On a reduction and solutions of non-linear wave equations with broken symmetry

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A generalised definition for invariance of partial differential equations is proposed. Exact solutions of the equations with broken symmetry are obtained.

Let us consider the non-linear wave equation

$$\Box u + F_1(u) = 0, \qquad u = u(x_0, x_1, x_2, x_3),$$
$$\Box = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2, \qquad \partial_\mu = \partial/\partial x_\mu, \qquad \mu = 0, 1, 2, 3, \tag{1}$$

where $F_1(u)$ is an arbitrary smooth function. The ansatz

$$u = f(x)\varphi(\omega) + g(x) \tag{2}$$

suggested by Fushchych [5] was used to construct the family of exact solutions of equations (1). $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is the function to be determined and $\omega = (\omega_1, \omega_2, \omega_3)$ are new invariant variable. Wide classes of exact solutions of equation (1) have been constructed by Fushchych and Serov [7, 8], Fushchych et al [10] and Fushchych and Shtelen [9]. It is important to note that Poincaré invariance of equation (1) was used.

The possibility of using an ansatz of type (2) to find exact solutions of the non-linear wave equations with broken symmetry naturally arises in connection with the fact that many equations of theoretical physics are not invariant with respect to the Poincaré, Galilei and Euclidean groups. A more specific formulation of this problem is as follows: are we able to construct the solutions of wave equations not invariant with respect to the Lorentz groups, for example, but nevertheless with the help of the Lorentz-invariant ansatz?

The present letter suggests an affirmative answer to this question, i.e. we construct the many-dimensional non-linear wave equations with broken symmetry. The multi-parametrical exact solutions of these equations are found with the help of ansatz (2), previously used to find exact solutions of Poincaré- and Galilei-invariant equations only. It is obvious that ansatz (2) cannot be applied to the equations with arbitrary breakdown of symmetry, which is why the equation with the breakdown of symmetry should have some hidden symmetry. The set of equations with such symmetry was considered by Fushchych and Nikitin [6]. We do not deal with the symmetry of all the solutions of the equations but only with a definite subset of solutions, which may be much wider that the symmetry of the equation itself. This idea will be used below.

Let us consider the wave equation with broken symmetry

$$Lu \equiv \Box u + F(x, \underset{1}{u}, u) = 0, \tag{3}$$

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where $F(x, \underset{1}{u}, u)$ is an arbitrary smooth function, depending on $x = (x_0, x_1, x_2, x_3)$, $\underset{1}{u} \equiv (\partial u/\partial x_0, \partial u/\partial x_1, \partial u/\partial x_2, \partial u/\partial x_3)$. Following Fushchych [3] we generalise the Lie definition of invariance of equation (3).

Definition. *We shall say that equation (3) is invariant with respect to some set of operators $\hat{Q} = \{\hat{Q}_A\}$, $A = 1, 2, \ldots, N$, a number of linearly independent operators, if the following condition is fulfilled:*

$$\hat{Q}_A Lu\Big|_{\substack{Lu = 0, \\ \{\hat{Q}_A u\} = 0}} = 0, \tag{4}$$

where $\{\hat{Q}_A u\} = 0$ is a set of equations

$$\hat{Q}_A u = 0, \qquad D\hat{Q}_A u = 0, \qquad D^2\hat{Q}_A u = 0, \quad \ldots, \quad D^n\hat{Q}_A u = 0, \tag{5}$$

where D is an operator of total differentiation. Condition (4) is a necessary condition for reduction of differential equations.

Definition (4) is a generalisation of the Lie definition (see, e.g., Ovsyannikov [12])

$$\hat{Q}_A Lu\Big|_{Lu=0} = 0, \tag{6}$$

where $\hat{Q}_A$ are a number of first-order differential operators forming a Lie algebra.

To demonstrate the efficacy of definition (4) and to find exact solutions of equation (3) we choose the function F in a form

$$\begin{aligned} F = -\left(\frac{\lambda_0}{x_0}\right)^2\left(\frac{\partial u}{\partial x_0}\right)^2 + \left(\frac{\lambda_1}{x_1}\right)^2\left(\frac{\partial u}{\partial x_1}\right)^2 + \\ + \left(\frac{\lambda_2}{x_2}\right)^2\left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\lambda_3}{x_3}\right)^2\left(\frac{\partial u}{\partial x_3}\right)^2, \end{aligned} \tag{7}$$

where λ_μ are arbitrary parameters and $x_\mu \neq 0$.

