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On a (r, s)-Analogue of Changhee and Daehee Numbers and Polynomials

Young-Ki Cho
Microwave & Antenna Lab., Kyungpook National University, Taegu 702-701, S. Korea
e-mail : ykcho@ee.knu.ac.kr

Taekyun Kim
Department of Mathematics, Kwangwoon University, Seoul 139-701, S. Korea
e-mail : tkkim@kw.ac.kr

Toufik Mansour*
Department of Mathematics, University of Haifa, 3498838 Haifa, Israel
e-mail : tmansour@univ.haifa.ac.il

Seog-Hoon Rim
Department of Mathematics Education, Kyungpook National University, Taegu 702-701, S. Korea
e-mail : shrim@knu.ac.kr

Abstract. We consider Witt-type formula for the extension of Changchee and Daehee numbers and polynomials. We derive some identities and properties of those numbers and polynomials which are related to special polynomials.

1. Introduction

Throughout this paper, we denote the rings of p-adic integers by $\mathbb{Z}_p$, the fields of p-adic numbers by $\mathbb{Q}_p$, and the completion of algebraic closure of $\mathbb{Q}_p$ by $\mathbb{C}_p$. The *p-adic norm* $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. Let q b an inderteminate in $\mathbb{C}_p$ with $|1-q|_p < p^{\frac{1}{p-1}}$. Let $UD[\mathbb{Z}_p]$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. The following q-Haar measure is defined by Kim in [5, 6] (see also

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[3]) $\mu_q(a+p^m\mathbb{Z}_p)\frac{q^a}{[p^m]_q}$, where $[x]_q=\frac{1-q^x}{1-q}$. For $f\in UD[\mathbb{Z}_p]$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim [6] to be

$$I_q(f)=\int_{\mathbb{Z}_p} f(x)d\mu_q(x)=\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{j=0}^{p^N-1} q^j f(j). \tag{1.1}$$

Note that the bosonic integral is considered as the bosonic limit $q\to 1$, $I_1(f)=\lim_{q\to 1} I_q(f)$. In [1, 7, 8, 9], the p-adic fermionic integration on $\mathbb{Z}_p$ defined as

$$I_{-q}(f)=\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x). \tag{1.2}$$

By (1.2), we have the following well-known integral identity

$$qI_{-q}(f_1)+I_{-q}(f)=[2]_q f(0), \tag{1.3}$$

$$-qI_q(f_1)+I_q(f)=(1-q)f(0)+\frac{1-q}{\log q}\frac{d}{dx}f(x)\mid_{x=0}, \tag{1.4}$$

where $f_1(x)=f(x+1)$.

The *Changhee polynomials* $Ch_n(x)$ are defined by the generating function to be

$$\frac{2}{2+t}(1+t)^x=\sum_{n\geq 0} Ch_n(x)\frac{t^n}{n!}. \tag{1.5}$$

When $x=0$, $Ch_n=Ch_n(0)$ are called Changhee numbers. The *Daehee polynomials* $D_{n,q}(x)$ are defined by the generating function to be

$$\frac{\log(1+t)}{t}(1+t)^x=\sum_{n\geq 0} D_n(x)\frac{t^n}{n!}. \tag{1.6}$$

When $x=0$, $D_n=D_n(0)$ are called Daehee numbers. Recently, Changhee and Daehee numbers and polynomials are introduced (see [10, 11]). Many interesting identities of those numbers and polynomials arise from umbral calculus.

In this paper, we consider (r,s)-generalizations for Changhee and Daehee numbers and polynomials and we present the Witt-type formula for each case. To state our main results, we introduce some notation from the q-calculus (see [2]). The q-Pochhammer symbol $(a;q)_n$ is defined as $\prod_{j=0}^{n-1}(1-aq^j)=(1-a)(1-aq)\cdots(1-aq^{n-1})$ with $(a;q)_0=1$. The q-factorial $[n]_q!$ is defined as $\frac{(q;q)_n}{(1-q)^n}$. More generally, the q-falling factorial is defined as $[x]_{n;q}=[x]_q[x-1]_q\cdots[x+1-n]_q$ with $[x]_{0;q}=1$. By the q-factorial, ones can define the q-binomial coefficients as $\binom{n}{k}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}$. The q-exponential function $e_q(x)$ is defined by $e_q(t)=\sum_{n\geq 0}\frac{t^n}{[n]_q!}=\sum_{n\geq 0}\frac{((1-q)t)^n}{(q;q)_n}$. The q-binomial theorem is given by $(-t;q)_n=\sum_{i=0}^n q^{\binom{i}{2}}\binom{n}{i}_q t^i$. More generally, we define $(1+t)_q^x$ to be $\sum_{i\geq 0} q^{\binom{i}{2}}\binom{x}{i}_q t^i$, where $\binom{x}{k}_q = \frac{[x]_{k;q}}{[k]_q!}$ for all $k \geq 0$.

