# On a renewal risk process with dependence under a Farlie - Gumbel - Morgenstern Copula

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**Abstract**

In this paper we consider an extension to the renewal or Sparre Andersen risk process by introducing a dependence structure between the claim sizes and the interclaim times through a Farlie-Gumbel-Morgenstern copula proposed by Cossette et al. for the classical compound Poisson risk model [H. Cossette, E. Marceau, F. Marri, Analysis of ruin measures for the classical compound Poisson risk model with dependence, Scandinavian Actuarial Journal, 2010, 3, 221-245]. We consider that the inter-arrival times follow the Erlang(n) distribution. By studying the roots of the generalized Lundberg equation, the Laplace transform of the expected discounted penalty function is derived and a detailed analysis of the Gerber - Shiu function is given when the initial surplus is zero. It is proved that this function satisfies a defective renewal equation and its solution is given through the compound geometric tail representation of the Laplace transform of the time to ruin. Explicit expressions for the discounted joint and marginal distribution functions of the surplus prior to the time of ruin and the deficit at the time of ruin are derived. Finally, for exponential claim sizes explicit expressions and numerical examples for the ruin probability and the Laplace transform of the time to ruin are given.

*Keywords*: Gerber-Shiu discounted penalty function; dependence; Farlie-Gumbel-Morgenstern copula; Integro-differential equation; Laplace Transform; defective renewal equation; ruin probability
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## 1 Introduction

In the actuarial literature, a lot of attention is concentrated on the classical risk model in which claims occur according to a Poisson process as well as on the renewal or Sparre Andersen risk model in which claims occur more generally as a renewal process. Ruin probabilities and many other ruin measures such as the marginal and the joint (defective or not) distributions of the time to ruin, the deficit at ruin and the surplus prior to ruin have been extensively studied. A unified approach to study together these fundamental ruin measures in just one function has been proposed by the Gerber and Shiu (1998) seminal paper, by introducing the expected discounted penalty function for the classical risk model. For a detailed study of these ruin measures we refer to Lin and Willmot (1999) and the references therein. Since then, many authors have studied several renewal risk models with specific interclaim times. The Erlang distribution is one of the most commonly used distributions in risk theory and queueing theory. Several results concerning the

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evaluation of the Gerber - Shiu function for renewal risk models have been obtained when the interclaim times occur according to an Erlang process. See, for example, Dickson (1998), Dickson and Hipp (1998, 2001), Cheng and Tang (2003), Tsai and Sun (2004), Li and Garrido (2004), Sun (2005), Gerber and Shiu (2005) and the references therein. In the study of these models, it is assumed that the claim sizes and the interclaim times are mutually independent. Although this hypothesis indeed simplifies the study of several ruin measures, it has been proved to be inappropriate and very restrictive in some real applications. For example, in modelling damages due to natural catastrophic events (e.g. earthquakes) the intensity of the catastrophe and the time elapsed since the last catastrophe are expected to be dependent. We refer to Boudreault (2003) and Nikoloulopoulos and Karlis (2008) for such a dependence structure in an earthquake context.

Recently, many authors have paid more and more attention to risk models with dependence between claim sizes and interclaim times. The modelling of the dependence structure has led to several generalizations of the classical risk model. Albrecher and Boxma (2004) extended the classical risk model by considering that the interclaim time depends on the previous claim size and studied the ruin probability. A generalization of this dependence structure is examined in Albrecher and Boxma (2005) who considered a Markov - dependent risk model. Albrecher and Teugels (2006) considered several copulas in order to model the dependence structure between the claim size and the interclaim time and studied the asymptotic behavior of the ruin probability in both finite and infinite time. Boudreault et al. (2006) extended the classical risk model by introducing a dependence structure where the distribution of the next claim size depends on the time elapsed since the last claim and examined the Gerber - Shiu function. Meng et al. (2008) studied the ruin probability for a risk model with a dependent setting where the time between the two claim occurrences determines the distribution of the next claim. Cossette et al. (2008, 2010) considered the classical compound Poisson risk process with a dependence structure based on the (generalized) Farlie - Gumbel - Morgenstern copula and evaluated the Gerber - Shiu function. Badescu et al. (2009) using fluid flow techniques studied the Gerber - Shiu function by modelling the dependence structure via bivariate phase - type distributions. Ambagaspitiya (2009) by means of Wiener - Hopf factorization technique obtained the ruin probabilities for two classes of bivariate distributions modelling the claim size and the interclaim time. Cheung at al. (2010) studied the structural properties for a generalized Gerber - Shiu function by assuming a general dependence structure for the renewal risk model. Albrecher et al. (2011) obtained explicit ruin formulas for models with dependence among claims only or among interclaim times only via copula. Zhang and Yang (2011) studied the Gerber - Shiu functions for a perturbed by diffusion compound Poisson risk model with dependence structure between the claim size and the interclaim time based on the Farlie - Gumbel - Morgenstern copula. Zhang et al. (2011) using a q-potential measure studied the Gerber - Shiu function for a perturbed by a jump - diffusion process renewal risk model where the claim size and the interclaim time follow some bivariate distribution. Woo (2011) considered a delayed renewal risk model with a general dependence structure and investigated a generalized Gerber - Shiu function.

In this paper, we consider a renewal or Sparre Andersen risk process with dependence between the claim size and the interclaim time, based on the Farlie - Gumbel - Morgenstern copula. We assume that the interclaim times are distributed according to an Erlang(n) distribution. Therefore our risk model is an extension of the classical compound Poisson risk process proposed by Cossette et al. (2010). Therefore our risk model is an extension of the classical compound Poisson risk process proposed by Cossette et al. (2010). The choice of an Erlang(n) distribution is motivated by the fact that it is more general than the Exponential distribution allowing a flexible modelling of the interarrival times. A further discussion why the proposed Erlang(n) risk model is more appropriate than the Poisson arrival process is given in Subsection 8.1. The rest of the paper is organized as follows. In Section 2, we describe the dependence structure of the proposed model. We derive the generalized Lundberg equation and analyze its roots in Section 3. In Section 4 we obtain the Laplace transform of the Gerber - Shiu function. The analysis of the Gerber - Shiu function with zero initial surplus is given in Section 5. The defective renewal equation for the Gerber - Shiu function, the Laplace transform of the time to ruin and their solutions are obtained in Section 6. In Section 7, explicit expressions for the discounted distributions of the surplus prior to ruin and the deficit at ruin are given. Finally, explicit expressions and numerical results are given for exponential claims in Section 8.

# 2 The risk model and the dependence structure

For an insurance portfolio, we consider the surplus process $\{U(t), t \geq 0\}$ defined by $U(t) = u + ct - S(t)$, where $u = U(0) \geq 0$ is the initial surplus and $c$ is the premium rate which is assumed to be a positive constant. $\{S(t), t \geq 0\}$ is the total claim amount process defined by $S(t) = \sum_{i=1}^{N(t)} X_i$ and $\sum_{\alpha}^{b} = 0$ if $\alpha > b$. The random variable (r.v.) $X_i$ represents the size of the i-th claim, and the r.v.'s $\{X_i\}_{i=1}^{\infty}$ are assumed to form a sequence of i.i.d. r.v.'s distributed as a generic r.v. X with probability density function (p.d.f.) $f_X$, cumulative distribution function (c.d.f.) $F_X(x) = 1 - \overline{F}_X(x)$ and Laplace transform (L.T.) $\widehat{f}_X(s) = \int_0^{\infty} e^{-sx} f_X(x)dx$. The claim number process $\{N(t), t \geq 0\}$ is a renewal process defined via a sequence of i.i.d. interclaim times $\{W_i\}_{i=1}^{\infty}$ with $W_1$ the time of the first claim and $W_i$ the time between the (i-1)-th and the i-th claim for $i \geq 2$. We assume that the r.v.'s $\{W_i\}_{i=1}^{\infty}$ are distributed as a generic r.v. W with p.d.f. $f_W$, c.d.f. $F_W(t) = 1 - \overline{F}_W(t)$ and L.T. $\widehat{f}_W(s) = \int_0^{\infty} e^{-st} f_W(t)dt$.

In this paper we consider that the r.v. $W$ has an Erlang(n) distribution with expectation $n/\lambda$, $n \in \mathbb{N}^+$, $\lambda > 0$ with p.d.f., c.d.f. and L.T. given by

$$f_W(t) = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t}, t \geq 0 \; n \in \mathbb{N}^+, \tag{1}$$

$$F_W(t) = 1 - e^{-\lambda t} \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!}, \; and \tag{2}$$

$$\widehat{f}_W(s) = E[e^{-sW}] = \left(\frac{\lambda}{\lambda + s}\right)^n. \tag{3}$$

As in Cossette et al. (2010), we assume that the pairs $\{(X_i, W_i)\}_{i=1}^{\infty}$ form a sequence of i.i.d. random vectors distributed as the generic random vector $(X, W)$, in which the components $X$ and $W$ may be dependent, so that $\{cW_i - X_i\}_{i=1}^{\infty}$ is also a sequence of i.i.d. r.v.'s. The joint p.d.f. of $(X, W)$ is denoted by $f_{X,W}$ and the joint c.d.f. is denoted by $F_{X,W}$.

Motivated by Cossette et al. (2010), we use the Farlie - Gumbel - Morgenstern (FGM) copula to define the joint distribution of $(X, W)$ and hence the dependence structure between the claim size and the interclaim time. The FGM copula is defined by

$$C_{\theta}^{FGM}(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2), \tag{4}$$

with $(u_1, u_2) \in [0, 1] \times [0, 1]$ and $-1 \leq \theta \leq 1$. The FGM copula allows positive and negative dependence, and it also includes the independence copula for $\theta = 0$. The FGM copula is often used in applications to describe dependence structures due to its tractability and simplicity. For more on the FGM copula we refer to Nelsen (2006) and for applications of this in risk theory, health insurance plans, financial risk management, stochastic frontiers and a stereological context see Cossette et al. (2010) and the references therein.

