}{\omega_1}, \dots, \frac{\omega_n}{\omega_1})$ and the proceeding implies that

$$\dim_{\mathbb{F}_p}\Big(G \bigcap \operatorname{Span}_{\mathbb{F}_p}\Big(\frac{\omega_2}{\omega_1}, \dots, \frac{\omega_n}{\omega_1}\Big)\Big) \leq \frac{n}{3} - 1. \tag{2.26}$$

Therefore, under our assumption on $|H_1|$, back to (2.24)

$$\begin{aligned}\Big|\sum_{x \in B} \chi(x)\Big| &< c(n)\Big((\log p) p^{-\frac{\varepsilon}{2}} |B| + p^{\frac{n}{3}}\Big) \\ &< \Big(c(n)(\log p) p^{-\frac{\varepsilon}{2}} + p^{-\frac{n}{13}}\Big)\,|B|,\end{aligned}$$

since $|B| > p^{\frac{2}{5}n}$. This proves our claim.

We now treat the case when $n$ is even. The analysis leading to the second part of Theorem 2 was kindly communicated by Andrew Granville to the author.

*Case 2. n is even.*

In view of the earlier discussion, our only concern is to bound

$$(*_2) = \Big| \sum_{x_1 \in I_1} \sum_{(x_2, \dots, x_n) \in D_2} \chi\Big(x_1 + x_2 \frac{\omega_2}{\omega_1} + \cdots + x_n \frac{\omega_n}{\omega_1}\Big)\Big| \tag{2.27}$$

with

$$D_2 = \Big\{(x_2, \dots, x_n) \in I_2 \times \cdots \times I_n : \Big(x_2 \frac{\omega_2}{\omega_1} + \cdots + x_n \frac{\omega_n}{\omega_1}\Big) \in F_2\Big\} \tag{2.28}$$

and $F_2$ the subfield of size $p^{n/2}$.

First, we note that since $1, \frac{\omega_2}{\omega_1}, \dots, \frac{\omega_n}{\omega_1}$ are independent, $\frac{\omega_j}{\omega_1} \in F_2$ for at most $\frac{n}{2} - 1$ many $j$'s. After reordering, we may assume that $\frac{\omega_j}{\omega_1} \in F_2$ for $2 \le j \le k$ and $\frac{\omega_j}{\omega_1} \notin F_2$ for $k+1 \le j \le n$, where $k \le \frac{n}{2}$. We also assume that $H_{k+1} \le \dots \le H_n$. Fix $x_2, \dots, x_{n-1}$. Obviously there is no more than one value of $x_n$ such that $x_2 \frac{\omega_2}{\omega_1} + \dots + x_n \frac{\omega_n}{\omega_1} \in F_2$, since otherwise $(x_n - x_n') \frac{\omega_n}{\omega_1} \in F_2$ with $x_n \ne x_n'$ contradicting the fact that $\frac{\omega_n}{\omega_1} \notin F_2$.

Therefore,

$$|D_2| \le |I_2| \cdots |I_{n-1}| \tag{2.29}$$

and

$$(*_2) \le \frac{|B|}{H_n}. \tag{2.30}$$

If $H_n > p^\tau$, we are done. Otherwise

$$H_{k+1} \cdots H_n \le p^{(n-k)\tau}. \tag{2.31}$$

Define

$$B_2 = \left\{ x_1 + x_2 \frac{\omega_2}{\omega_1} + \dots + x_k \frac{\omega_k}{\omega_1} : x_i \in I_i, 1 \le i \le k \right\}.$$

Hence $B_2 \subset F_2$ and by (2.31)

$$|B_2| > \frac{|B|}{H_{k+1} \cdots H_n} > p^{(\frac{2}{5} - \frac{\tau}{2})n} > p^{\frac{n}{3}}. \tag{2.32}$$

(We can assume $\tau < \frac{2}{15}$.)

Clearly, if $(x_2, \dots, x_n) \in D_2$, then $z = x_{k+1} \frac{\omega_{k+1}}{\omega_1} + \dots + x_n \frac{\omega_n}{\omega_1} \in F_2$. Assume $\chi|_{F_2}$ is non-principal, it follows from the generalized Polya-Vinogradov inequality and (2.32 ) that

$$\left| \sum_{y \in B_2} \chi(y+z) \right| \le (\log p)^{\frac{n}{2}} \max_\psi \left| \sum_{x \in F_2} \psi(x) \chi(x) \right| \le (\log p)^{\frac{n}{2}} \cdot |F_2|^{\frac{1}{2}} \le p^{-\frac{n}{13}} |B_2|, \tag{2.33}$$

where $\psi$ runs over all additive characters. Therefore, clearly

$$(*_2) \le H_{k+1} \cdots H_n p^{-\frac{n}{13}} |B_2| = p^{-\frac{n}{13}} |B| \tag{2.34}$$

providing the required estimate.

If $\chi|_{F_2}$ is principal, then obviously

$$(*_2) = H_1 \cdot |D_2| = \Big|F_2 \cap \frac{1}{\omega_1} B\Big| \tag{2.35}$$

and

$$\Big|\sum_{x\in B} \chi(x)\Big| = \big|\omega_1 F_2 \cap B\big| + cnp^{-\tau}|B|. \tag{2.36}$$

This complete the proof of Theorem 2. $\square$

**Remark 2.1.** The conclusion of Theorem 2 certainly holds, if we replace the assumption of $\prod_{j=1}^{n} H_j > p^{(\frac{2}{5}+\varepsilon)n}$ by the stronger assumption

$$p^{\frac{2}{5}+\varepsilon} < H_j \text{ for all } j. \tag{2.37}$$

This improves on Theorem 2 of [DL] for $n > 4$. In [DL], the condition $H_j > p^{\frac{n}{2(n+1)}+\varepsilon}$ is required. Our assumption (2.37) is independent of $n$, while, in the [DL] result, when $n$ goes to $\infty$, the exponent $\frac{n}{2(n+1)}$ goes to $\frac{1}{2}$.

**Remark 2.2.** In the case of a prime field ($n = 1$), Burgess theorem (see [Bu1]) requires the assumption $H > p^{\frac{1}{4}+\varepsilon}$, for some $\varepsilon > 0$, which seems to be the limit of this method. For $n > 1$, the exact counterpart of Burgess' estimate seems unknown in the generality of an arbitrary basis $\omega_1, \dots, \omega_n$ of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$, as considered in [DL] and here. Higher dimensional results of the strength of Burgess seem only known for certain special basis, in particular, basis of the form $\omega_j = g^j$ with given $g$ generating $\mathbb{F}_{p^n}$. (See [Bu3], [Bu4] and [Kar2].)

