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ON A QUESTION OF N. SALINAS

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ABSTRACT. In [5], Salinas asked the following question: If $T = (T_1, \ldots, T_n)$ consists of commuting hyponormal operators, is it true that (1) $\delta(T - \lambda) = d(\lambda, \sigma_\pi(T))$ and (2) $r_\pi(T) = \|D_T\|$? He proved that, for a doubly commuting n-tuple of quasinormal operators, (2) was true and (1) was true for $\lambda = 0$. In this paper we shall show that (2) holds for a doubly commuting n-tuple of hyponormal operators and give an example of a subnormal operator which does not satisfy (1).

1. Introduction. Let H be a complex Hilbert space, and all operators on H will be assumed to be linear and bounded. $\mathcal{L}$ will denote the algebra of all operators on H.

Let $T = (T_1, \ldots, T_n) \in \mathcal{L}^n$. We shall say that a point $z = (z_1, \ldots, z_n)$ of $\mathbf{C}^n$ is in the *joint approximate point spectrum* $\sigma_\pi(T)$ of T if there exists a sequence $\{x_k\}$ of unit vectors in H such that

$$\|(T_i - z_i)^* x_k\| \to 0 (k \to \infty), \qquad i = 1, \ldots, n.$$

A point $z = (z_1, \ldots, z_n)$ of $\mathbf{C}^n$ is said to be in the *joint defect spectrum* $\sigma_\delta(T)$ of T if there exists a sequence $\{x_k\}$ of unit vectors in H such that

$$\|(T_i - z_i)^* x_k\| \to 0 (k \to \infty), \ i = 1, \ldots, n.$$

The *joint numerical range* of T is the subset $W(T)$ of $\mathbf{C}^n$ such that

$$W(T) = \{((T_1 x, x), \ldots, (T_n x, x)) \colon x \in H, \|x\| = 1\}.$$

The *joint operator norm* and the *joint approximate spectral radius* of T, denoted by $\|D_T\|$ and $r_\pi(T)$, respectively, are defined by

$$\|D_T\| = \sup\left\{ \left(\sum_{i=1}^{n} \|T_i x\|^2 \right)^{1/2} : x \in H, \|x\| = 1 \right\}$$

and

$$r_\pi(T) = \sup\left\{ \left(\sum_{i=1}^{n} |z_i|^2 \right)^{1/2} : (z_1, \cdots, z_n) \in \sigma_\pi(T) \right\},$$

respectively. Let

$$\delta(T - \lambda) = \inf\left\{ \left(\sum_{i=1}^{n} \|(T_i - \lambda_i)x\|^2 \right)^{1/2} : x \in H, \|x\| = 1 \right\}.$$

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Let $\sigma_T(T)$ be *Taylor's joint spectrum* of T. We refer the reader to [**7**] for the definition of $\sigma_T(T)$.

In the sequel, we let T^*T be the n-tuple $(T_1^*T_1, \ldots, T_n^*T_n)$.

2. Theorem.

THEOREM. *Let $T = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple of hyponormal operators.*

Then $\|D_T\| = r_\pi(T)$.

We shall need the following three facts.

THEOREM A (CURTO [**2**]). *Let $T = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple of hyponormal operators.*

Then $\sigma_T(T) = \sigma_\delta(T)$.

THEOREM B (DASH [**3**]). *Let $T = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple of normal operators.*

Then $\overline{W(T)} = \operatorname{co}\sigma_T(T)$ ($=$ convex hull of $\sigma_T(T)$).

THEOREM C (CHŌ AND TAKAGUCHI [**1**]). *Let $T = (T_1, \ldots, T_n)$ be a doubly commuting n-tuple of hyponormal operators. If $r = (r_1, \ldots, r_n) \in \sigma_T(T^*T)$, then there exists a point $z = (z_1, \ldots, z_n) \in \sigma_T(T)$ such that $z_i = \sqrt{r_i} (i = 1, \ldots, n)$.*

PROOF OF THE THEOREM.

$$\|D_T\|^2 = \sup\left\{\sum_{i=1}^n \|T_i x\|^2 \colon \|x\| = 1\right\} = \sup\left\{\sum_{i=1}^n (T_i^* T_i x, x) \colon \|x\| = 1\right\}.$$

So there exists $r = (r_1, \ldots, r_n)$ in $\overline{W(T^*T)}$ such that $\|D_T\|^2 = \sum_{i=1}^n r_i$. By Theorem B, it follows that $\overline{W(T^*T)} = \operatorname{co}\sigma_T(T^*T)$. If there exist $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ in $\sigma_T(T^*T)$ and $0 < t < 1$ such that $r = t\alpha + (1-t)\beta$, then we have

$$\sum_{i=1}^n r_i < \max\left\{\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \beta_i\right\},$$

which is a contradiction to the choice of r. Therefore, $r = (r_1, \ldots, r_n)$ belongs to $\sigma_T(T^*T)$ and there exists $z = (z_1, \ldots, z_n)$ in $\sigma_\delta(T)$ such that $\sum_{i=1}^n r_i = |z|^2$ by Theorems C and A.

Let $\{x_k\}$ be a sequence of unit vectors in H such that

$$\|(T_i - z_i)^* x_k\| \to 0 (k \to \infty), \qquad i = 1, \ldots, n.$$

Hence,

$$\sum_{i=1}^n \|(T_i - z_i) x_k\|^2 \le |z|^2 - 2 \cdot \sum_{i=1}^n \operatorname{Re} \bar{z}_i (T_i x_k, x_k) + |z|^2,$$

and, since $(T_i x_k, x_k) = (x_k, T_i^* x_k) = (x_k, (T_i - z_i)^* x_k) + z_i \to z_i (k \to \infty)$, $i = 1, \ldots, n$, we have

$$\sum_{i=1}^n \|(T_i - z_i) x_k\|^2 \to 0, \qquad \text{as } k \to \infty.$$

Therefore, $z = (z_1, \ldots, z_n)$ belongs to $\sigma_\pi(T)$.

So the proof is complete.

3. Examples. Let H be an infinite-dimensional Hilbert space with orthonormal basis $\{e_k\}_{k=1}^{\infty}$. Consider the weighted shift T on H such that

$$Te_1 = \frac{1}{2} \cdot e_2, \qquad Te_k = e_{k+1} \quad (k = 2, 3, \ldots).$$

Then T is subnormal (cf. p. 379 in Stampfli [**6**]), $\sigma_\pi(T) = \{z : |z| = 1\}$ (cf. Furuta [**4**]) and evidently $\delta(T) = \frac{1}{2}$. Therefore, $\delta(T) < 1 = d(0, \sigma_\pi(T))$, so that (1) is in general false for a subnormal n-tuple.

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