The bivariate c.d.f. of $F_{X,W}$ based on the FGM copula is defined by

$$\begin{aligned} F_{X,W}(x,t) &= C_{\theta}^{FGM}\Big(F_X(x), F_W(t)\Big) \\ &= F_X(x)F_W(t) + \theta F_X(x)\overline{F}_X(x)F_W(t)\overline{F}_W(t), \; x, t \in \mathbb{R}^+. \end{aligned}$$

The p.d.f. associated to Eq. (4) is given by

$$\begin{aligned} c_\theta^{FGM} &= \frac{\partial^2}{\partial u_1 \partial u_2} C_\theta^{FGM} \\ &= 1 + \theta\left(1 - 2u_1\right)\left(1 - 2u_2\right), \end{aligned}$$

and with this, the bivariate p.d.f. of $(X, W)$ is given by

$$\begin{aligned} f_{X,W}(x,t) &= c_\theta^{FGM}\Big(F_X(x), F_W(t)\Big) f_X(x) f_W(t) \\ &= f_X(x) f_W(t) + \theta h(x) f_W(t)\left[2\overline{F}_W(t) - 1\right],\ x, t \in \mathbb{R}^+, \end{aligned} \tag{5}$$

where $h(x) = f_X(x)\left[1 - 2F_X(x)\right]$ with L.T. $\widehat{h}(s) = \int_0^\infty e^{-sx} h(x) dx$. Using Eqs (1) and (2), Eq. (5) can be written as

$$\begin{aligned} f_{X,W}(x,t) &= f_X(x) \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} + \theta h(x)\Bigg[2 \frac{\lambda^n}{(n-1)!} t^{n-1} \Bigg(\sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!}\Bigg) e^{-2\lambda t} \\ &- \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t}\Bigg],\ x, t \in \mathbb{R}^+. \end{aligned} \tag{6}$$

In particular, from Eq. (6) the conditional p.d.f. of the claim size X is given by

$$f_{X|W}(x|t) = f_X(x) + \theta h(x)\Bigg[2\Bigg(\sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!}\Bigg) e^{-\lambda t} - 1\Bigg],\ x, t \in \mathbb{R}^+.$$

In the sequel we assume that $\theta \neq 0$, since otherwise our model reduces to the renewal risk model with Erlang(n) interclaim times.

Let $\tau = inf_{t \geq 0}\{t, U(t) < 0\}$ be the time of ruin with $\tau = \infty$ if $U(t) \geq 0$ for all $t \geq 0$ (i.e. ruin does not occur) and $\psi(u) = Pr(\tau < \infty | U(0) = u)$ be the ultimate ruin probability. To guarantee that ruin will not almost surely occur, the premium rate $c$ is such that

$$E[cW_i - X_i] > 0,\ i = 1, 2, \cdots,$$

providing a positive safety loading. This condition is equivalent to

$$c > \frac{\lambda}{n} E(X). \tag{7}$$

The main goal of this paper is the evaluation of the expected discounted penalty function introduced by Gerber and Shiu (1998) for the classical risk model and by Gerber and Shiu (2005) for the Sparre Andersen or renewal risk model. This function includes many other ruin measures and is defined by

$$m_\delta(u) = E[e^{-\delta\tau} w(U(\tau^-), |U(\tau)|) I(\tau < \infty) | U(0) = u],\ u \geq 0, \tag{8}$$

where $\delta \geq 0$ is interpreted as the force of interest or the Laplace argument, $U(\tau^-)$ is the surplus prior to ruin, $|U(\tau)|$ is the deficit at ruin, $w(x, y)$ is a non-negative bivariate function of $0 \leq x, y < \infty$ and $I(A)$ represents the indicator function of the event $A$. Note that when $w(x, y) = 1$ for all $x, y \geq 0$, then $m_\delta(u)$ reduces to the L.T. of the time to ruin, denoted by $m_\tau(u)$, i.e. $m_\tau(u) = E[e^{-\delta\tau} I(\tau < \infty) | U(0) = u]$ and in addition if $\delta = 0$ the $m_\delta(u)$ and hence $m_\tau(u)$ becomes the ruin probability $\psi(u) = E[I(\tau < \infty) | U(0) = u]$.

## 3 Analysis of a generalized Lundberg equation

In this Section we introduce a generalized version of the Lundberg equation for the Erlang$(n)$ risk process with dependence based on a FGM copula, and then we analyze the number of its roots in the right-half complex plane. These roots are needed to derive the defective renewal equation for the Gerber-Shiu function $m_\delta(u)$ as well as to evaluate several ruin quantities.

In order to derive Lundberg's generalized equation, we consider the corresponding discrete-time process embedded in the continuous-time surplus process $\{U(t); t \geq 0\}$. Let $V_0 = 0$ and $V_k = \sum_{i=1}^{k} W_i$, $k \geq 1$, be the arrival time of the k-th claim. Define the discrete-time process by $U_0 = u$ and for $k = 1, 2, \cdots$,

$$U_k = U(V_k) = u + cV_k - \sum_{i=1}^{k} X_i = u + \sum_{i=1}^{k} \Big( cW_i - X_i \Big),$$

to be the surplus immediately after the k-th claim. We seek a number $s$ such that the process $\{e^{-\delta V_k + sU_k}; k = 0, 1, 2, \cdots\}$ is a martingale. This process is a martingale if and only if

$$E[e^{-\delta W} e^{s(cW-X)}] = E[e^{(cs-\delta)W} e^{-sX}] = 1, \tag{9}$$

which is called the *generalized Lundberg equation* associated with our risk model. Note that by Eq. (5), the left-hand side of Eq. (9) can be written as

$$\begin{aligned} E\Big[e^{-\delta W} e^{s(cW-X)}\Big] &= \int_0^\infty \int_0^\infty e^{t(cs-\delta)} e^{-sx} f_{X,W}(x,t)\,dx\,dt \\ &= \int_0^\infty \int_0^\infty e^{t(cs-\delta)} e^{-sx} \left[ f_X(x) - \theta h(x) \right] f_W(t)\,dx\,dt \\ &+ 2\theta \int_0^\infty \int_0^\infty e^{t(cs-\delta)} e^{-sx} h(x) f_W(t) \overline{F}_W(t)\,dx\,dt \\ &= \Big[\widehat{f}(s) - \theta \widehat{h}(s)\Big] \widehat{f}_W(\delta - cs) + 2\theta \widehat{h}(s) \int_0^\infty e^{-t(\delta-cs)} f_W(t) \overline{F}_W(t)\,dt. \end{aligned} \tag{10}$$

From Eqs (1) and (2) we have

$$\begin{aligned} \int_0^\infty e^{-t(\delta-cs)} f_W(t)\overline{F}_W(t)\,dt &= \int_0^\infty e^{-t(\delta-cs)} \frac{\lambda^n}{(n-1)!} t^{n-1} \left( \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!} \right) e^{-2\lambda t}\,dt \\ &= \frac{\lambda^n}{(n-1)!} \sum_{i=0}^{n-1} \frac{\lambda^i}{i!} \int_0^\infty e^{-t(\delta+2\lambda-cs)} t^{n+i-1}\,dt \\ &= \frac{\lambda^n}{(n-1)!} \sum_{i=0}^{n-1} \frac{\lambda^i}{i!} \cdot \frac{(n+i-1)!}{(\delta+2\lambda-cs)^{n+i}} \\ &= \lambda^n \sum_{i=0}^{n-1} \binom{n+i-1}{i} \frac{\lambda^i}{(\delta+2\lambda-cs)^{n+i}}, \end{aligned}$$

and thus using this and Eq. (3), Eq. (10) becomes

$$E[e^{-\delta W} e^{s(cW-X)}] = \Big[\widehat{f}_X(s) - \theta \widehat{h}(s)\Big] \left( \frac{\lambda}{\delta+\lambda-cs} \right)^n + 2\theta \widehat{h}(s) \sum_{i=0}^{n-1} \binom{n+i-1}{i} \left( \frac{\lambda}{\delta+2\lambda-cs} \right)^{n+i}.$$

Then, Lundberg's generalized equation Eq. (9) reduces to

$$\widehat{f}_X(s)\left(\frac{\lambda}{\delta+\lambda-cs}\right)^n+\theta\widehat{h}(s)\left[2\sum_{i=0}^{n-1}\binom{n+i-1}{i}\left(\frac{\lambda}{\delta+2\lambda-cs}\right)^{n+i}-\left(\frac{\lambda}{\delta+\lambda-cs}\right)^n\right]=1. \quad (11)$$

When $n=1$, Eq. (11) simplifies to the Lundberg's generalized equation (Eq. 16) of Cossete et al. (2010), and it is shown that the Lundberg's generalized equation has exactly two positive roots with $Re(s)\geq 0$ when $\delta>0$ and $\theta\neq 0$.

**Proposition 1.** *For $\delta>0$ and $\theta\neq 0$, Lundberg's generalized equation in Eq. (11) has exactly $3n-1$ roots, say $\rho_1(\delta)$,$\rho_2(\delta)$,$\cdots$,$\rho_{3n-1}(\delta)$ in the right - half complex plane, i.e. with $Re(\rho_i(\delta))>0$, $i=1,2,\cdots,3n-1$.*

*Proof.* Since the generalized Lundberg equation (11) also becomes

$$\lambda^n\widehat{f}_X(s)\left(\delta+2\lambda-cs\right)^{2n-1} + \theta\lambda^n\widehat{h}(s)\Bigg[2\sum_{i=0}^{n-1}\binom{n+i-1}{i}\lambda^i\left(\delta+\lambda-cs\right)^n\left(\delta+2\lambda-cs\right)^{n-i-1}$$
$$- \left(\delta+2\lambda-cs\right)^{2n-1}\Bigg]=\left(\delta+\lambda-cs\right)^n\left(\delta+2\lambda-cs\right)^{2n-1}, \quad (12)$$

it suffices to show that the above Eq. (12) has exactly $3n-1$ roots with positive real parts. Let $r>0$ be a sufficiently large number, and denote by $C_r$ the contour containing the imaginary axis running from $-ir$ to $ir$ and a semicircle with radius $r$ running clockwise from $ir$ to $-ir$, i.e. $C_r=\{s\in\mathbb{C}:|s|=r, Re(s)\geq 0, r>0$ is fixed $\}$. We let $r\to\infty$ and denote by $C$ the limiting contour. To prove the result, we apply Rouché's theorem on the closed contour $C$. We distinguish two cases according to as $Re(s)>0$ or $Re(s)=0$.

For $s$ on the semicircle, i.e. for $Re(s)>0$, we have $|\delta+\lambda-cs|\to\infty$ and $|\delta+2\lambda-cs|\to\infty$ as $r\to\infty$, and thus

$$\left|\widehat{f}_X(s)\left(\frac{\lambda}{\delta+\lambda-cs}\right)^n+\theta\widehat{h}(s)\left[2\sum_{i=0}^{n-1}\binom{n+i-1}{i}\left(\frac{\lambda}{\delta+2\lambda-cs}\right)^{n+i}-\left(\frac{\lambda}{\delta+\lambda-cs}\right)^n\right]\right|$$

$$\leq\left|\widehat{f}_X(s)\right|\frac{\lambda^n}{|\delta+\lambda-cs|^n}+\left|\theta\widehat{h}(s)\right|\left|\left[2\sum_{i=0}^{n-1}\binom{n+i-1}{i}\left(\frac{\lambda}{\delta+2\lambda-cs}\right)^{n+i}-\left(\frac{\lambda}{\delta+\lambda-cs}\right)^n\right]\right|$$

$$\leq\left|\widehat{f}_X(s)\right|\frac{\lambda^n}{|\delta+\lambda-cs|^n}+\left|\theta\widehat{h}(s)\right|\left[2\sum_{i=0}^{n-1}\binom{n+i-1}{i}\frac{\lambda^{n+i}}{|\delta+2\lambda-cs|^{n+i}}+\frac{\lambda^n}{|\delta+\lambda-cs|^n}\right]\to 0$$

on $C$, i.e. for $r\to\infty$, and hence it holds that

$$\left|\widehat{f}_X(s)\left(\frac{\lambda}{\delta+\lambda-cs}\right)^n+\theta\widehat{h}(s)\left[2\sum_{i=0}^{n-1}\binom{n+i-1}{i}\left(\frac{\lambda}{\delta+2\lambda-cs}\right)^{n+i}-\left(\frac{\lambda}{\delta+\lambda-cs}\right)^n\right]\right|<1, \quad (13)$$

on $C$.