2. (r,s)-Changhee Numbers and Polynomials

We define the n-th (r,s)-Changhee number as

$$Ch_n(r,s) = \frac{(-r)^n [n]_s!}{(1+rs)(1+rs^2)\cdots(1+rs^n)} = \frac{r^n (s;s)_n}{(s-1)^n(-rs;s)_n},$$

for all $n \geq 0$. For instance, $Ch_0(r,s) = 1$, $Ch_1(r,s) = -\frac{r}{1+rs}$ and $Ch_2(r,s) = \frac{r^2(1+s)}{(1+rs)(1+rs^2)}$.

Theorem 2.1. *For all* $n \geq 0$.

$$\int_{\mathbb{Z}_p} [x]_{n;s} d\mu_{-r}(x) = Ch_n(r,s).$$

Proof. Let $L_n = \int_{\mathbb{Z}_p} [x]_{n;s} d\mu_{-r}(x)$. Then

$$\begin{aligned}\int_{\mathbb{Z}_p} [x+1]_{n;s} d\mu_{-r}(x) &= \int_{\mathbb{Z}_p} \left(\frac{1-s^n+s^n-s^{x+1}}{1-s}[x]_{n-1;s}\right) d\mu_{-r}(x)\\ &= [n]_s L_{n-1} + s^n \int_{\mathbb{Z}_p} [x]_{n;s} d\mu_{-r}(x)\\ &= [n]_s L_{n-1} + s^n L_n.\end{aligned}$$

On the other hand, by (1.3), we have $rI_{-r}(f_1) + I_{-r}(f) = (1+r)f(0)$. Thus $r([n]_s L_{n-1} + s^n L_n) + L_n = 0$, which implies $L_n = \frac{-r[n]_s}{1+rs^n} L_{n-1}$, for all $n \geq 1$. By the initial condition $L_0 = 1$, and induction on n, we obtain that $L_n = Ch_n(r,s)$, as claimed. □

Example 2.1. Theorem 2.1 with $s = 1$ gives

$$\int_{\mathbb{Z}_p} x(x-1)\cdots(x+1-n) d\mu_{-r}(x) = \frac{(-r)^n n!}{(1+r)^n},$$

which agrees with the generalization of Changhee numbers in [13] (for the case $r = s = 1$, see [10]).

The generating function for the (r,s)-Changhee numbers is given by

$$\sum_{n\geq 0} Ch_n(r,s)\frac{t^n}{[n]_s!} = \sum_{n\geq 0} \frac{(-rt)^n}{(-rs;s)_n}.$$

Corollary 2.1. *We have*

$$\int_{\mathbb{Z}_p} (1+t)_s^x d\mu_{-r}(x) = \sum_{n\geq 0} s^{\binom{n}{2}} Ch_n(r,s)\frac{t^n}{[n]_s!}.$$

Proof. By Theorem 2.1 we have

$$\int_{\mathbb{Z}_p}(1+t)_s^x d\mu_{-r}(x)=\sum_{i\geq 0}\left(\int_{\mathbb{Z}_p}[x]_{i;s}d\mu_{-r}(x)\right)\frac{s^{\binom{i}{2}}t^i}{[i]_s!}=\sum_{i\geq 0}\frac{s^{\binom{i}{2}}(-rt)^i}{(-rs,s)_i},$$

which completes the proof. □

Now, we define the (r,s)-Changhee polynomials by the generating function

$$(1+t)_s^x\sum_{i\geq 0}\frac{s^{\binom{i}{2}}(-rt)^i}{(-rs,s)_i}=\sum_{n\geq 0}s^{\binom{n}{2}}Ch_n(x|r,s)\frac{t^n}{[n]_s!}.$$

For instance, $Ch_0(x|r,s)=1$, $Ch_1(x|r,s)=[x]_s-\frac{r}{1+rs}$, and

$$Ch_2(x|r,s)=[x]_{2;s}+\frac{r^2[2]_s!}{(1+rs)(1+rs^2)}-\frac{r[x]_s[2]_s!}{s(1+rs)}.$$