Theorem 2 allows us to estimate the number of primitive roots of $\mathbb{F}_{p^n}$ that fall into $B$.

We denote the Euler function by $\phi$.

**Corollary 3.** *Let $B \subset \mathbb{F}_{p^n}$ be as in Theorem 2 and satisfying $\max_\xi \big|B \cap \xi F_2\big| < p^{-\varepsilon}|B|$ if $n$ even. The number of primitive roots of $\mathbb{F}_{p^n}$ belonging to $B$ is*

$$\frac{\varphi(p^n-1)}{p^n-1}|B|(1 + O(p^{-\tau'}))$$

*where $\tau' = \tau'(\varepsilon) > 0$ and assuming $n \ll \log\log p$.*

## §3. Some further implications of the method.

In what follows, we only consider for simplicity the case of a prime field (several statements below have variants over a general finite field, possibly with worse exponents).

**3.1.** Recall that a generalized $d$-dimensional arithmetic progression in $\mathbb{F}_p$ is a set of the form

$$\mathcal{P} = a_0 + \Big\{ \sum_{j=1}^{d} x_j a_j : x_j \in [0, N_j - 1] \Big\} \tag{3.1}$$

for some elements $a_0, a_1, \dots, a_d \in \mathbb{F}_p$. If the representation of elements of $\mathcal{P}$ in (3.1) is unique, we call $\mathcal{P}$ *proper.* Hence $\mathcal{P}$ is proper if and only if $|\mathcal{P}| = N_1 \cdots N_d$ (which we assume in the sequel).

Assume $|\mathcal{P}| < 10^{-d}\sqrt{p}$, hence $\mathbb{F}_p \neq \frac{\mathcal{P}-\mathcal{P}}{\mathcal{P}-\mathcal{P}}$ (in the considerations below, $|\mathcal{P}| \ll p^{1/2}$ so that there is no need to consider the alternative $|\mathcal{P}| \gg p^{1/2}$). Following the argument in [KS1] (or the proof of Proposition 1), we have

$$E(\mathcal{P}, \mathcal{P}) < c^d (\log\ p) |\mathcal{P}|^{11/4}. \tag{3.2}$$

Also, repeating the proof of Theorem 2, we obtain

**Theorem 4.** *Let $\mathcal{P}$ be a proper $d$-dimensional generalized arithmetic progression in $\mathbb{F}_p$ with*

$$|\mathcal{P}| > p^{2/5+\varepsilon} \tag{3.3}$$

*for some $\varepsilon > 0$. If $\mathcal{X}$ is a non-principal multiplicative character of $\mathbb{F}_p$, we have*

$$\Big| \sum_{x \in \mathcal{P}} \mathcal{X}(x) \Big| < p^{-\tau} |\mathcal{P}| \tag{3.4}$$

*where $\tau = \tau(\varepsilon, d) > 0$ and assuming $p > p(\varepsilon, d)$.*

Theorem 4 is another extension of Burgess' inequality. A natural problem is to try to improve the exponent $\frac{2}{5}$ in (3.3) to $\frac{1}{4}$.

Let us point out one consequence of Theorem 4 which gives an improvement of a result in [HIS]. (See [HIS], Corollary 1.3.)

**Corollary 5.** *Given $C > 0$ and $\varepsilon > 0$, there is a constant $c = c(C, \varepsilon) > 0$ and a positive integer $k < k(\varepsilon)$, such that if $A \subset \mathbb{F}_p$ satisfies*

(i) $|A + A| < C|A|$

(ii) $|A| > p^{\frac{2}{5}+\varepsilon}$.

*Then we have*

$$|A^k| > cp.$$

*Proof.*

According to Freiman's structural theorem for sets with small doubling constants (see [TV]), under assumption (i), there is a proper generalized $d$-dimensional progression $\mathcal{P}$ such that $A \subset \mathcal{P}$ and

$$d \leq C \tag{3.5}$$

$$\log \frac{|\mathcal{P}|}{|A|} < C^2(\log C)^3 \tag{3.6}$$

By assumption (ii), Theorem 4 applies to $\mathcal{P}$. Let $\tau$ be as given in Theorem 4. We fix

$$k \in \mathbb{Z}_+, \quad k > \frac{1}{\tau}. \tag{3.7}$$

(Hence $k > k(\varepsilon)$.) Denote by $\nu$ the probability measure on $\mathbb{F}_p$ obtained as the image measure of the normalized counting measure on the $k$-fold product $\mathcal{P}^k$ under the product map

$$\begin{aligned} \mathcal{P} \times \cdots \times \mathcal{P} &\longrightarrow \mathbb{F}_p \\ (x_1, \ldots, x_k) &\longmapsto x_1 \ldots x_k. \end{aligned}$$

Hence by the Fourier inversion formula, we have

$$\begin{aligned} \nu(x) &= \frac{1}{p-1} \sum_\chi \chi(x)\hat{\nu}(\chi) = \frac{1}{p-1} \sum_\chi \chi(x)\big(\sum_t \nu(t)\overline{\chi(t)}\ \big) \\ &= \frac{|\mathcal{P}|^{-k}}{p-1} \sum_\chi \chi(x)\Big(\sum_{y\in\mathcal{P}} \bar{\chi}(y)\Big)^k \leq \frac{|\mathcal{P}|^{-k}}{p-1} \sum_\chi \Big|\sum_{y\in\mathcal{P}} \chi(y)\Big|^k, \end{aligned}$$

$\chi$ denoting a multiplicative character, and we get

$$\max_{x\in\mathbb{F}_p^*} \nu(x) \leq \frac{1}{p-1} + \max_{\chi \text{ non-principal}} |\mathcal{P}|^{-k}\Big|\sum_{x\in\mathcal{P}} \chi(x)\Big|^k < \frac{1}{p-1} + p^{-\tau k} < \frac{2}{p}. \tag{3.8}$$