Theorem. *The maximal local (in the Lie sense) invariance group of equations (3) and (7) is the two-parametrical group of the transformations*

$$x_\mu = e^a x_\mu, \qquad u' = e^{2a} u \tag{8}$$

and

$$u' = u + c, \qquad c = \text{constant},$$

where a is real parameter.

The proof of the theorem is reduced to application of the well known Lie algorithm and we do not present it here. One can make sure non-linearity breaks the rotational and translational symmetry.

Now we show that the Lorentz-non-invariant equations (3) and (7) are reduced to an ordinary differential equation with the help of the Lorentz-invariant ansatz

$$u = \varphi(\omega), \qquad \omega = x_\mu x^\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{9}$$

Substituting (9) into (3) and (7) we obtain the ordinary differential equation

$$\omega\frac{d^2\varphi}{d\omega^2}+2\frac{d\varphi}{d\omega}=-\lambda^2\left(\frac{d\varphi}{d\omega}\right)^2, \qquad \lambda^2=\lambda_0^2-\lambda_1^2-\lambda_2^2-\lambda_3^2. \tag{10}$$

Solving equation (10), we obtain

$$\varphi(\omega)=2\left(-\lambda^2\right)^{-1/2}\tan^{-1}\left[\omega\left(-\lambda^2\right)^{-1/2}\right], \qquad -\lambda^2>0, \tag{11}$$

$$\varphi(\omega)=-\left(\lambda^2\right)^{-1/2}\ln\left(\frac{\left(\lambda^2\right)^{1/2}+\omega}{\left(\lambda^2\right)^{1/2}-\omega}\right), \qquad -\lambda^2<0. \tag{12}$$

Thus the Lorentz-non-invariant (in the Lie sense) equations (3) and (7) are reduced to an ordinary differential equation.

Formulae (11) and (12) give a Lorentz-invariant family of solutions of equations (3) and (7). It means that the following set of conditions is fulfilled:

$$J_{\mu\nu}u(x)=0, \qquad \mu,\nu=0,1,2,3, \tag{13}$$

$$J_{0a}=x_0\frac{\partial}{\partial x_a}+x_a\frac{\partial}{\partial x_0}, \qquad J_{ab}=x_a\frac{\partial}{\partial x_b}-x_b\frac{\partial}{\partial x_a}, \qquad a,b=1,2,3 \tag{14}$$

for the set of solutions (11) and (12).

The operators (14) generate Lorentz transformations. Equations (13) are the concrete realisation of the first equation of (5). In this case the index A varies from 1 to 6.

Thus, equations (13) pick out a Lorentz-invariant subset of the set of all solutions of equations (3) and (7). In other words, equations (3) and (7) are Lorentz-invariant in the sense of definition (4).

Now let us consider the equation

$$\frac{\partial^2 u}{\partial t^2}=\lambda\Delta u(\nabla u)^2, \qquad \lambda=\frac{1}{3}m^2. \tag{15}$$

It is simple to verify that equation (15) is not invariant with respect to Galilean transformations, generated by operators

$$G_a=t\frac{\partial}{\partial x_a}+mx_a, \qquad a=1,2,3. \tag{16}$$

In this case equations $\{\hat{Q}_A u\}=0$ are

$$G_a u=t\frac{\partial u}{\partial x_a}-mx_a u=0, \tag{17}$$

$$\frac{\partial}{\partial t}(G_a u)=0. \tag{18}$$

Thus equation (15) is invariant under transformations generated by operators (16) in the sense of definition (4). It means that the subset of solutions of equations (15) picked out by means of conditions (17) and (18) is invariant under Galilean transformations while equation (15) is not invariant under these transformations.