Let us define the random variable Z with probability density function given by $g_z(x)=2f_X(x)F_X(x)$, and thus $h(x)=f_X(x)-g_z(x)$. Let also $\widehat{g}_z(s)=\int_0^\infty e^{-sx}g_z(x)dx$ and

$$\widehat{d}_\delta(s)=2\sum_{i=0}^{n-1}\binom{n+i-1}{i}\left(\frac{\lambda}{\delta+2\lambda-cs}\right)^{n+i}-\left(\frac{\lambda}{\delta+\lambda-cs}\right)^n.$$

For $s$ on the imaginary axis, i.e., for $Re(s) = 0$, and for $\delta > 0$, similar to Cossette et al. (2008), we have

$$
\begin{aligned}
&\left| \widehat{f}_X(s) \left( \frac{\lambda}{\delta + \lambda - cs} \right)^n + \theta \widehat{h}(s) \left[ 2 \sum_{i=0}^{n-1} \binom{n+i-1}{i} \left( \frac{\lambda}{\delta + 2\lambda - cs} \right)^{n+i} - \left( \frac{\lambda}{\delta + \lambda - cs} \right)^n \right] \right| \\
&= \left| \widehat{f}_X(s) \left( \frac{\lambda}{\delta + \lambda - cs} \right)^n + \theta \left[ \widehat{f}_X(s) - \widehat{g}_z(s) \right] \widehat{d}_\delta(s) \right| \\
&\leq \left| \widehat{f}_X(s) \left( \frac{\lambda}{\delta + \lambda - cs} \right)^n \right| + \left| \theta \left[ \widehat{f}_X(s) - \widehat{g}_z(s) \right] \widehat{d}_\delta(s) \right| \\
&= \left| \widehat{f}_X(s) \right| \frac{\lambda^n}{|\delta + \lambda - cs|^n} + |\theta| \left| \widehat{f}_X(s) - \widehat{g}_z(s) \right| \left| \widehat{d}_\delta(s) \right| \\
&\leq \frac{\lambda^n}{|\delta + \lambda - cs|^n} + |\theta| \left| \widehat{d}_\delta(s) \right| \leq \left( \frac{\lambda}{\delta + \lambda} \right)^n + |\theta| \left| \widehat{d}_\delta(0) \right| \leq \left( \frac{\lambda}{\delta + \lambda} \right)^n + \left| \widehat{d}_\delta(0) \right|. \qquad (14)
\end{aligned}
$$

For $\delta > 0$, it holds $\widehat{d}_\delta(0) > 0$. Indeed,

$$
\begin{aligned}
\widehat{d}_\delta(0) &= 2 \sum_{i=0}^{n-1} \binom{n+i-1}{i} \left( \frac{\lambda}{\delta + 2\lambda} \right)^{n+i} - \left( \frac{\lambda}{\delta + \lambda} \right)^n \\
&= \frac{1}{2^{n-1}} \sum_{i=0}^{n-1} \binom{n+i-1}{i} \left( \frac{1}{2} \right)^i \left( \frac{2\lambda}{\delta + 2\lambda} \right)^{n+i} - \left( \frac{\lambda}{\delta + \lambda} \right)^n \\
&\geq \frac{1}{2^{n-1}} \left( \frac{2\lambda}{\delta + 2\lambda} \right)^n \sum_{i=0}^{n-1} \binom{n+i-1}{i} \left( \frac{1}{2} \right)^i - \left( \frac{\lambda}{\delta + \lambda} \right)^n \\
&= \left( \frac{2\lambda}{\delta + 2\lambda} \right)^n - \left( \frac{\lambda}{\delta + \lambda} \right)^n > 0,
\end{aligned}
$$

for $\delta > 0$, $n \geq 1$, where we use the well-known combinatorial identity

$$
\sum_{i=0}^{n} \binom{n+i}{i} \left( \frac{1}{2} \right)^i = 2^n, n \geq 0. \qquad (15)
$$

Therefore, for s on the imaginary axis, Eq. (14) becomes

$$
\begin{aligned}
&\left| \widehat{f}_X(s) \left( \frac{\lambda}{\delta + \lambda - cs} \right)^n + \theta \widehat{h}(s) \left[ 2 \sum_{i=0}^{n-1} \binom{n+i-1}{i} \left( \frac{\lambda}{\delta + 2\lambda - cs} \right)^{n+i} - \left( \frac{\lambda}{\delta + \lambda - cs} \right)^n \right] \right| \\
&\leq \left( \frac{\lambda}{\delta + \lambda} \right)^n + \widehat{d}_\delta(0) \\
&= \left( \frac{\lambda}{\delta + \lambda} \right)^n + 2 \sum_{i=0}^{n-1} \binom{n+i-1}{i} \left( \frac{\lambda}{\delta + 2\lambda} \right)^{n+i} - \left( \frac{\lambda}{\delta + \lambda} \right)^n
\end{aligned}
$$

$$= 2\left(\frac{\lambda}{\delta+2\lambda}\right)^n \sum_{i=0}^{n-1}\binom{n+i-1}{i}\left(\frac{\lambda}{\delta+2\lambda}\right)^i$$
$$< 2\left(\frac{\lambda}{\delta+2\lambda}\right)^n \sum_{i=0}^{n-1}\binom{n+i-1}{i}\left(\frac{1}{2}\right)^i, \text{ (since } \lambda/(\delta+2\lambda) < 1/2 \text{ for } \delta > 0)$$
$$= 2\left(\frac{\lambda}{\delta+2\lambda}\right)^n 2^{n-1}, \text{ (from Eq. (15))}$$
$$= \left(\frac{2\lambda}{\delta+2\lambda}\right)^n < 1, \text{ for } \delta > 0.$$

Therefore, in each case we proved that

$$\left|\widehat{f}_X(s)\left(\frac{\lambda}{\delta+\lambda-cs}\right)^n + \theta\widehat{h}(s)\left[2\sum_{i=0}^{n-1}\binom{n+i-1}{i}\left(\frac{\lambda}{\delta+2\lambda-cs}\right)^{n+i} - \left(\frac{\lambda}{\delta+\lambda-cs}\right)^n\right]\right| < 1, \tag{16}$$

or equivalently

$$\Bigg|\lambda^n\widehat{f}_X(s)\,(\delta+\lambda-cs)^{2n-1} + \theta\lambda^n\widehat{h}(s)\Bigg[2\sum_{i=0}^{n-1}\binom{n+i-1}{i}\lambda^i\,(\delta+\lambda-cs)^n\,(\delta+2\lambda-cs)^{n-i-1} - (\delta+2\lambda-cs)^{2n-1}\Bigg]\Bigg| < \left|(\delta+\lambda-cs)^n\,(\delta+2\lambda-cs)^{2n-1}\right|,$$

and thus by Rouché's theorem, it follows that Eq. (12) has the same number of roots as the following equation $(\delta+\lambda-cs)^n\,(\delta+2\lambda-cs)^{2n-1} = 0$ inside $C_r$. Since the latter equation has exactly $3n-1$ positive roots inside $C_r$, we deduce that Eq. (12) and equivalently Eq. (11) has exactly $3n-1$ roots, say $\rho_1(\delta), \ldots, \rho_{3n-1}(\delta)$ with positive real parts. Finally we complete the proof by letting $r \to \infty$. □

In the sequel, for simplicity we write $\rho_j$ for $\rho_j(\delta)$, $j = 1, 2, \cdots, 3n-1$, when $\delta > 0$.
**Remark**. For $\delta = 0$, the conditions to Rouché's theorem are not satisfied, since

$$\left|\widehat{f}_X(s)\left(\frac{\lambda}{\delta+\lambda-cs}\right)^n + \theta\widehat{h}(s)\left[2\sum_{i=0}^{n-1}\binom{n+i-1}{i}\left(\frac{\lambda}{\delta+2\lambda-cs}\right)^{n+i} - \left(\frac{\lambda}{\delta+\lambda-cs}\right)^n\right]\right|$$
$$= \left|1+\theta\left[2\left(\frac{1}{2}\right)^n\sum_{i=0}^{n-1}\binom{n+i-1}{i}\left(\frac{1}{2}\right)^i - 1\right]\right|$$
$$= \left|1+\theta\left[\left(\frac{1}{2}\right)^{n-1}2^{n-1} - 1\right]\right| = 1,$$

for $s = 0$ due to Eq. (15). Also, this shows that a trivial root to Lundberg's generalized equation (13) equals zero for $\delta = 0$. The importance of the case $\delta = 0$ is due to the evaluation among others of several ruin related quantities, such as the ruin probability, the defective joint distribution of the surplus prior to ruin and the deficit at ruin, as well as their joint moments, being special cases of the Gerber - Shiu penalty function at $\delta = 0$.

**Proposition 2.** *For $\delta = 0$ and for $\theta \neq 0$, Lundberg's generalized equation in (11) has exactly $3n-2$ roots in the right-half plane with positive real parts and one root equal to zero.*

*Proof.* Let $z = 1 - \frac{s}{k}$ and define $D_k = \{s : |z| = 1\}$, i.e. in terms of $s$ the contour $D_k$ is a circle with origin at $k$ and radius $k$. Similarly as in Proposition 1, we let $k \to \infty$ and denote by $D$ the limiting contour. Using identical arguments as in the proof of Proposition 1, one can show that Eq. (13) also holds on $D$ (excluding $s = 0$ or equivalently $z = 1$) for $\delta = 0$. Also note that the functions $\lambda^n \left\{ \widehat{f}_X(s)\,(2\lambda - cs)^{2n-1} + \theta \widehat{h}(s) \left[ \sum_{i=0}^{n-1} \binom{n+i-1}{i} \lambda^i (\lambda - cs)^n (2\lambda - cs)^{n-i-1} - (2\lambda - cs)^{2n-1} \right] \right\}$ and $(\lambda - cs)^n (2\lambda - cs)^{2n-1}$ are continuous on $D$. As in Cossette et al. (2010) in order to apply Theorem 1 of Klimenok (2001), we must prove that

$$\frac{d}{dz}\left\{1 - \widehat{f}_X(k-kz)\left(\frac{\lambda}{\lambda - ck(1-z)}\right)^n - \theta \widehat{h}(k-kz)\left[2\sum_{i=0}^{n-1}\binom{n+i-1}{i}\left(\frac{\lambda}{2\lambda - c(k-kz)}\right)^{n+i} - \left(\frac{\lambda}{\lambda - c(k-kz)}\right)^n\right]\right\}\Bigg|_{z=1} > 0.$$

The left-hand side of this relation is equal to

$$\frac{d}{dz}\left\{1 - E\left[e^{(k-kz)(cW-X)}\right]\right\}\Bigg|_{z=1} = kE\left[cW - X\right]$$

which is always positive under the security loading condition (see Eq. (7)).