The last inequality is by (3.7). Assuming $A \subset \mathbb{F}_p^*$, we write

$$\begin{aligned}|A|^k &\leq |A^k| \max_{x\in\mathbb{F}_p^*} \big|\{(x_1,\dots,x_k)\in A\times\cdots\times A : x_1\dots x_k = x\}\big|\\ &\leq |A^k|\ |\mathcal{P}|^k \max_{x\in\mathbb{F}_p^*}\nu(x)\end{aligned}$$

implying by (3.6) and (3.8)

$$|A^k| > \Big(\frac{|A|}{|\mathcal{P}|}\Big)^k \frac{p}{2} > \frac{p}{2}\ \exp\big(-kC^2(\log C)^3\big) > c(C,\varepsilon)p.$$

This proves Corollary 5. $\square$

**3.2.** Recall the well-known conjecture stating that if $A, B \subset \mathbb{F}_p, |A| > p^\varepsilon, |B| > p^\varepsilon$, then

$$\Big|\sum_{x\in A, y\in B}\chi(x+y)\Big| < p^{-\delta}|A|\ |B| \tag{3.9}$$

where $\delta = \delta(\varepsilon) > 0$ and $\chi$ a non-principal multiplicative character.

An affirmative answer is only known in the case $|A| > p^{\frac{1}{2}+\varepsilon}, |B| > p^\varepsilon$ for some $\varepsilon > 0$ (as a consequence of Weil's inequality (2.14)). Even for $|A| > p^{1/2}, |B| > p^{1/2}$, an inequality of the form (3.9) seems unknown. On the other hand, for more structured sets $A$ and $B$, better results can be obtained (See in particular [Kar1] and [FI].) In the rest of this section and the next section, we will establish further estimates in this vein.

Our first result provides a statement of this type, assuming $A$ or $B$ has a small doubling constant.

**Theorem 6.** *Assume $A, B \subset \mathbb{F}_p$ such that*

(a) $|A| > p^{\frac{4}{9}+\varepsilon}, |B| > p^{\frac{4}{9}+\varepsilon}$

(b) $|B+B| < K|B|$.

*Then*

$$\Big|\sum_{x\in A, y\in B}\chi(x+y)\Big| < p^{-\tau}|A|\ |B|,$$

*where $\tau = \tau(\varepsilon, K) > 0$, $p > p(\varepsilon, K)$ and $\chi$ is a non-principal multiplicative character of $\mathbb{F}_p$.*

*Proof.*

The argument is a variant of the proof of Theorem 2, so we will be brief. The case $|B| > p^{\frac{1}{2}+\varepsilon}$ is taken care of by Weil's estimate (2.14). Since we can dissect $B$ into $\leq p^\varepsilon$ subsets satisfying assumptions (a) and (b), we may assume that $|B| < \frac{1}{2}(\sqrt{p}-1)$. We denote the various constants $\big($possibly depending on the constant $K$ in assumption (b)$\big)$ by $C$.

Let $\mathcal{B}_1$ be a generalized $d$-dimensional proper arithmetic progression in $\mathbb{F}_p$ satisfying $B \subset \mathcal{B}_1$ and

$$d \leq K \tag{3.10}$$

$$\log \frac{|\mathcal{B}_1|}{|B|} < C. \tag{3.11}$$

Let

$$\mathcal{B}_2 = (-\mathcal{B}_1) \cup \mathcal{B}_1.$$

We take

$$\delta = \frac{\varepsilon}{4d}, \quad r = \left[\frac{10}{\delta}\right]. \tag{3.12}$$

Similar to the proof of Theorem 2, we take a proper progression $\mathcal{B}_0 \subset \mathcal{B}_2 \subset \mathbb{F}_p$ and an integral interval $I = [1, p^\delta]$ with the following properties

$$|B_0| > p^{-2d\delta}|\mathcal{B}_2|$$

$$B - \mathcal{B}_0 I \subset \mathcal{B}_2. \tag{3.13}$$

Therefore,

$$|\mathcal{B}| \leq |\mathcal{B}_1| \leq e^{C(K)}|\mathcal{B}| \ \text{ and } \ |\mathcal{B}_2| = 2|\mathcal{B}_1| - 1. \tag{3.14}$$

Estimate

$$\begin{aligned}\Big|\sum_{x\in A, y\in B} \chi(x+y)\Big| \leq& \sum_{y\in B}\Big|\sum_{x\in A}\chi(x+y)\Big| \\ \leq& |\mathcal{B}_0|^{-1}|I|^{-1} \sum_{\substack{y\in\mathcal{B}_2\\ z\in\mathcal{B}_0, t\in I}} \Big|\sum_{x\in A}\chi(x+y+zt)\Big|.\end{aligned} \tag{3.15}$$

The second inequality is by (3.13). Write

$$\sum_{\substack{y\in\mathcal{B}_2\\ z\in\mathcal{B}_0, t\in I}} \Big|\sum_{x\in A}\chi(x+y+zt)\Big| \leq (|\mathcal{B}_2|\,|\mathcal{B}_0|\,|I|)^{\frac{1}{2}} \ \Bigg|\sum_{\substack{y\in\mathcal{B}_2, z\in\mathcal{B}_0, t\in I\\ x_1, x_2\in A}} \chi\Big(\frac{(x_1+y)z^{-1}+t}{(x_2+y)z^{-1}+t}\Big)\Bigg|^{\frac{1}{2}}. \tag{3.16}$$

The sum on the right-hand side of (3.16) equals

$$\Big|\sum_{u_1,u_2\in\mathbb{F}_p}\nu(u_1,u_2)\sum_{t\in I}\chi\Big(\frac{u_1+t}{u_2+t}\Big)\Big|$$

$$\leq\Big[\sum_{u_1,u_2}\nu(u_1,u_2)^{\frac{2r}{2r-1}}\Big]^{1-\frac{1}{2r}}\Big[\sum_{u_1,u_2}\Big|\sum_{t\in I}\chi\Big(\frac{u_1+t}{u_2+t}\Big)\Big|^{2r}\Big]^{\frac{1}{2r}} \tag{3.17}$$

where for $(u_1,u_2)\in\mathbb{F}_p^2$ we define

$$\nu(u_1,u_2)=|\{(x_1,x_2,y,z)\in A\times A\times\mathcal{B}_2\times\mathcal{B}_0:\frac{x_1+y}{z}=u_1\text{ and }\frac{x_2+y}{z}=u_2\}|. \tag{3.18}$$