Theorem 2.2. *For all $n\geq 0$,*

$$\int_{\mathbb{Z}_p}[x+y]_{n;s}d\mu_{-r}(y)=Ch_n(x|r,s).$$

Proof. By the definitions, we have

$$\begin{aligned}\sum_{n\geq 0}\left(\int_{\mathbb{Z}_p}[x+y]_{n;s}d\mu_{-r}(y)\right)\frac{s^{\binom{n}{2}}t^n}{[n]_s!}&=\int_{\mathbb{Z}_p}(1+t)_s^{x+y}d\mu_{-r}(y)\\&=(1+t)_s^x\int_{\mathbb{Z}_p}(1+t)_s^y d\mu_{-r}(y)\\&=(1+t)_s^x\sum_{n\geq 0}\frac{s^{\binom{n}{2}}(-rt)^n}{(-rs,s)_n}\\&=\sum_{n\geq 0}s^{\binom{n}{2}}Ch_n(x|r,s)\frac{t^n}{[n]_s!},\end{aligned}$$

By comparing the coefficient of t^n, we complete the proof. □

Example 2.2. Theorem 2.1 with $s=1$ gives

$$\sum_{n\geq 0}\left(\int_{\mathbb{Z}_p}(x+y)(x+y-1)\cdots(x+y+1-n)d\mu_{-r}(x)\right)\frac{t^n}{n!}=\frac{1+r}{1+r+rt}(1+t)^x,$$

which agrees with Theorem 2.1 in [13] (for the case $r=s=1$, see [10]).

3. (r,s)-Daehee Numbers and Polynomials

Let $(r,s)\neq(1,1)$. We define the n-th (r,s)-Daehee number as

$$D_n(r,s)=\frac{r^n[n]_s!}{(rs,s)_n}\left(1-\frac{(1-r)\log s}{r(1-s)\log r}\sum_{j=0}^{n-1}\frac{(-1)^j(rs;s)_j}{r^js^{\binom{j+1}{2}}[j+1]_s}\right),$$

for all $n\geq 0$. For instance, $D_0(r,s)=1$, $D_1(r,s)=\frac{r}{1-rs}-\frac{(1-r)\log s}{(1-rs)(1-s)\log r}$ and $D_2(r,s)=\frac{r^2(1+s)}{(1-rs)(1-rs^2)}+\frac{(1-r)(1-2rs-rs^2)\log s}{s(1-s)(1-rs)(1-rs^2)\log r}$.

Theorem 3.3. *For all* $n\geq 0$.

$$\int_{\mathbb{Z}_p}[x]_{n;s}d\mu_r(x)=D_n(r,s).$$

Proof. Let $L_n=\int_{\mathbb{Z}_p}[x]_{n;s}d\mu_r(x)$. Then

$$\begin{aligned}\int_{\mathbb{Z}_p}[x+1]_{n;s}d\mu_r(x)&=\int_{\mathbb{Z}_p}\left(\frac{1-s^n+s^n-s^{x+1}}{1-s}[x]_{n-1;s}\right)d\mu_r(x)\\&=[n]_sL_{n-1}+s^n\int_{\mathbb{Z}_p}[x]_{n;s}d\mu_r(x)\\&=[n]_sL_{n-1}+s^nL_n.\end{aligned}$$

On the other hand, by (1.4), we have

$$-r([n]_sL_{n-1}+s^nL_n)+L_n=\frac{(-1)^n[n-1]_s!}{s^{\binom{n}{2}}}\frac{(1-r)\log s}{(1-s)\log r},$$

which implies

$$L_n=\frac{r[n]_s}{1-rs^n}L_{n-1}+\frac{(-1)^n[n-1]_s!}{s^{\binom{n}{2}}(1-rs^n)}\frac{(1-r)\log s}{(1-s)\log r}.$$