The Galilean-invariant ansatz has the form

$$u=\varphi(t)+m\left(x_1^2+x_2^2+x_3^2\right)/2t, \qquad \omega=t, \qquad f=1. \tag{19}$$

Substituting (19) into (15), we obtain

$$\frac{d^2\varphi}{\partial t^2}=0 \quad \leftrightarrow \quad u=m\left(x_1^2+x_2^2+x_3^2\right)/2t+At+C, \tag{20}$$

where A and C are arbitrary constants.

A generalised definition of the invariance (4) can be applied to the system of partial differential equations.

Let us consider, for example, a non-linear Dirac system of equations:

$$\begin{gathered}\gamma_\mu\partial^\mu\psi+g\left[2\bar\psi(x_\mu\partial^\mu)\psi-\left(x^2/c_\alpha x^\alpha\right)\bar\psi(c_\mu\partial^\mu)\psi\right]M^{-1}(x)(\bar\psi\psi)^{1/3}\psi=0,\\ M(x)=2(c_\alpha x^\alpha)^{-1}\bar\psi S_{\mu\nu}c^\mu\beta^\nu\psi+\bar\psi\psi,\\ S_{\mu\nu}=\frac{1}{4}i(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu) \qquad \mu,\nu,\alpha=0,1,2,3,\end{gathered} \tag{21}$$

where g, β_μ, c_α are arbitrary parameters.

Equation (21) is not invariant under conformal transformations. Nevertheless, it is reduced to the system of ordinary differential equations

$$i\gamma_\mu\beta^\mu\frac{d\varphi}{d\omega}+g(\bar\varphi\varphi)^{1/3}\varphi=0 \tag{22}$$

with the help of the conformally invariant ansatz (4)

$$\psi(x)=\left[\gamma_\mu x^\mu/(x^2)^2\right]\varphi(\omega), \quad \omega=\beta_\mu x^\mu/x^2, \quad \beta^2\neq0, \quad x^2=x_\mu x^\mu\neq0, \tag{23}$$

where $\varphi(\omega)$ is the four-component spinor depending on a variable ω. The general solution of equation (22) is the vector function

$$\varphi=\exp\left[-i\frac{\gamma_\mu\beta^\mu}{\beta^2}g(\bar\chi\chi)^{1/3}\omega\right]\chi, \tag{24}$$

where χ is a constant spinor.

Equation (21) is invariant under the transformations generated by the operator $c_\mu K^\mu$ on a set of solutions of the equations

$$\begin{gathered}c_\mu K^\mu\psi=0,\\ c^\mu K_\mu=2(cx)(x\partial)-x^2(c\partial)+2(cx)-(\gamma c)(\gamma x).\end{gathered} \tag{25}$$

In conclusion we note that an idea like the one set forward here was used by Bluman and Cole [2], Ames [1], Fokas [3] and Olver and Rosenau [11], as was kindly indicated by the referee.

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1. Ames W.F., Nonlinear partial differential equations in engineering, Vol.2, New York, Academic, 1972.
2. Bluman G.W., Cole I.D., *J. Math. Mech.*, 1969, **18**, 1025–1042.
3. Fokas A.S., PhD thesis, California Institute of Technology, 1979.

4. Fushchych W.I., *Dokl. Akad. Nauk*, 1979, **246**, 846–850.
5. Fushchych W.I., The symmetry of mathematical physics problems, in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Mathematical Institute, 1981, 6–28.
6. Fushchych W.I., Nikitin A.G., Symmetry of Maxwell Equations, Kiev, Naukova Dumka, 1983.
7. Fushchych W.I., Serov N.I., *J. Phys. A: Math Gen.*, 1983, **16**, 3645–3656.
8. Fushchych W.I., Serov N.I., *Dokl. Akad. Nauk*, 1983, **273**, 543.
9. Fushchych W.I., Shtelen W.M., *J. Phys. A: Math Gen.*, 1983, **16**, 271–277.
10. Fushchych W.I., Shtelen W.M., Zhdanov R.Z., *Phys. Lett. B*, 1985, **159**, 189–191.
11. Olver P.J., Rosenau P., *Phys. Lett. A*, 1986, **114**, 107–112.
12. Ovsyannikov L.V., The group analysis of differential equations, Moscow, Nauka, 1978.