Hence

$$\sum_{u_1,u_2}v(u_1,u_2)=|A|^2|\mathcal{B}_2|\,|\mathcal{B}_0| \tag{3.19}$$

and

$$\begin{aligned}
&\sum_{u_1,u_2}\nu(u_1,u_2)^2\\
=&\ \Big|\{(x_1,x_2,x_1',x_2',y,y',z,z')\in A^4\times\mathcal{B}_2^2\times\mathcal{B}_0^2:\frac{x_i+y}{z}=\frac{x_i'+y'}{z'}\text{ for }i=1,2\}\Big|\\
\leq&\ |A|^3\max_{x_1,x_1'}\Big|\Big\{(y,y',z,z')\in\mathcal{B}_2^2\times\mathcal{B}_0^2:\frac{x_1+y}{z}=\frac{x_1'+y'}{z'}\Big\}\Big|\\
\leq&\ |A|^3E(\mathcal{B}_0,\mathcal{B}_0)^{\frac{1}{2}}\max_x E(x+\mathcal{B}_2,x+\mathcal{B}_2)^{\frac{1}{2}}\\
<&\ |A|^3\log p\,|\mathcal{B}_0|^{\frac{11}{8}}|\mathcal{B}_2|^{\frac{11}{8}}\\
<&\ C|A|^3\,|\mathcal{B}_2|^{\frac{11}{4}}
\end{aligned} \tag{3.20}$$

by Proposition 1, Fact 1 and several applications of the Cauchy-Schwarz inequality. Therefore, by Fact 5 (after (2.12)), (4,19) and (3.20) , the first factor of (3.17) is bounded by

$$\Big[\sum\nu(u_1,u_2)\Big]^{1-\frac{1}{r}}\Big[\sum\nu(u_1,u_2)^2\Big]^{\frac{1}{2r}}$$
$$\leq C|A|^2|\mathcal{B}_2|\,|\mathcal{B}_0|(|A|^{-\frac{1}{2}}|\mathcal{B}_2|^{-\frac{5}{8}}p^{2d\delta})^{\frac{1}{r}}. \tag{3.21}$$

Next, write using Weil's inequality (2.14)

$$\sum_{u_1,u_2\in\mathbb{F}_p}\Big|\sum_{t\in I}\chi\Big(\frac{u_1+t}{u_2+t}\Big)\Big|^{2r}\leq\sum_{t_1,\dots,t_{2r}\in I}\Big|\sum_{u\in\mathbb{F}_p}\chi\Big(\frac{(u+t_1)\cdots(u+t_r)}{(u+t_{r+1})\cdots(u+t_{2r})}\Big)\Big|^2$$

$$\leq p^2\,|I|^r\,r^{2r}+Cr^2p\,|I|^{2r}, \tag{3.22}$$

so that the second factor in (3.17) is bounded by

$$Crp^{\frac{1}{r}}\ |I|^{\frac{1}{2}} + Cp^{\frac{1}{2r}}\ |I|. \tag{3.23}$$

Applying (3.14) and collecting estimates (3.16), (3.17), (3.21), (3.23) and assumption (a), we bound (3.15) by

$$\begin{aligned}\Big|\sum_{x\in A, y\in B}\chi(x+y)\Big| &< C|A|\ |B|\ |I|^{-\frac{1}{2}}(|A|^{-\frac{1}{2}}|B|^{-\frac{5}{8}}p^{2d\delta})^{\frac{1}{2r}}(\sqrt{r}\ p^{\frac{1}{2r}}\ |I|^{\frac{1}{4}} + p^{\frac{1}{4r}}\ |I|^{\frac{1}{2}})\\ &< C\sqrt{r}\ |A|\ |B|\ (p^{-(\frac{4}{9}+\varepsilon)\frac{9}{8}+2d\delta})^{\frac{1}{2r}}(p^{\frac{1}{2r}-\frac{\delta}{4}} + p^{\frac{1}{4r}})\\ &< C\sqrt{r}\ |A|\ |B|\ (p^{\frac{1}{2}-\frac{9}{8}\varepsilon+2d\delta-\frac{\delta}{2}r} + p^{-\frac{9}{8}\varepsilon+2d\delta})^{\frac{1}{2r}}. \end{aligned}\tag{3.24}$$

Recall (3.12). The theorem follows by taking $\tau(\varepsilon) = \frac{\varepsilon^2}{128K}$ $\square$.

## §4. The case of an interval.

Next, we consider the special case $\sum_{x\in A, y\in I}\chi(x+y)$, where $A\subset\mathbb{F}_p$ is arbitrary and $I\subset\mathbb{F}_p$ is an interval. We begin with the following technical lemma.

**Lemma 7.** *Let $A\subset\mathbb{F}_p^*$ and let $I_1,\dots,I_s$ be intervals such that $I_i\subset[1,p^{\frac{1}{k_i}}]$. Denote*

$$w(u) = \Big|\big\{(y,z_1,\dots,z_s)\in A\times I_1\times\cdots\times I_s:\ y\equiv uz_1\dots z_s\ \ (mod\ p)\big\}\Big| \tag{4.1}$$

*and*

$$\gamma = \frac{1}{k_1}+\cdots+\frac{1}{k_s}. \tag{4.2}$$

*Then*

$$\sum_u w(u)^2 < |A|^{1+\gamma}\prod_{i=1}^{s}|I_i|\ p^{\frac{s}{\log\log p}} < |A|^{1+\gamma}p^{\gamma+\frac{s}{\log\log p}}.$$

*Proof.* Using multiplicative characters and Plancherel, we have

$$\sum_u w(u)^2 = \frac{1}{p-1}\sum_\chi\langle w,\chi\rangle^2, \tag{4.3}$$

where

$$\langle w,\chi\rangle = \sum_u w(u)\overline{\chi(u)} = \sum_{\substack{y\in A\\ z_i\in I_i}}\overline{\chi(y)}\chi(z_1)\dots\chi(z_s).$$

Hence

$$\left|\langle w,\chi\rangle\right| = \Big|\sum_{y\in A}\chi(y)\Big| \ \prod_i \Big|\sum_{z_i\in I_i}\chi(z_i)\Big|.$$