Thus, from Klimenok (2001), we conclude that inside $D$, the number of roots of Eq. (12) and thus of Eq. (11) is equal to $3n - 2$, i.e. the number of roots of $(\lambda - cs)^n (2\lambda - cs)^{2n-1}$inside $D$ minus 1. Finally we have seen that a trivial root to Lundberg's generalized equation (11) equals zero. □

# 4 Laplace transform of $m_\delta(u)$

The main goal of this section is to derive the Laplace transform $\widehat{m}_\delta(s) = \int_0^\infty e^{-su} m_\delta(u) du$ of the Gerber-Shiu expected discounted penalty function $m_\delta(u)$ defined by Eq. (8). For $u \geq 0$, we define the following functions

$$\gamma_1(u) = \int_u^\infty w(u, x-u) f_X(x) dx, \quad \gamma_2(u) = \int_u^\infty w(u, x-u) h(x) dx \tag{17}$$

$$\sigma_{1,\delta}(u) = \int_0^u m_\delta(u-x) f_X(x) dx + \gamma_1(u), \quad \sigma_{2,\delta}(u) = \int_0^u m_\delta(u-x) h(x) dx + \gamma_2(u). \tag{18}$$

By conditioning on the time and the amount of the first claim, and making use of Eq. (5) we have

$$\begin{aligned} m_\delta(u) &= \int_0^\infty e^{-\delta t}\left\{\int_0^{u+ct} m_\delta(u+ct-x) f_{X,W}(x,t) dx \right. \\ &+ \left. \int_{u+ct}^\infty w(u+ct, x-u-ct) f_{X,W}(x,t) dx\right\} dt \\ &= \int_0^\infty e^{-\delta t} f_W(t)\Big[\sigma_{1,\delta}(u+ct) - \theta \sigma_{2,\delta}(u+ct)\Big] dt \\ &+ 2\theta \int_0^\infty e^{-\delta t} f_W(t) \overline{F}_W(t) \sigma_{2,\delta}(u+ct) dt. \end{aligned} \tag{19}$$

Setting $y = u + ct$, Eq. (19) yields

$$\begin{aligned} cm_\delta(u) &= \int_u^\infty e^{-\frac{\delta(y-u)}{c}} f_W\left(\frac{y-u}{c}\right)\Big[\sigma_{1,\delta}(y) - \theta \sigma_{2,\delta}(y)\Big] dy \\ &+ 2\theta \int_u^\infty e^{-\frac{\delta(y-u)}{c}} f_W\left(\frac{y-u}{c}\right) \overline{F}_W\left(\frac{y-u}{c}\right) \sigma_{2,\delta}(y) dy, \end{aligned}$$

from which by substituting $f_W(t)$ and $\overline{F}_W(t)$ from Eq. (1) and Eq. (2), we obtain

$$\begin{aligned} c^n m_\delta(u) &= \frac{\lambda^n}{(n-1)!}\int_u^\infty e^{-\frac{(\delta+\lambda)(y-u)}{c}}(y-u)^{n-1}\Big[\sigma_{1,\delta}(y)-\theta\sigma_{2,\delta}(y)\Big]dy \\ &+ \frac{2\theta\lambda^n}{(n-1)!}\int_u^\infty e^{-\frac{(\delta+2\lambda)(y-u)}{c}}(y-u)^{n-1}\Bigg[\sum_{i=0}^{n-1}\frac{\lambda^i}{i!}\left(\frac{y-u}{c}\right)^i\Bigg]\sigma_{2,\delta}(y)dy. \end{aligned}$$

Taking LTs gives

$$\begin{aligned} c^n \widehat{m}_\delta(s) &= \frac{\lambda^n}{(n-1)!}\int_0^\infty e^{-su}\int_u^\infty e^{-\frac{(\delta+\lambda)(y-u)}{c}}(y-u)^{n-1}\Big[\sigma_{1,\delta}(y)-\theta\sigma_{2,\delta}(y)\Big]dydu \\ &+ \frac{2\theta\lambda^n}{(n-1)!}\int_0^\infty e^{-su}\int_u^\infty e^{-\frac{(\delta+2\lambda)(y-u)}{c}}(y-u)^{n-1}\Bigg[\sum_{i=0}^{n-1}\frac{\lambda^i}{i!}\left(\frac{y-u}{c}\right)^i\Bigg]\sigma_{2,\delta}(y)dydu \\ &= \frac{\lambda^n}{(n-1)!}\int_0^\infty e^{-\frac{(\delta+\lambda)(y-u)}{c}}\Big[\sigma_{1,\delta}(y)-\theta\sigma_{2,\delta}(y)\Big]\int_0^y (y-u)^{n-1}e^{-(s-\frac{\delta+\lambda}{c})u}dudy \\ &+ \frac{2\theta\lambda^n}{(n-1)!}\int_0^\infty e^{-\frac{(\delta+2\lambda)y}{c}}\sigma_{2,\delta}(y)\sum_{i=0}^{n-1}\frac{\lambda^i}{c^i i!}\int_0^y (y-u)^{n-i-1}e^{-(s-\frac{\delta+2\lambda}{c})u}dudy. \end{aligned} \tag{20}$$

It can be easily proved (e.g., by induction) that the following equality holds for $\alpha>0$, $k=0,1,2,\cdots$

$$\int_0^y (y-u)^k e^{-\alpha u}du = \sum_{j=0}^{k}(-1)^j j!\binom{k}{j}\frac{y^{k-j}}{\alpha^{j+1}} + (-1)^{k+1}\frac{k!}{\alpha^{k+1}}e^{-\alpha y}. \tag{21}$$

Therefore, using Eq. (21), the relation (20) can be written in the form

$$\begin{aligned} c^n \widehat{m}_\delta(s) &= (-1)^n\frac{\lambda^n}{\left(s-\frac{\delta+\lambda}{c}\right)^n}\Big[\widehat{\sigma}_{1,\delta}(s)-\theta\widehat{\sigma}_{2,\delta}(s)\Big] \\ &+ 2\theta\lambda^n\sum_{i=0}^{n-1}(-1)^{n+i}\binom{n+i-1}{i}\frac{\lambda^i}{c^i\left(s-\frac{\delta+2\lambda}{c}\right)^{n+i}}\widehat{\sigma}_{2,\delta}(s)+\widehat{B}_\delta(s) \\ &= \frac{\lambda^n}{\left(\frac{\delta+\lambda}{c}-s\right)^n}\Big[\widehat{\sigma}_{1,\delta}(s)-\theta\widehat{\sigma}_{2,\delta}(s)\Big] \\ &+ 2\theta\lambda^n\sum_{i=0}^{n-1}\binom{n+i-1}{i}\frac{\lambda^i}{c^i\left(\frac{\delta+2\lambda}{c}-s\right)^{n+i}}\widehat{\sigma}_{2,\delta}(s)+\widehat{B}_\delta(s), \end{aligned} \tag{22}$$

where

$$\widehat{\sigma}_{i,\delta}(s) = \int_0^\infty e^{-su}\sigma_{i,\delta}(u)du,\ i=1,2,$$

and

$$\begin{aligned} \widehat{B}_\delta(s) &= \frac{\lambda^n}{(n-1)!}\sum_{j=0}^{n-1}(-1)^j\frac{j!\binom{n-1}{j}}{\left(s-\frac{\delta+\lambda}{c}\right)^{j+1}}\int_0^\infty y^{n-1-j}e^{-\frac{(\delta+\lambda)y}{c}}\left[\sigma_{1,\delta}(y)-\theta\sigma_{2,\delta}(y)\right]dy \\ &+ \frac{2\theta\lambda^n}{(n-1)!}\sum_{i=0}^{n-1}\frac{\lambda^i}{c^i i!}\sum_{j=0}^{n+i-1}(-1)^j\frac{j!\binom{n+i-1}{j}}{\left(s-\frac{\delta+2\lambda}{c}\right)^{j+1}}\int_0^\infty y^{n+i-1-j}e^{-\frac{(\delta+2\lambda)y}{c}}\sigma_{2,\delta}(y)dy. \end{aligned}$$

Let $\widehat{\gamma}_i(s) = \int_0^\infty e^{-su}\gamma_i(u)du$, $i = 1, 2$. Since from Eq. (18) it holds $\widehat{\sigma}_{1,\delta}(s) = \widehat{m}_\delta(s)\widehat{f}_X(s) + \widehat{\gamma}_1(s)$ and $\widehat{\sigma}_{2,\delta}(s) = \widehat{m}_\delta(s)\widehat{h}(s) + \widehat{\gamma}_2(s)$, the above equation (22) reduces to

$$\widehat{m}_\delta(s)\Bigg\{c^n - \frac{\lambda^n}{\left(\frac{\delta+\lambda}{c}-s\right)^n}\left[\widehat{f}_X(s)-\theta\widehat{h}(s)\right] - 2\theta\lambda^n\sum_{i=0}^{n-1}\binom{n+i-1}{i}\frac{\lambda^i}{c^i\left(\frac{\delta+2\lambda}{c}-s\right)^{n+i}}\widehat{h}(s)\Bigg\}$$
$$= \frac{\lambda^n}{\left(\frac{\delta+\lambda}{c}-s\right)^n}\left[\widehat{\gamma}_1(s)-\theta\widehat{\gamma}_2(s)\right] + 2\theta\lambda^n\sum_{i=0}^{n-1}\binom{n+i-1}{i}\frac{\lambda^i}{c^i\left(\frac{\delta+2\lambda}{c}-s\right)^{n+i}}\widehat{\gamma}_2(s) + \widehat{B}_\delta(s). \qquad (23)$$

Now using Eq. (23) we give in the following theorem an expression for $\widehat{m}_\delta(s)$.