By induction on n with using the initial value $L_0=1$, we obtain

$$L_n=\prod_{i=1}^n\frac{r[i]_s}{1-rs^i}+\sum_{j=1}^n\frac{(-1)^j[j-1]_s!}{s^{\binom{j}{2}}(1-rs^j)}\frac{(1-r)\log s}{(1-s)\log r}\prod_{i=j+1}^n\frac{r[i]_s}{1-rs^i},$$

which is equivalent to

$$L_n=\frac{r^n[n]_s!}{(rs,s)_n}\left(1-\frac{(1-r)\log s}{r(1-s)\log r}\sum_{j=0}^{n-1}\frac{(-1)^j(rs;s)_j}{r^js^{\binom{j+1}{2}}[j+1]_s}\right)=D_n(r,s),$$

as required. □

Example 3.3. Theorem 3.3 with $s = 1$ gives (see [14] and [10])

$$\int_{\mathbb{Z}_p} x(x-1)\cdots(x+1-n)d\mu_r(x) = \frac{(-1)^n n!}{(1-\frac{1}{r})^n}\left(1+\log\frac{1}{r}\sum_{j=1}^{n}\frac{(1-\frac{1}{r})^j}{j}\right).$$

Corollary 3.2. *We have*

$$\int_{\mathbb{Z}_p}(1+t)_s^x d\mu_r(x) = \sum_{n\geq 0} s^{\binom{n}{2}} D_n(r,s)\frac{t^n}{[n]_s!}.$$

Proof. Direct calculations show

$$\int_{\mathbb{Z}_p}(1+t)_s^x d\mu_r(x) = \sum_{i\geq 0}\left(\int_{\mathbb{Z}_p}[x]_{n;s}d\mu_r(x)\right)\frac{s^{\binom{n}{2}}t^n}{[n]_s!},$$

which, by Theorem 3.3, completes the proof. □

Now, we define the (r, s)-Daehee polynomials by the generating function

$$(1+t)_s^x\sum_{n\geq 0}\left(1-\frac{(1-r)\log s}{r(1-s)\log r}\sum_{j=0}^{n-1}\frac{(-1)^j(rs;s)_j}{r^j s^{\binom{j+1}{2}}[j+1]_s}\right)\frac{(rt)^n}{(rs;s)_n}$$
$$= \sum_{n\geq 0} s^{\binom{n}{2}} D_n(x|r,s)\frac{t^n}{[n]_s!}.$$

Theorem 3.4. *For all $n \geq 0$,*

$$\int_{\mathbb{Z}_p}[x+y]_{n;s}d\mu_r(y) = D_n(x|r,s).$$

Proof. By the definitions, we have

$$\begin{aligned}
&\sum_{n\geq 0}\left(\int_{\mathbb{Z}_p}[x+y]_{n;s}d\mu_r(y)\right)\frac{s^{\binom{n}{2}}t^n}{[n]_s!}\\
&\quad= \int_{\mathbb{Z}_p}(1+t)_s^{x+y}d\mu_r(y) = (1+t)_s^x\int_{\mathbb{Z}_p}(1+t)_s^y d\mu_r(y)\\
&\quad= (1+t)_s^x\sum_{n\geq 0}\left(\frac{r^n[n]_s!}{(rs,s)_n}\left(1-\frac{(1-r)\log s}{r(1-s)\log r}\sum_{j=0}^{n-1}\frac{(-1)^j(rs;s)_j}{r^j s^{\binom{j+1}{2}}[j+1]_s}\right)\right)\frac{s^{\binom{n}{2}}t^n}{[n]_s!}\\
&\quad= \sum_{n\geq 0}s^{\binom{n}{2}}D_n(x|r,s)\frac{t^n}{[n]_s!},
\end{aligned}$$

By comparing the coefficient of t^n, we complete the proof. □

Example 3.4. Theorem 2.1 with $s=1$ gives (see [14] and [10])

$$\begin{aligned}
&\sum_{n\geq 0}\left(\int_{\mathbb{Z}_p}(x+y)(x+y-1)\cdots(x+y+1-n)d\mu_r(x)\right)\frac{t^n}{n!}\\
&=(1+t)^x\sum_{n\geq 0}\left(\frac{(-1)^n}{(1-\frac{1}{r})^n}\left(1+\log\frac{1}{r}\sum_{j=1}^{n}\frac{(1-\frac{1}{r})^j}{j}\right)\right)t^n\\
&=\left(\frac{r-1}{r-1+rt}+\log\frac{1}{r}\sum_{j\geq 0}\sum_{n\geq j}\frac{(-t)^n}{j(1-\frac{1}{r})^{n-j}}\right)(1+t)^x\\
&=\frac{1-r}{r-1+rt}(\log\frac{1}{r}\log(1+t)-1)(1+t)^x.
\end{aligned}$$

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