Using generalized Hölder inequality with $1=(1-\gamma)+\frac{1}{k_1}+\cdots+\frac{1}{k_s}$, we have

$$\begin{aligned}\sum_u w(u)^2 &= \frac{1}{p-1}\sum_\chi \ \Big|\sum_{y\in A}\chi(y)\Big|^2 \ \prod_i \Big|\sum_{z_i\in I_i}\chi(z_i)\Big|^2\\ &\le \frac{1}{p-1}\Big(\sum_\chi\Big|\sum_{y\in A}\chi(y)\Big|^{\frac{2}{1-\gamma}}\Big)^{1-\gamma} \ \prod_i\Big(\sum_\chi\Big|\sum_{z_i\in I_i}\chi(z_i)\Big|^{2k_i}\Big)^{\frac{1}{k_i}}. \end{aligned}\tag{4.4}$$

Now we estimate different factors. Writing the exponent as $\frac{2}{1-\gamma}=\frac{2\gamma}{1-\gamma}+2$ and using the trivial bound, we have

$$\sum_\chi\Big|\sum_{y\in A}\chi(y)\Big|^{\frac{2}{1-\gamma}} \le |A|^{\frac{2\gamma}{1-\gamma}}\sum_\chi\Big|\sum_{y\in A}\chi(y)\Big|^2 = |A|^{\frac{2\gamma}{1-\gamma}}\sum_{y,z\in A}\sum_\chi\chi(yz^{-1}) = p|A|^{\frac{1+\gamma}{1-\gamma}}. \tag{4.5}$$

For an interval $I\subset[1,p^{\frac{1}{k}}]$, we define

$$\eta(u) = \Big|\{(z_1,\ldots,z_k)\in I\times\cdots\times I:\ z_1\ldots z_k\equiv u \pmod p\}\Big|.$$

Since $z_1\ldots z_k\equiv z_1'\ldots z_k' \pmod p$ implies $z_1\ldots z_k=z_1'\ldots z_k'$ in $\mathbb{Z}$, $\eta(u)<\big(\exp(\frac{\log p}{\log\log p})\big)^k$. On the other hand $\sum_u\eta(u)=|I|^k$. Therefore,

$$\sum_\chi\Big|\sum_{z\in I}\chi(z)\Big|^{2k} = \sum_\chi\Big(\sum_u\eta(u)\chi(u)\Big)^2 = \sum_\chi\langle\eta,\chi\rangle^2 = (p-1)\sum_u\eta(u)^2 < p^{1+\frac{k}{\log\log p}}|I|^k. \tag{4.6}$$

Putting (4.4)-(4.6) together, we have the lemma. $\square$

We may state Lemma 7 in the following sharper version.

**Lemma 7'.** *Under the same assumption as Lemma 7, we have*

$$\sum_u w(u)^2 < |A|^{1-2\gamma}E(A,A)^\gamma p^{\frac{s}{\log\log p}}\prod_{i=1}^s|I_i| < |A|^{1-2\gamma}E(A,A)^\gamma p^{\gamma+\frac{s}{\log\log p}},$$

*where $E(A,A)$ is defined as in (1.1).*

*Proof.* Proceeding as in the proof of Lemma 7, we replace (4.5) by the estimate

$$\begin{aligned}\sum_\chi\Big|\sum_{y\in A}\chi(y)\Big|^{\frac{2}{1-\gamma}} &\le \Big[\sum_\chi\Big|\sum_{y\in A}\chi(y)\Big|^2\Big]^{\frac{1-2\gamma}{1-\gamma}}\Big[\sum_\chi\Big|\sum_{y\in A}\chi(y)\Big|^4\Big]^{\frac{\gamma}{1-\gamma}}\\ &\le (p|A|)^{\frac{1-2\gamma}{1-\gamma}}\big(p\,E(A,A)\big)^{\frac{\gamma}{1-\gamma}}. \qquad \square\end{aligned}$$

**Theorem 8.** *Let $A \subset \mathbb{F}_p$ be a subset with $|A| = p^\alpha$ and let $I \subset [1,p]$ be an arbitrary interval with $|I| = p^\beta$, where*

$$(1-\alpha)(1-\beta) < \frac{1}{2} - \delta \tag{4.7}$$

*and $\beta > \delta > 0$. Then for a non-principal multiplicative character $\chi$, we have*

$$\Big|\sum_{\substack{x\in I\\ y\in A}} \chi(x+y)\Big| < p^{-\frac{\delta^2}{13}}|A|\,|I|.$$

*Proof.* Let

$$\tau = \frac{\delta}{6} \tag{4.8}$$

and

$$R = \Big\lfloor \frac{1}{2\tau}\Big\rfloor. \tag{4.9}$$

Choose $k_1, \dots, k_s \in \mathbb{Z}^+$ such that

$$2\tau < \beta - \sum_i \frac{1}{k_i} < 3\tau. \tag{4.10}$$

Denote

$$I_0 = \big[1, p^\tau\big], \quad I_i = \big[1, p^{\frac{1}{k_i}}\big] \qquad (1 \le i \le s).$$

We perform the Burgess amplification as follows. First, for any $z_0 \in I_0, \dots, z_s \in I_s$,

$$\sum_{\substack{x\in I\\ y\in A}} \chi(x+y) = \sum_{\substack{x\in I\\ y\in A}} \chi(x+y+z_0z_1\dots z_s) + O(|A|p^{\beta-\tau}).$$

Letting $\gamma = \sum_i \frac{1}{k_i}$, we have (up to the error term)

$$\begin{aligned}
\Big|\sum_{\substack{x\in I\\ y\in A}} \chi(x+y)\Big| &= p^{-\gamma-\tau}\Big|\sum_{\substack{x\in I,\ y\in A\\ z_0\in I_0,\dots,z_s\in I_s}} \chi(x+y+z_0z_1\dots z_s)\Big| \\
&\le p^{-\gamma-\tau}\sum_{\substack{x\in I,\ y\in A\\ z_1\in I_1,\dots,z_s\in I_s}} \Big|\sum_{z_0\in I_0}\chi(x+y+z_0z_1\dots z_s)\Big| \\
&\le p^{\beta-\gamma-\tau}\max_{x\in I}\sum_{\substack{y\in A\\ z_1\in I_1,\dots,z_s\in I_s}} \Big|\sum_{z_0\in I_0}\chi\Big(\frac{x+y}{z_1\dots z_s}+z_0\Big)\Big|.
\end{aligned} \tag{4.11}$$

Fix $x \in I$ achieving maximum in (4.11), and replace $A$ by $A_1 = A + x$. Denote $w(u)$ the function (4.1) with $A$ replaced by $A_1$. Hence (4.11) is

$$p^{\beta-\gamma-\tau} \sum_u w(u) \Big| \sum_{z\in I_0} \chi(u+z) \Big|. \tag{4.12}$$