**Theorem 1.** *In the Erlang(n) risk process with a dependence structure based on the FGM copula, the Laplace transform $\widehat{m}_\delta(s)$ of the Gerber-Shiu discounted penalty function $m_\delta(u)$, is given by*

$$\widehat{m}_\delta(s) = \frac{\widehat{\beta}_{1,\delta}(s)+\widehat{\beta}_{2,\delta}(s)}{\widehat{h}_{1,\delta}(s)-\widehat{h}_{2,\delta}(s)}, \qquad (24)$$

*where*

$$\widehat{h}_{1,\delta}(s) = \left(\frac{\delta+\lambda}{c}-s\right)^n\left(\frac{\delta+2\lambda}{c}-s\right)^{2n-1}, \qquad (25)$$

$$\widehat{h}_{2,\delta}(s) = \frac{\lambda^n}{c^n}\widehat{f}_X(s)\left(\frac{\delta+2\lambda}{c}-s\right)^{2n-1} + \theta\frac{\lambda^n}{c^n}\widehat{h}(s)\Bigg[2\left(\frac{\delta+\lambda}{c}-s\right)^n$$
$$\times\sum_{i=0}^{n-1}\frac{\lambda^i}{c^i}\binom{n+i-1}{i}\left(\frac{\delta+2\lambda}{c}-s\right)^{n-i-1} - \left(\frac{\delta+2\lambda}{c}-s\right)^{2n-1}\Bigg], \qquad (26)$$

$$\widehat{\beta}_{1,\delta}(s) = \frac{\lambda^n}{c^n}\widehat{\gamma}_1(s)\left(\frac{\delta+2\lambda}{c}-s\right)^{2n-1} + \theta\frac{\lambda^n}{c^n}\widehat{\gamma}_2(s)\Bigg[2\left(\frac{\delta+\lambda}{c}-s\right)^n$$
$$\times\sum_{i=0}^{n-1}\frac{\lambda^i}{c^i}\binom{n+i-1}{i}\left(\frac{\delta+2\lambda}{c}-s\right)^{n-i-1} - \left(\frac{\delta+2\lambda}{c}-s\right)^{2n-1}\Bigg], \qquad (27)$$

*and $\widehat{\beta}_{2,\delta}(s)$ is a polynomial in s of degree $3n-2$ or less, given by*

$$\widehat{\beta}_{2,\delta}(s) = -\sum_{j=1}^{3n-1}\widehat{\beta}_{1,\delta}(\rho_j)\prod_{k=1,k\neq j}^{3n-1}\frac{s-\rho_k}{\rho_j-\rho_k}.$$

*Proof.* Multiplying both sides of Eq. (23) by $\left(\frac{\delta+\lambda}{c}-s\right)^n\left(\frac{\delta+2\lambda}{c}-s\right)^{2n-1}/c^n$ and then solving the resulting equation for $\widehat{m}_\delta(s)$ we get immediately the Eq. (24), with

$$\begin{aligned}
\widehat{\beta}_{2,\delta}(s) &= \frac{1}{c^n}\widehat{h}_{1,\delta}(s)\widehat{B}_\delta(s) \\
&= -\Big\{\frac{\lambda^n}{c^n(n-1)!}\sum_{j=0}^{n-1} j!\binom{n-1}{j}\left(\frac{\delta+\lambda}{c}-s\right)^{n-j-1}\left(\frac{\delta+2\lambda}{c}-s\right)^{2n-1}\widehat{\mu}_j(\frac{\delta+\lambda}{c}) \\
&+ \frac{2\theta\lambda^n}{c^n(n-1)!}\sum_{i=0}^{n-1}\frac{\lambda^i}{c^i i!}\sum_{j=0}^{n+i-1} j!\binom{n+i-1}{j}\left(\frac{\delta+\lambda}{c}-s\right)^{n}\left(\frac{\delta+2\lambda}{c}-s\right)^{2n-j-2}\widehat{\delta}_{i,j}(\frac{\delta+2\lambda}{c})\Big\} \\
&= -\Big\{\frac{\lambda^n}{c^n(n-1)!}\sum_{j=0}^{n-1} j!\binom{n-1}{j}\left(\frac{\delta+\lambda}{c}-s\right)^{n-j-1}\left(\frac{\delta+2\lambda}{c}-s\right)^{2n-1}\widehat{\mu}_j(\frac{\delta+\lambda}{c}) \\
&+ \frac{2\theta\lambda^n}{c^n(n-1)!}\sum_{j=0}^{2n-2}\left(\sum_{i=max(0,j+1-n)}^{n-1}\frac{\lambda^i}{c^i i!}j!\binom{n+i-1}{j}\widehat{\delta}_{i,j}(\frac{\delta+2\lambda}{c})\right) \\
&\left(\frac{\delta+\lambda}{c}-s\right)^{n}\left(\frac{\delta+2\lambda}{c}-s\right)^{2n-j-2}\Big\},
\end{aligned}$$

which is a polynomial in s of degree $3n-2$ or less, where

$$\begin{aligned}
\widehat{\mu}_j(\frac{\delta+\lambda}{c}) &= \int_0^\infty y^{n-1-j}e^{-\frac{(\delta+\lambda)y}{c}}\left[\sigma_{1,\delta}(y)-\theta\sigma_{2,\delta}(y)\right]dy, \\
\widehat{\delta}_{i,j}(\frac{\delta+2\lambda}{c}) &= \int_0^\infty y^{n+i-1-j}e^{-\frac{(\delta+2\lambda)y}{c}}\sigma_{2,\delta}(y)dy.
\end{aligned}$$

It is easy to see that the Lundberg's generalized equation (11) can also be written in the form $\widehat{h}_{1,\delta}(s)-\widehat{h}_{2,\delta}(s)=0$, which means that $\rho_i$'s, $i=1,2,\cdots,3n-1$ are roots of the denominator in Eq. (24). Since $\widehat{m}_\delta(s)$ is analytic for $Re(s)\geq 0$ this implies that $\rho_i$'s, $i=1,2,\cdots,3n-1$ are also roots of the numerator in Eq. (24), and thus $\widehat{\beta}_{2,\delta}(\rho_i)=-\widehat{\beta}_{1,\delta}(\rho_i)$, $i=1,2,\cdots,3n-1$. Since $\widehat{\beta}_{2,\delta}(s)$ is a polynomial in s of degree 3n-2, by the Lagrange interpolation formula at the $3n-1$ points $\rho_1,\rho_2,\cdots,\rho_{3n-1}$, we have

$$\widehat{\beta}_{2,\delta}(s)=\sum_{j=1}^{3n-1}\widehat{\beta}_{2,\delta}(\rho_j)\prod_{k=1,k\neq j}^{3n-1}\frac{s-\rho_k}{\rho_j-\rho_k}=-\sum_{j=1}^{3n-1}\widehat{\beta}_{1,\delta}(\rho_j)\prod_{k=1,k\neq j}^{3n-1}\frac{s-\rho_k}{\rho_j-\rho_k},$$

and hence the proof is completed. □

Note that for $n=1$, our Theorem 1 is reduced to Proposition 6.1 of Cossette et al. (2010).

## 5 Analysis of the Gerber - Shiu penalty function when $u=0$

In this section we examine some ruin quantities by considering the case of $u=0$. The roots of the Lundberg's generalized equation, studied in Section 3, play an important role in the rest of this paper. In what follows, we only consider the case that the roots $\rho_1$,$\rho_2$,$\cdots$,$\rho_{3n-1}$ are all distinct since the analysis of the other case (which can be made in a similar manner) will not impose any technical obstacle, but the computational procedure will become more tedious and does not offer further insight except the complexity of the results. By applying the initial value theorem, we get

$$
\begin{aligned}
m_\delta(0) &= \lim_{s\to\infty} s\widehat{m}(s) = \lim_{s\to\infty} \frac{\frac{\widehat{\beta}_{1,\delta}(s)}{s^{3n-2}} - \frac{1}{s^{3n-2}}\sum_{j=1}^{3n-1}\widehat{\beta}_{1,\delta}(\rho_j)\prod_{k=1,k\neq j}^{3n-1}\frac{s-\rho_k}{\rho_j-\rho_k}}{\frac{\widehat{h}_{1,\delta}(s)}{s^{3n-1}} - \frac{\widehat{h}_{2,\delta}(s)}{s^{3n-1}}} \\
&= \frac{-\lim_{s\to\infty}\frac{1}{s^{3n-2}}\sum_{j=1}^{3n-1}\widehat{\beta}_{1,\delta}(\rho_j)\prod_{k=1,k\neq j}^{3n-1}\frac{s-\rho_k}{\rho_j-\rho_k}}{(-1)^{3n-1}} \\
&= \frac{-\sum_{j=1}^{3n-1}\widehat{\beta}_{1,\delta}(\rho_j)\prod_{k=1,k\neq j}^{3n-1}\frac{1}{\rho_j-\rho_k}}{(-1)^{3n-1}} \\
&= \sum_{j=1}^{3n-1}\frac{\widehat{\beta}_{1,\delta}(\rho_j)}{\prod_{k=1,k\neq j}^{3n-1}(\rho_k-\rho_j)}. \qquad (28)
\end{aligned}
$$

If $n = 1$, then Eq. (28) simplifies to Corollary 6.1 in Cossette et al. (2010).
Let

$$
b_{1,\delta}(s) = \frac{\lambda^n}{c^n}\left(\frac{\delta+2\lambda}{c} - s\right)^{2n-1}, \qquad (29)
$$

and

$$
b_{2,\delta}(s) = \theta\frac{\lambda^n}{c^n}\left[2\left(\frac{\delta+\lambda}{c} - s\right)^n \sum_{i=0}^{n-1}\frac{\lambda^i}{c^i}\binom{n+i-1}{n-1}\left(\frac{\delta+2\lambda}{c} - s\right)^{n-i-1} - \left(\frac{\delta+2\lambda}{c} - s\right)^{2n-1}\right]. \qquad (30)
$$

Then from Eq. (27) we have

$$
\widehat{\beta}_{1,\delta}(s) = b_{1,\delta}(s)\widehat{\gamma}_1(s) + b_{2,\delta}(s)\widehat{\gamma}_2(s). \qquad (31)
$$

Let also

$$
b_{ij} = \frac{b_{i,\delta}(\rho_j)}{\prod_{k=1\,k\neq j}^{3n-1}(\rho_k-\rho_j)}, \qquad (32)
$$

for $i = 1, 2$ and $j = 1, 2, ..., 3n - 1$. Then Eq. (28), using Eqs (31) and (32), is written as

$$
m_\delta(0) = \sum_{j=1}^{3n-1}\frac{b_{1,\delta}(\rho_j)\widehat{\gamma}_1(\rho_j) + b_{2,\delta}(\rho_j)\widehat{\gamma}_2(\rho_j)}{\prod_{k=1\,k\neq j}^{3n-1}(\rho_k-\rho_j)} = \sum_{i=1}^{2}\sum_{j=1}^{3n-1} b_{i,j}\widehat{\gamma}_i(\rho_j). \qquad (33)
$$