By (4.12), Hölder inequality, Fact 5 and Weil estimate $\big($cf (2.16)$\big)$, (4.11) is bounded by

$$\begin{aligned}
&p^{\beta-\gamma-\tau} \Big( \sum_u w(u)^{\frac{2R}{2R-1}} \Big)^{1-\frac{1}{2R}} \Big( \sum_u \Big| \sum_{z\in I_0} \chi(u+z) \Big|^{2R} \Big)^{\frac{1}{2R}} \\
\le &p^{\beta-\gamma-\tau} \Big[ \sum_u w(u) \Big]^{1-\frac{1}{R}} \Big[ \sum_u w(u)^2 \Big]^{\frac{1}{2R}} \Big( R|I_0|^{\frac{1}{2}} p^{\frac{1}{2R}} + 2|I_0| p^{\frac{1}{4R}} \Big) \\
\ll &p^{\alpha+\beta-\frac{1}{2R}(\delta-3\tau-\frac{1}{\log\log p})} < |A||I|p^{-\frac{\delta^2}{13}}.
\end{aligned}$$

In the last inequalities, we use $|\sum_u w(u)| = |A|p^\gamma$, (4.7)-(4.10) and Lemma 7. $\square$

Next we consider the sum

$$\sum_{x\in I, y\in\mathcal{D}} \chi(x+y), \tag{4.13}$$

where $I \subset \mathbb{F}_p$ is an interval with $|I| = p^\beta$ and $\mathcal{D}$ is $p^\beta$-spaced modulo $p$. Such sums were estimated in [FI]. In particular, Theorem 2' of [FI] gives a non-trivial estimate for (4.13) under the following assumptions

(*) $\mathcal{D}$ lies in an interval of length $D$. Moreover, for some $r \in \mathbb{Z}_+$ and $\varepsilon > 0$

$$|I|D < p^{1+\frac{1}{2r}} \quad \text{and} \quad |I||\mathcal{D}|^{\frac{1}{2}} > p^{\frac{1}{4}+\frac{1}{4r}+\varepsilon}. \tag{4.14}$$

Note that if we do not specify $\mathcal{D}$ to be contained in an interval of size $D$, (hence $D = p$), the restriction (4.14) forces $I$ and $\mathcal{D}$ to satisfy

$$|\mathcal{D} + I| \sim |I||\mathcal{D}| > p^{\frac{1}{2}+2\varepsilon}, \tag{4.15}$$

which can be dealt with in an elementary way.

In what follows we give new estimates without any restriction on the $|I|$-spaced set.

Observe that any sum as considered in Theorem 8 may be replaced by a sum of the form (4.13). Conversely, Theorem 8 may be used to bound (4.13) as follows. Denote

$I' = [1, p^{\beta-\tau}]$ for some $\tau > 0$ and $A = \mathcal{D} + I'$. Hence $|A| = |\mathcal{D}| \cdot |I'|$ by the separation assumption. Also,

$$\sum_{x\in I, y\in\mathcal{D}} \chi(x+y) = \frac{1}{|I'|} \sum_{\substack{x\in I, t\in I'\\ y\in\mathcal{D}}} \chi(x+y+t) + O(p^{-\tau}|I||\mathcal{D}|)$$
$$= \frac{1}{|I'|} \sum_{x\in I, z\in A} \chi(x+z) + O(p^{-\tau}|I||\mathcal{D}|). \tag{4.16}$$

If $|\mathcal{D}| = p^\sigma$, then $|A| = p^\alpha$ with $\alpha = \sigma + \beta - \tau$ and condition (4.7) becomes (for $\tau$ small enough)

$$\sigma + (2 - \beta - \sigma)\beta > \frac{1}{2}, \tag{4.17}$$

which improves over (4.15). One has in fact a stronger statement if $\beta > \sigma$ (when Lemma 7' is an improvement over Lemma 7).

**Theorem 9.** *Let $I \subset \mathbb{F}_p$ be an interval with $|I| = p^\beta$ and let $\mathcal{D} \subset \mathbb{F}_p$ be a $p^\beta$-spaced set with $|\mathcal{D}| = p^\sigma$. Assume*

$$(1 - 2\beta)(1 - \sigma) < \frac{1}{2} - \delta \tag{4.18}$$

*for some $\delta > 0$. Then*

$$\Big| \sum_{x\in I, y\in\mathcal{D}} \chi(x+y) \Big| < p^{-\frac{\delta^2}{17}} |I| \cdot |\mathcal{D}|$$

*for a non-principal multiplicative character $\chi$.*

*Sketch of the Proof.* The argument is a technical refinement of that of Theorem 8 based on Lemma 7'. We use the same notation as above and assume $\beta < \frac{1}{2}$. We choose $\tau = \frac{\delta}{8}$ and $R$, $\gamma$ the same as in Theorem 8. (See (4.8)-(4.10).)

Let $A = \mathcal{D} + I'$. As in (4.11), we write

$$\sum_{x\in I, y\in\mathcal{D}} \chi(x+y) = \frac{1}{|I'|} \sum_{x\in I, z\in A} \chi(x+z) + O(p^{-\tau}|I||\mathcal{D}|)$$
$$\le \frac{p^{-\gamma-\tau}}{|I'|} \Big| \sum_{\substack{x\in I,\ y\in A\\ z_0\in I_0, \dots, z_s\in I_s}} \chi(x + y + z_0 z_1 \dots z_s) \Big| + O(p^{-\tau}|I||\mathcal{D}|)$$
$$\le p^{-\gamma} \max_{x\in I} \sum_{\substack{y\in A\\ z_1\in I_1, \dots, z_s\in I_s}} \Big| \sum_{z_0\in I_0} \chi\Big(\frac{x+y}{z_1 \dots z_s} + z_0\Big) \Big| + O(p^{-\tau}|I||\mathcal{D}|).$$

To use Lemma 7', we bound $E(A, A)$ as follows. Write

$$E(A,A) = E(\mathcal{D} + I', \mathcal{D} + I') \le p^{4\sigma} \max_{d_1, d_2 \in \mathcal{D}} E(d_1 + I', d_2 + I')$$
$$< p^{4\sigma + o(1)} |I'|^2 < p^{2\sigma + o(1)} |A|^2. \tag{4.19}$$

Here we use the well-known estimate (e.g. see [FI] p.369).

$$E(I_1, I_2) < p^{o(1)} |I_1| \cdot |I_2| \tag{4.20}$$

for the multiplicative energy of intervals $I_1, I_2 \subset \mathbb{F}_p$ such that $|I_1| \cdot |I_2| < p$. Substitution of (4.19) in Lemma 7' gives

$$\sum_u w(u)^2 < |A| p^{\gamma(1+2\sigma)+o(1)}$$

and the proof is completed as in Theorem 8. $\square$

Finally we establish some improvement over Karacuba's theorem [Ka1]. Recall the statement of [Ka1]. Let $I \subset [1, p]$ be an interval with $|I| = p^\beta$ and $S \subset [1, p]$ be an arbitrary set with $|S| = p^\alpha$. If for some $\varepsilon > 0$

$$\alpha > \varepsilon, \beta > \varepsilon \text{ and } \alpha + 2\beta > 1 + \varepsilon$$

then for some $\varepsilon' > 0$

$$\sum_{y \in I} \Big| \sum_{x \in S} \chi(x+y) \Big| < p^{-\varepsilon'} |I| \ |S|. \tag{4.21}$$

We will prove the following

**Theorem 10.** *In the above setting, assume that $\alpha, \beta$ satisfy*

$$\varepsilon < \beta \le \frac{1}{k} \text{ and } \Big(1 - \frac{2}{3k}\Big)\alpha + \frac{2}{3}\Big(1 + \frac{2}{k}\Big)\beta > \frac{1}{2} + \frac{1}{3k} + \varepsilon. \tag{4.22}$$

*for some $\varepsilon > 0$ and $k \in \mathbb{Z}_+$. Then (4.21) holds for some $\varepsilon' = \varepsilon'(\varepsilon) > 0$.*

To see the strength of Theorem 10, for example, we take $\alpha = \beta$, and let $k = 3$, then estimate (4.21) is valid, provided

$$\alpha, \beta > \frac{11}{34} + \varepsilon$$

which is a slight improvement over the condition $\alpha, \beta > \frac{1}{3}$ gotten from [Ka1].

The proof of Theorem 10 is a combination of variants of arguments used in [FI] (Theorem 3) and [Ka2], together with Lemma 7'.

**Proof of Theorem 10.**

Take $\beta_1 = \beta - \tau$ with $\tau > 0$ and $\tau = o(1)$.

We partition $[1, p]$ in intervals $I_j$ of size $p^{\beta_1}$ and consider the intersections $S \cap I_j$. Up to a factor of $\log p$, one may clearly replace $S$ by sets of the form

$$S = \bigcup_{\xi_r \in \mathcal{D}} (\xi_r + S_r), \tag{4.24}$$

where $\mathcal{D}$ is a $p^{\beta_1}$-spaced set with $|\mathcal{D}| = p^\gamma$ and $S_r \subset [0, p^{\beta_1}]$ satisfying $|S_r| \sim p^{\beta_1 - \sigma}$ (for some $\sigma$ independent of $r$) and $|\mathcal{D}|.p^{\beta_1-\sigma} > p^{-o(1)}|S|$. Hence

$$\alpha \geq \gamma + \beta_1 - \sigma > \alpha - o(1). \tag{4.25}$$

We will carry out two estimates.

*Case 1.* $\alpha + \beta - \sigma - \frac{2\gamma}{k} > \frac{1}{2} + \delta$ for some $\delta > 0$.

We assume $\sigma < \beta_1 - \tau$ (more restrictive conditions will appear later).

By (4.24) and Cauchy-Schwarz, we have

$$\begin{aligned} \sum_{y \in I} \Big| \sum_{x \in S} \chi(x+y) \Big| &\leq \sum_{\xi_r \in \mathcal{D}} \sum_{y \in I} \Big| \sum_{x \in S_r} \chi(\xi_r + x + y) \Big| \\ &\leq |\mathcal{D}|^{\frac{1}{2}} |I|^{\frac{1}{2}} \Big| \sum_{\xi_r \in \mathcal{D}, y \in I, x_1, x_2 \in S_r} \chi\Big(\frac{\xi_r + x_1 + y}{\xi_r + x_2 + y}\Big) \Big|^{\frac{1}{2}}. \end{aligned}$$

It will suffice to establish a non-trivial bound on the inner sum

$$\sum_{\substack{\xi_r \in \mathcal{D}, y \in I \\ x_1 \neq x_2 \in S_r}} \chi\Big(1 + \frac{x_1 - x_2}{\xi_r + x_2 + y}\Big). \tag{4.26}$$

Denote $V$ the interval $[0, p^{\frac{\tau}{2}}]$. We recall that $x_1 - x_2 \in [-p^{\beta-\tau}, p^{\beta-\tau}]$. After fixing $r$ and $x_1, x_2 \in S_r$ in the summation (4.26), we may translate $y \in I$ by a product $t.(x_1 - x_2)$ with $t \in V$. The error is $O\big(p^{-\frac{\tau}{2}} |I| (\sum_{\mathcal{D}} |S_r|^2)\big)$.

Hence we obtain

$$\frac{1}{|V|} \sum_{\substack{\xi_r \in \mathcal{D}, y \in I, t \in V \\ x_1 \neq x_2 \in S_r}} \chi\Big(1 + \frac{1}{\frac{\xi_r + y + x_2}{x_1 - x_2} + t}\Big),$$

which we bound by

$$\frac{1}{|V|}\sum_{u\in\mathbb{F}_p}\eta(u)\Big|\sum_{t\in V}\chi\Big(1+\frac{1}{u+t}\Big)\Big|. \tag{4.27}$$

Here

$$\eta(u)=\Big|\{(\xi_r,y,x_1,x_2)\in\mathcal{D}\times I\times S_r^2 : x_1\neq x_2 \text{ and } u=\frac{\xi_r+y+x_2}{x_1-x_2}\}\Big|.$$

Under the assumption of the case, we claim

$$\Big(\sum_u\eta(u)\Big)^2>p^{\frac{1}{2}+\delta}\Big(\sum_u\eta(u)^2\Big). \tag{4.28}$$

It is obvious from the construction that

$$\sum_u\eta(u)\sim|\mathcal{D}|.|I|.p^{2(\beta_1-\sigma)}\sim p^{\beta+\gamma+2(\beta_1-\sigma)}. \tag{4.29}$$