Since (see Eq. (17)) $\gamma_1(x) = \int_x^\infty w(x, y-x)f_X(y)dy = \int_0^\infty w(x,y)f_X(x+y)dy$ we have that $\widehat{\gamma}_1(s) = \int_0^\infty e^{-sx}\gamma_1(x)dx = \int_0^\infty\int_0^\infty e^{-sx}w(x,y)f_X(x+y)dydx$ and similarly that $\widehat{\gamma}_2(s) = \int_0^\infty\int_0^\infty e^{-sx}w(x,y)h(x+y)dydx$. Therefore from Eq. (33) we obtain

$$
m_\delta(0) = \int_0^\infty\int_0^\infty w(x,y)\left[f_X(x+y)\sum_{j=1}^{3n-1} b_{1,j}e^{-\rho_j x} + h(x+y)\sum_{j=1}^{3n-1} b_{2,j}e^{-\rho_j x}\right]dydx. \qquad (34)
$$

Let $f(x, y, t|0)$ be the joint defective probability density function of the surplus prior to ruin $(x)$, the deficit at ruin $(y)$ and the time of ruin $(t)$ given $U(0) = 0$, and $f_\delta(x, y|0)$ be the discounted (marginal if $\delta \to 0$) probability density function of the surplus prior to ruin and the deficit at ruin given $U(0) = 0$. Then we have

$$
f_\delta(x,y|0) = \int_0^\infty e^{-\delta t} f(x,y,t|0)dt.
$$

From Eq. (16) of Cheung et al. (2010) and for $u = 0$, it follows that

$$m_\delta(0) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta t} w(x,y) f(x,y,t|0) dt dy dx = \int_0^\infty \int_0^\infty w(x,y) f_\delta(x,y|0) dy dx, \quad (35)$$

which combined with Eq. (34) yields

$$f_\delta(x,y|0) = f_X(x+y) \sum_{j=1}^{3n-1} b_{1,j} e^{-\rho_j x} + h(x+y) \sum_{j=1}^{3n-1} b_{2,j} e^{-\rho_j x}. \quad (36)$$

In the sequel, we need to introduce the well-known Dickson-Hipp operator $T_r$ of a real-valued integrable function g defined as

$$T_r g(x) = \int_x^\infty e^{-r(y-x)} g(y) dy, x \geq 0, r \in \mathbb{C},$$

where r has a non-negative real part, i.e. $Re(r) \geq 0$. Some useful properties of the operator $T_r$ needed in the paper are listed below:

1. $T_r g(0) = \int_0^\infty e^{-ry} g(y) dy = \widehat{g}(r), r \in \mathbb{C}.$
2. $T_{r_1} T_{r_2} g(x) = T_{r_2} T_{r_1} g(x) = \dfrac{T_{r_1} g(x) - T_{r_2} g(x)}{r_2 - r_1}, x \geq 0, r_1 \neq r_2 \in \mathbb{C}.$
3. $(\widehat{T_r g})(s) = T_r \widehat{g}(s) = T_s T_r g(0) = \dfrac{\widehat{g}(s) - \widehat{g}(r)}{r - s}, r \neq s \in \mathbb{C}.$
4. If $r_1, r_2, \cdots, r_m$ are distinct complex numbers and $\pi_m(s) = \prod_{i=1}^m (s - r_i)$, then

$$T_{r_1} \cdots T_{r_m} g(x) = (-1)^m \sum_{i=1}^m \frac{T_{r_i} g(x)}{\pi'_m(r_i)}, \ x \geq 0, \quad (37)$$

and the corresponding Laplace Transform is

$$T_s T_{r_1} \cdots T_{r_m} g(0) = (-1)^m \left[ \frac{\widehat{g}(s)}{\pi_m(s)} - \sum_{i=1}^m \frac{\widehat{g}(r_i)}{(s - r_i)\pi'_m(r_i)} \right], \ s \in \mathbb{C}. \quad (38)$$

Further properties of this operator can be found in Li and Garrido (2004) and the references therein.

Let $f_{1,\delta}(x|0) = \int_0^\infty f_\delta(x,y|0) dy$ be the discounted (marginal if $\delta \to 0$) probability density function of the surplus prior to ruin and $f_{2,\delta}(y|0) = \int_0^\infty f_\delta(x,y|0) dx$ be the discounted (marginal if $\delta \to 0$) probability density function of the deficit at ruin given $U(0) = 0$. Since

$$\int_0^\infty h(x+y) dy = \int_x^\infty h(y) dy = \int_x^\infty f_X(y) \Big[ 1 - 2F_X(y) \Big] dy = -F_X(x) \bar{F}_X(x),$$

from Eq. (36) we get

$$f_{1,\delta}(x|0) = \int_0^\infty f_\delta(x,y|0) dy = \bar{F}_X(x) \Bigg[ \sum_{j=1}^{3n-1} b_{1,j} e^{-\rho_j x} - F_X(x) \sum_{j=1}^{3n-1} b_{2,j} e^{-\rho_j x} \Bigg],$$

and

$$
\begin{aligned}
f_{2,\delta}(y|0) &= \int_0^\infty f_\delta(x,y|0)dx \\
&= \sum_{j=1}^{3n-1} b_{1,j}\int_0^\infty e^{-\rho_j x} f_X(x+y)dx + \sum_{j=1}^{3n-1} b_{2,j}\int_0^\infty e^{-\rho_j x} h(x+y)dx \\
&= \sum_{j=1}^{3n-1} b_{1,j} T_{\rho_j} f_X(y) + \sum_{j=1}^{3n-1} b_{2,j} T_{\rho_j} h(y).
\end{aligned}
$$

The LT of $f_{2,\delta}(y|0)$ is given by

$$
\begin{aligned}
\widehat{f}_{2,\delta}(s) &= \int_0^\infty e^{-sy} f_{2,\delta}(y|0)dy = T_s f_{2,\delta}(0|0) \\
&= \sum_{j=1}^{3n-1} b_{1,j} T_s T_{\rho_j} f_X(0) + \sum_{j=1}^{3n-1} b_{2,j} T_s T_{\rho_j} h(0) \\
&= \sum_{j=1}^{3n-1} \frac{b_{1,j}\widehat{f}_X(\rho_j) + b_{2,j}\widehat{h}(\rho_j)}{s-\rho_j} - \widehat{f}_X(s)\sum_{j=1}^{3n-1}\frac{b_{1,j}}{s-\rho_j} - \widehat{h}(s)\sum_{j=1}^{3n-1}\frac{b_{2,j}}{s-\rho_j}. \qquad (39)
\end{aligned}
$$

Using Eqs (25), (29) and (30) it follows that $\widehat{h}_{2,\delta}(s) = b_{1,\delta}(s)\widehat{f}_X(s) + b_{2,\delta}(s)\widehat{h}(s)$, and thus for $j = 1, \cdots, 3n-1$ it holds

$$
\begin{aligned}
b_{1,j}\widehat{f}_X(\rho_j) + b_{2,j}\widehat{h}(\rho_j) &= \frac{b_{1,\delta}(\rho_j)\widehat{f}_X(\rho_j) + b_{2,\delta}(\rho_j)\widehat{h}(\rho_j)}{\prod_{k=1\, k\neq j}^{3n-1}(\rho_k-\rho_j)} = \frac{\widehat{h}_{2,\delta}(\rho_j)}{\prod_{k=1\, k\neq j}^{3n-1}(\rho_k-\rho_j)} \\
&= \frac{\widehat{h}_{1,\delta}(\rho_j)}{\prod_{k=1\, k\neq j}^{3n-1}(\rho_k-\rho_j)},
\end{aligned}
$$

and hence from Eqs (25) and (39) we have that

$$
\widehat{f}_{2,\delta}(s) = \sum_{j=1}^{3n-1} \frac{(\delta+\lambda-c\rho_j)^n\,(\delta+2\lambda-c\rho_j)^{2n-1}}{c^{3n-1}(s-\rho_j)\prod_{k=1\, k\neq j}^{3n-1}(\rho_k-\rho_j)} - \widehat{f}(s)\sum_{j=1}^{3n-1}\frac{b_{1,j}}{s-\rho_j} - \widehat{h}(s)\sum_{j=1}^{3n-1}\frac{b_{2,j}}{s-\rho_j}. \qquad (40)
$$

Using a similar argument from interpolation theory as in Li and Garrido (2005, see their Eqs (17) and (18)), it can be easily proved that the following identities hold

$$
\sum_{j=1}^{3n-1} \frac{(\delta+\lambda-c\rho_j)^n\,(\delta+2\lambda-c\rho_j)^{2n-1}}{c^{3n-1}(s-\rho_j)\prod_{k=1\, k\neq j}^{3n-1}(\rho_k-\rho_j)} = 1 - \frac{(\delta+\lambda-cs)^n\,(\delta+2\lambda-cs)^{2n-1}}{c^{3n-1}\prod_{i=1}^{3n-1}(\rho_i-s)}, \qquad (41)
$$

$$
\begin{aligned}
\sum_{j=1}^{3n-1}\frac{b_{1,j}}{s-\rho_j} &= \sum_{j=1}^{3n-1}\frac{b_{1,\delta}(\rho_j)}{(s-\rho_j)\prod_{k=1,k\neq j}^{3n-1}(\rho_k-\rho_j)} = \sum_{j=1}^{3n-1}\frac{\lambda^n\,(\delta+2\lambda-c\rho_j)^{2n-1}}{c^{3n-1}(s-\rho_j)\prod_{k=1,k\neq j}^{3n-1}(\rho_k-\rho_j)} \\
&= -\frac{\lambda^n\,(\delta+2\lambda-cs)^{2n-1}}{c^{3n-1}\prod_{i=1}^{3n-1}(\rho_i-s)}, \qquad (42)
\end{aligned}
$$

and

$$\begin{aligned}\sum_{j=1}^{3n-1} \frac{b_{2,j}}{s-\rho_j} &= \sum_{j=1}^{3n-1} \frac{b_{2,\delta}(\rho_j)}{(s-\rho_j)\prod_{k=1,k\neq j}^{3n-1}(\rho_k-\rho_j)} \\ &= \sum_{j=1}^{3n-1} \frac{\theta\lambda^n\left[2\left(\delta+\lambda-c\rho_j\right)^n\sum_{i=0}^{n-1}\binom{n+i-1}{n-1}\lambda^i\left(\delta+2\lambda-c\rho_j\right)^{n-i-1}-\left(\delta+2\lambda-c\rho_j\right)^{2n-1}\right]}{c^{3n-1}(s-\rho_j)\prod_{k=1\,k\neq j}^{3n-1}(\rho_k-\rho_j)} \\ &= -\frac{\theta\lambda^n\left[2\left(\delta+\lambda-cs\right)^n\sum_{i=0}^{n-1}\binom{n+i-1}{n-1}\lambda^i\left(\delta+2\lambda-cs\right)^{n-i-1}-\left(\delta+2\lambda-cs\right)^{2n-1}\right]}{c^{3n-1}\prod_{i=1}^{3n-1}(\rho_i-s)}. \quad (43)\end{aligned}$$