Also

$$\sum_u\eta(u)^2$$
$$=\Big|\Big\{(\xi_r,\xi_{r'},y,y',x_1,x_2,x_1',x_2') : x_1\neq x_2, x_1'\neq x_2' \text{ and } \frac{\xi_r+y+x_2}{x_1-x_2}=\frac{\xi_{r'}+y'+x_2'}{x_1'-x_2'}\Big\}\Big|$$
$$\leq p^{2(\beta_1-\sigma)}\Big|\Big\{(\xi_r,\xi_{r'},\bar{y},\bar{y}',z,z')\in\mathcal{D}^2\times[0,2p^\beta]^2\times[-p^{\beta_1},p^{\beta_1}]^2 : \frac{\xi_r+\bar{y}}{z}=\frac{\xi_{r'}+\bar{y}'}{z'}\Big\}\Big|$$
$$=p^{2(\beta_1-\sigma)}E(\mathcal{D}+[0,2p^\beta],[-p^{\beta_1},p^{\beta_1}]).$$

Applying Lemma 7′ with $A=\mathcal{D}+[0,2p^\beta]$, $s=1,\gamma=\frac{1}{k}$ and $I=[0,2p^{\beta_1}]$ where $\beta_1<\beta\leq\frac{1}{k}$, we get $E(A,A)\ll|\mathcal{D}|^4p^{2\beta+o(1)}$ by (4.21), and

$$E(A,I)<p^{o(1)}|A|^{1-\frac{2}{k}}E(A,A)^{\frac{1}{k}}|I|<p^{\beta+\beta_1+(1+\frac{2}{k})\gamma+o(1)}. \tag{4.30}$$

Hence

$$\sum_u\eta(u)^2<p^{\beta+3\beta_1-2\sigma+(1+\frac{2}{k})\gamma+o(1)}. \tag{4.31}$$

and (4.28) holds by (4.29), (4.31) and recalling (4.25).

We follow the usual procedure (e.g. see the bounding of (4.11)), we have the bound $|I|\ |S|\ p^{-\frac{\delta^2}{4}}$.

Note that since we may assume $\alpha < \frac{1}{2} + o(1)$, the condition $\sigma < \beta_1 - \tau$ for $\tau$ small enough, is automatically satisfied under the assumption of this case.

*Case 2.* $2\alpha + \beta + \sigma - \frac{2\gamma}{k} > 1 + \delta$ for some $\delta > 0$.

Since

$$\sum_{y \in I} \Big| \sum_{x \in S} \chi(x+y) \Big| \le |I|^{\frac{1}{2}} \Big| \sum_{\substack{x_1, x_2 \in S \\ y \in I}} \chi\Big(\frac{x_1 + y}{x_2 + y}\Big) \Big|^{\frac{1}{2}},$$

we need a nontrivial estimate on

$$\sum_{\substack{x_1, x_2 \in S \\ y \in I}} \chi\Big(\frac{x_1 + y}{x_2 + y}\Big).$$

Making a translation $y \to y + zt$ with $z \in [1, p^{\beta_1}] = I_1, t \in V = [0, p^{\frac{\tau}{2}}]$ leads to

$$\frac{1}{|V|} \sum_{u_1, u_2 \in \mathbb{F}_p} \eta(u_1, u_2) \Big| \sum_{t \in V} \chi\Big(\frac{u_1 + t}{u_2 + t}\Big) \Big|, \tag{4.32}$$

where

$$\eta(u_1, u_2) = \Big| \Big\{ (x_1, x_2, y, z) \in S^2 \times I \times I_1 : u_i = \frac{x_i + y}{z}, \text{ for } i = 1, 2 \Big\} \Big|.$$

Let $\eta(u) = \eta(u_1, u_2)$. We will show that the assumption of this case implies

$$\Big( \sum_u \eta(u) \Big)^2 > p^{1+\delta} \Big( \sum_u \eta(u)^2 \Big). \tag{4.33}$$

Here

$$\sum_u \eta(u) = p^{2\alpha + \beta + \beta_1}.$$

Clearly, using the bound (4.30), we have

$$\begin{aligned}
&\sum_u \eta(u)^2 \\
&= \Big| \Big\{ (x_1, x_2, x_1', x_2', y, y', z, z') \in S^4 \times I^2 \times I_1^2 : \frac{x_i + y}{z} = \frac{x_i' + y'}{z'}, i = 1, 2 \Big\} \Big| \\
&\le |S| \, \Big| \Big\{ (x, x', y, y', z, z') \in S^2 \times I^2 \times I_1^2 : \frac{x + y}{z} = \frac{x' + y'}{z'} \Big\} \Big| \\
&< p^\alpha \, \Big| \Big\{ (\xi_r, \xi_{r'}, x, x', y, y', z, z') \in \mathcal{D}^2 \times S^2 \times I^2 \times I_1^2 : \frac{\xi_r + x + y}{z} = \frac{\xi_{r'} + x' + y'}{z'} \Big\} \Big| \\
&< p^\alpha p^{2(\beta_1 - \sigma)} E(\mathcal{D} + [0, 2p^\beta], [0, p^{\beta_1}]) \\
&< p^{\alpha + \beta + 3\beta_1 - 2\sigma + (1 + \frac{2}{k})\gamma + o(1)}.
\end{aligned}$$

Proceeding in the same way as before, we obtain the bound $|I|\ |S|\ p^{-\frac{1}{2}(\frac{\delta^2}{2}-\beta_1)}$.

To reach condition (4.22), we assume Case 1 fails. Hence

$$\alpha+\beta-\sigma-\frac{2\gamma}{k}<\frac{1}{2}+o(1)$$

and recalling (4.25), i.e.

$$\alpha+o(1)>\gamma+\beta-\sigma>\alpha-o(1)$$

(letting $\tau$ be small enough), it follows that

$$\Big(1+\frac{2}{k}\Big)\sigma>\Big(1-\frac{2}{k}\Big)\alpha+\Big(1+\frac{2}{k}\Big)\beta-\frac{1}{2}-o(1).$$

Therefore the assumption of Case 2 will be satisfied if

$$\Big(1-\frac{2}{3k}\Big)\alpha+\frac{2}{3}\Big(1+\frac{2}{k}\Big)\beta>\frac{1}{2}+\frac{1}{3k}+(\frac{1}{3}+\frac{2}{3k})\delta.$$

This proves Theorem 10.

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