Therefore, Eq. (40) becomes

$$\begin{aligned}\widehat{f}_{2,\delta}(s) &= 1-\frac{1}{c^{3n-1}\prod_{i=1}^{3n-1}(\rho_i-s)}\Bigg\{(\delta+\lambda-cs)^n(\delta+2\lambda-cs)^{2n-1}-\lambda^n(\delta+2\lambda-cs)^{2n-1}\widehat{f}(s) \\ &- \theta\lambda^n\left[2(\delta+\lambda-cs)^n\sum_{i=0}^{n-1}\binom{n+i-1}{n-1}\lambda^i(\delta+2\lambda-cs)^{n-i-1}-(\delta+2\lambda-cs)^{2n-1}\right]\widehat{h}(s)\Bigg\},\end{aligned}$$

i.e., the LT of $f_{2,\delta}(y|0)$ reduces to

$$\widehat{f}_{2,\delta}(s) = 1-\frac{\widehat{h}_{1,\delta}(s)-\widehat{h}_{2,\delta}(s)}{\prod_{i=1}^{3n-1}(\rho_i-s)}. \quad (44)$$

Setting $w(x,y)=1$, Eq. (35) implies that the LT of the time of ruin $m_\tau$ given $U(0)=0$, is

$$\begin{aligned}m_\tau(0) &= E\left[e^{-\delta\tau}I(\tau<\infty)|U(0)=0\right] = \int_0^\infty\int_0^\infty f_\delta(x,y|0)dydx \\ &= \int_0^\infty f_{2,\delta}(y|0)dy = \lim_{s\to 0}\widehat{f}_{2,\delta}(s) = 1-\frac{\widehat{h}_{1,\delta}(0)-\widehat{h}_{2,\delta}(0)}{\rho_1\rho_2\ldots\rho_{3n-1}} \\ &= 1-\frac{(\delta+2\lambda)^{2n-1}\left[(\delta+\lambda)^n-\lambda^n\right]}{c^{3n-1}\prod_{i=1}^{3n-1}\rho_i}, \quad (45)\end{aligned}$$

since $\widehat{f}(0)=1$, $\widehat{h}(0)=0$. Also from Eq. (45) it follows that $m_\tau(0)<1$ due to $\delta>0$.
Now for the ruin probability $\psi(0)$ given the initial surplus $U(0)=0$, we have

$$\begin{aligned}\psi(0) &= \lim_{\delta\to 0^+} E\left[e^{-\delta\tau}I(\tau<\infty)|U(0)=u\right] \\ &= 1-\lim_{\delta\to 0^+}\frac{(\delta+2\lambda)^{2n-1}\left[(\delta+\lambda)^n-\lambda^n\right]}{c^{3n-1}\prod_{i=1}^{3n-1}\rho_i} \\ &= 1-\frac{n2^{2n-1}\lambda^{3n-2}}{c^{3n-1}\rho_1^{'}(0)\rho^*(0)},\end{aligned}$$

where $\rho^*(0)=\prod_{i=2}^{3n-1}\rho_i(0)$ and $\rho_1^{'}(0)=\frac{d}{d\delta}\rho_1(\delta)\Big|_{\delta\to 0^+}$. In order to find the quantity $\rho_1^{'}(0)$, we shall use the fact that $\rho_1(\delta)$ is a root of the denominator of Eq. (24). Thus we have $\widehat{h}_1(\rho_1(\delta))=\widehat{h}_2(\rho_1(\delta))$ from which by differentiating w.r.t. $\delta$ and then letting $\delta \to 0^+$, we get that

$$\begin{aligned} n2^{2n-1}\lambda^{3n-2}[1-c\rho_1'(0)] &+ (2n-1)2^{2n-2}\lambda^{3n-2}[1-c\rho_1'(0)] \\ &= (2n-1)2^{2n-2}\lambda^{3n-2} + 2^{2n-1}\lambda^{3n-1}\left[\rho_1'(0)\widehat{f}\,'(0) - \theta\rho_1'(0)\widehat{h}'(0)\right] \\ &+ \theta 2^n\lambda^{3n-1}\rho_1'(0)\widehat{h}'(0)\sum_{i=0}^{n-1}\binom{n+i-1}{n-1}\left(\frac{1}{2}\right)^i, \end{aligned}$$

and since $\widehat{f}\,'(0) = -E(X)$, the above relation with the help of Eq. (15) finally yields that

$$\rho_1'(0) = \frac{n}{nc-\lambda E(X)} = \frac{E(W)}{cE(W)-E(X)},$$

which is always positive due to the positive loading condition (see Eq. (7)). Therefore it holds that

$$\psi(0) = 1 - \frac{2^{2n-1}\lambda^{3n-2}\left[nc-\lambda E(X)\right]}{c^{3n-1}\rho^*(0)} < 1.$$

# 6 Defective Renewal Equation

Gerber and Shiu (2005) and Li and Garrido (2005) show that the defective renewal equation approach of Gerber and Shiu (1998) in the classical compound Poisson risk model, can be extended to the ordinary renewal risk process. Using similar arguments, i.e. by conditioning on the first drop in the surplus below its initial level $u \geq 0$ and whether ruin occurs on the first claim or not, Cheung et al. (2010) obtained an integral equation (see their Eq. (15)) for the generalized expected discounted penalty function including the surplus prior to ruin, the deficit at ruin, the minimum surplus before ruin time and the surplus immediately after the second last claim before ruin occurs. Therefore by taking $w(x,y,z,v) = w(x,y)$ in Eqs (15) and (12) of Cheung et al. (2010) we obtain that $m_\delta(u)$ satisfies the following integral equation

$$\begin{aligned} m_\delta(u) &= \int_0^u m_\delta(u-y)\left\{\int_0^\infty f_\delta(x,y|0)dx\right\}dy + G_\delta(u) \\ &= \int_0^u m_\delta(u-y)f_{2,\delta}(y|0)dy + G_\delta(u),\ u \geq 0. \end{aligned} \tag{46}$$

where $f_{2,\delta}(y|0)$ is given by Eq. (39) and

$$\begin{aligned} G_\delta(u) &= \int_u^\infty\int_0^\infty w(x+u,y-u)f_\delta(x,y|0)dxdy \\ &= \int_0^\infty\int_u^\infty w(s,t)f_\delta(s-u,t+u|0)dsdt. \end{aligned} \tag{47}$$

Since $\int_0^\infty f_{2,\delta}(y|0)dy = m_\tau(0) < 1$ (from Eq. (45)), Eq. (46) is a defective renewal equation. In the sequel, at first we give an alternative expression for $f_{2,\delta}(y|0)$. From Eq. (25), $\widehat{h}_{1,\delta}(s)$ is a polynomial of degree $3n-1$ in $s$. Using the Lagrange interpolating formula for $\widehat{h}_{1,\delta}(s)$ (which is a polynomial that passes through the $3n$ points $(0,\widehat{h}_{1,\delta}(0))$, $(\rho_j,\widehat{h}_{1,\delta}(\rho_j))$, $j=1,\cdots,3n-1$), we get that

$$\widehat{h}_{1,\delta}(s) = \widehat{h}_{1,\delta}(0)\prod_{k=1}^{3n-1}\frac{s-\rho_k}{(-\rho_k)} + s\sum_{j=1}^{3n-1}\frac{\widehat{h}_{1,\delta}(\rho_j)}{\rho_j}\prod_{k=1,k\neq j}^{3n-1}\frac{s-\rho_k}{\rho_j-\rho_k}.$$

Using Eq. (38) and similar arguments as in the page 15 of Cossette et al. (2010), the aforementioned relation implies that

$$\widehat{h}_{1,\delta}(s)-\widehat{h}_{2,\delta}(s)=\pi_{3n-1}(s)\Big[\frac{\widehat{h}_{1,\delta}(0)}{\pi_{3n-1}(0)}-\sum_{j=1}^{3n-1}\frac{\widehat{h}_{2,\delta}(\rho_j)}{(-\rho_j)\pi'_{3n-1}(\rho_j)}+\sum_{j=1}^{3n-1}\frac{\widehat{h}_{2,\delta}(\rho_j)}{(s-\rho_j)\pi'_{3n-1}(\rho_j)}-\frac{\widehat{h}_{2,\delta}(s)}{\pi_{3n-1}(s)}\Big], \tag{48}$$

where $\pi_{3n-1}(s)=\prod_{i=1}^{3n-1}(s-\rho_i)$. Since $\widehat{h}_{2,\delta}(\rho_j)=\widehat{h}_{1,\delta}(\rho_j)$, $j=1,2,\cdots,3n-1$, from Eqs (25) and (41) for $s=0$, we have that

$$\begin{aligned}
\frac{\widehat{h}_{1,\delta}(0)}{\tau(0)}+\sum_{j=1}^{3n-1}\frac{\widehat{h}_{2,\delta}(\rho_j)}{\rho_j\tau'(\rho_j)} &= \frac{\left(\frac{\delta+\lambda}{c}\right)^n\left(\frac{\delta+2\lambda}{c}\right)^{2n-1}}{\prod_{i=1}^{3n-1}(-\rho_i)}+\sum_{j=1}^{3n-1}\frac{\left(\frac{\delta+\lambda}{c}-\rho_j\right)^n\left(\frac{\delta+2\lambda}{c}-\rho_j\right)^{2n-1}}{\rho_j\prod_{k=1,k\neq j}^{3n-1}(\rho_j-\rho_k)}\\
&= \frac{(\delta+\lambda)^n(\delta+2\lambda)^{2n-1}}{c^{3n-1}\prod_{i=1}^{3n-1}(-\rho_i)}+(-1)^{3n-1}\Big\{1-\frac{(\delta+\lambda)^n(\delta+2\lambda)^{2n-1}}{c^{3n-1}\prod_{i=1}^{3n-1}\rho_i}\Big\}\\
&= (-1)^{3n-1}.
\end{aligned}$$

Therefore Eq. (48) becomes

$$\widehat{h}_{1,\delta}(s)-\widehat{h}_{2,\delta}(s)=(-1)^{3n-1}\pi_{3n-1}(s)\Big[1-T_sT_{\rho_1}\cdots T_{\rho_{3n-1}}h_{2,\delta}(0)\Big]. \tag{49}$$

Furthermore from Eqs (44) and (49) we obtain that

$$\begin{aligned}
\widehat{f}_{2,\delta}(s) &= 1-\frac{\widehat{h}_{1,\delta}(s)-\widehat{h}_{2,\delta}(s)}{\prod_{i=1}^{3n-1}(\rho_i-s)}\\
&= 1-\frac{(-1)^{3n-1}\pi_{3n-1}(s)\Big[1-T_sT_{\rho_1}\cdots T_{\rho_{3n-1}}h_{2,\delta}(0)\Big]}{(-1)^{3n-1}\pi_{3n-1}(s)}\\
&= T_sT_{\rho_1}\cdots T_{\rho_{3n-1}}h_{2,\delta}(0),
\end{aligned} \tag{50}$$

and thus by inverting this we get the following alternative expression for $f_{2,\delta}(y|0)$,

$$f_{2,\delta}(y|0)=T_{\rho_1}\cdots T_{\rho_{3n-1}}h_{2,\delta}(y),$$

which can be easily computed using Eq. (37) from the fourth property of the Dickson - Hipp operator. Also from Eqs (47) and (36) we have that

$$\begin{aligned}
G_\delta(u) &= \int_0^\infty \int_u^\infty w(s,t)\Big[f_X(s+t)\sum_{j=1}^{3n-1} b_{1,j}e^{-\rho_j(s-u)} + h(s+t)\sum_{j=1}^{3n-1} b_{2,j}e^{-\rho_j(s-u)}\Big]dsdt \\
&= \sum_{j=1}^{3n-1} b_{1,j}\int_u^\infty e^{-\rho_j(s-u)}\int_0^\infty w(s,t)f_X(s+t)dtds \\
&+ \sum_{j=1}^{3n-1} b_{2,j}\int_u^\infty e^{-\rho_j(s-u)}\int_0^\infty w(s,t)h(s+t)dtds \\
&= \sum_{j=1}^{3n-1} b_{1,j}\int_u^\infty e^{-\rho_j(s-u)}\gamma_1(s)ds + \sum_{j=1}^{3n-1} b_{2,j}\int_u^\infty e^{-\rho_j(s-u)}\gamma_2(s)ds \\
&= \sum_{i=1}^{2}\sum_{j=1}^{3n-1} b_{i,j}T_{\rho_j}\gamma_i(u). \qquad (51)
\end{aligned}$$

Now we shall give an alternative expression for the function $G_\delta(u)$. From Eq. (51) it follows that

$$\begin{aligned}
\widehat{G}_\delta(s) &= \int_0^\infty e^{-su}G_\delta(u)du = T_sG_\delta(0) = \sum_{i=1}^{2}\sum_{j=1}^{3n-1} b_{i,j}T_sT_{\rho_j}\gamma_i(0) \\
&= \sum_{j=1}^{3n-1}\frac{b_{1,j}\widehat{\gamma}_1(\rho_j)+b_{2,j}\widehat{\gamma}_2(\rho_j)}{(s-\rho_j)} - \widehat{\gamma}_1(s)\sum_{j=1}^{3n-1}\frac{b_{1,j}}{s-\rho_j} - \widehat{\gamma}_2(s)\sum_{j=1}^{3n-1}\frac{b_{2,j}}{s-\rho_j} \\
&= \sum_{j=1}^{3n-1}\frac{b_{1,\delta}(\rho_j)\widehat{\gamma}_1(\rho_j)+b_{2,\delta}(\rho_j)\widehat{\gamma}_2(\rho_j)}{(s-\rho_j)\prod_{k=1,k\neq j}^{3n-1}(\rho_k-\rho_j)} - \widehat{\gamma}_1(s)\sum_{j=1}^{3n-1}\frac{b_{1,j}}{s-\rho_j} - \widehat{\gamma}_2(s)\sum_{j=1}^{3n-1}\frac{b_{2,j}}{s-\rho_j},
\end{aligned}$$

where we use Eq. (32). Now from Eqs (31), (42) and (43) we get that

$$\begin{aligned}
\widehat{G}_\delta(s) &= \sum_{j=1}^{3n-1}\frac{\widehat{\beta}_{1,\delta}(\rho_j)}{(s-\rho_j)\prod_{k=1,k\neq j}^{3n-1}(\rho_k-\rho_j)} + \frac{\lambda^n(\delta+2\lambda-cs)^{2n-1}}{c^{3n-1}\prod_{i=1}^{3n-1}(\rho_i-s)} \\
&+ \theta\frac{\lambda^n\Big[2(\delta+\lambda-cs)^n\sum_{i=0}^{n-1}\binom{n+i-1}{n-i}\lambda^i(\delta+2\lambda-cs)^{n-i-1}-(\delta+2\lambda-cs)^{2n-1}\Big]}{c^{3n-1}\prod_{i=1}^{3n-1}(\rho_i-s)}\widehat{\gamma}_2(s) \\
&= \sum_{j=1}^{3n-1}\frac{\widehat{\beta}_{1,\delta}(\rho_j)}{(s-\rho_j)\prod_{k=1,k\neq j}^{3n-1}(\rho_k-\rho_j)} + \frac{b_{1,\delta}(s)\widehat{\gamma}_1(s)+b_{2,\delta}(s)\widehat{\gamma}_2(s)}{\prod_{i=1}^{3n-1}(\rho_i-s)} \\
&= \sum_{j=1}^{3n-1}\frac{\widehat{\beta}_{1,\delta}(\rho_j)}{(s-\rho_j)\prod_{k=1,k\neq j}^{3n-1}(\rho_k-\rho_j)} + \frac{\widehat{\beta}_{1,\delta}(s)}{\prod_{i=1}^{3n-1}(\rho_i-s)} \\
&= (-1)^{3n-1}\Big[\frac{\widehat{\beta}_{1,\delta}(s)}{\tau(s)} - \sum_{j=1}^{3n-1}\frac{\widehat{\beta}_{1,\delta}(\rho_j)}{(s-\rho_j)\tau'(\rho_j)}\Big] \\
&= T_sT_{\rho_1}...T_{\rho_{3n-1}}\beta_{1,\delta}(0), \qquad (52)
\end{aligned}$$

where the last equality follows from Eq. (38) for the Disckson-Hipp operator. Thus, by inverting Eq. (52) we also get the following alternative expression for $G_\delta(u)$,

$$G_\delta(u) = T_{\rho_1}...T_{\rho_{3n-1}}\beta_{1,\delta}(u),$$

which again can be easily computed using Eq. (37) from the fourth property of the Disckson - Hipp operator. Therefore from Eq. (46) and recalling that $\int_0^\infty f_{2,\delta}(y|0)dy = m_\tau(0) < 1$, we get the following proposition.

**Proposition 3.** *The Gerber-Shiu discounted penalty function $m_\delta(u)$ admits a defective renewal equation*

$$m_\delta(u) = \int_0^u m_\delta(u-y)f_{2,\delta}(y|0)dy + G_\delta(u), u \geq 0 \tag{53}$$

*where*

$$f_{2,\delta}(y|0) = T_{\rho_1}\cdots T_{\rho_{3n-1}}h_{2,\delta}(y),$$

*and*

$$G_\delta(u) = T_{\rho_1}...T_{\rho_{3n-1}}\beta_{1,\delta}(u).$$

*Furthermore, Eq. (53) admits the following alternative representation*

$$m_\delta(u) = \frac{1}{1+\kappa_\delta}\int_0^u m_\delta(u-y)\theta_\delta(y)dy + \frac{1}{1+\kappa_\delta}\Lambda_\delta(u), u \geq 0 \tag{54}$$

*where $\kappa_\delta$ is defined such that*

$$\frac{1}{1+\kappa_\delta} = T_0T_{\rho_1}...T_{\rho_{3n-1}}h_{2,\delta}(0) = m_\tau(0).$$

*In addition we have*

$$\Lambda_\delta(u) = (1+\kappa_\delta)G_\delta(u), \tag{55}$$

*and*

$$\theta_\delta(y) = (1+\kappa_\delta)f_{2,\delta}(y|0),$$

*which is a proper density function.*

The next result shows that the Laplace transform of the time to ruin $m_\tau(u)$ (and hence the ruin probability $\psi(u)$) is the tail of a compound geometric distribution.

**Proposition 4.** *The Laplace transform of the time to ruin $m_\tau(u)$ satisfies the defective renewal equation*

$$\begin{aligned} m_\tau(u) &= \int_0^u m_\tau(u-y)f_{2,\delta}(y|0)dy + \int_u^\infty f_{2,\delta}(y|0)dy \\ &= \frac{1}{1+\kappa_\delta}\int_0^u m_\tau(u-y)\theta_\delta(y)dy + \frac{1}{1+\kappa_\delta}\overline{\Theta}_\delta(u),\ u \geq 0, \end{aligned} \tag{56}$$

*which has the following compound geometric representation:*

$$m_\tau(u) = \frac{\kappa_\delta}{1+\kappa_\delta}\sum_{j=1}^\infty \left(\frac{1}{1+\kappa_\delta}\right)^j \overline{\Theta}_\delta^{*j}(u)\ u \geq 0,$$

*where $\overline{\Theta}_\delta(u) = \int_u^\infty \theta_\delta(y)dy$ and $\overline{\Theta}_\delta^{*j}(u)$ is the j-fold convolution of the survival distribution $\overline{\Theta}_\delta(u)$.*

*Proof.* For $w(x_1, x_2) = 1$, Eq (17) gives that $\gamma_1(u) = T_0 f(u)$ and $\gamma_2(u) = T_0 h(u)$ and thus from Eq. (51) we obtain

$$G_\delta(u) = \sum_{j=1}^{3n-1} b_{1,j} T_{\rho_j} T_0 f_X(u) + \sum_{j=1}^{3n-1} b_{2,j} T_{\rho_j} T_0 h(u).$$

Therefore using this and Eq. (39) it follows that

$$G_\delta(u) = T_0 f_{2,\delta}(u|0) = \int_u^\infty f_{2,\delta}(y|0)dy,$$

and hence the result follows directly from Proposition 3. □

An explicit solution of the defective renewal equation (54) can be derived directly by applying Theorem 2.1 of Lin and Willmot (1999).

**Proposition 5.** *The solution $m_\delta(u)$ to Eq. (54) may be written as*

$$m_\delta(u) = -\frac{1}{\kappa_\delta}\int_0^u m_\tau(u-x)d\Lambda_\delta(x) + \frac{1}{\kappa_\delta}\Lambda_\delta(u) - \frac{1}{\kappa_\delta}\Lambda_\delta(0)m_\tau(u). \tag{57}$$

# 7 Discounted distributions of $U(\tau^-)$ and $|U(\tau)|$

In this section, we derive the discounted joint and marginal distributions of the surplus prior to ruin $U(\tau^-)$ and the deficit at ruin $|U(\tau)|$ from the expected discounted penalty function using Proposition 5.

At first, we derive the discounted joint distribution function, say $F_\delta(x, y|u)$, of $U(\tau^-)$ and $|U(\tau)|$ given $U(0) = 0$, which can be obtained from $m_\delta(u)$ by letting $w(x_1, x_2) = I(x_1 \le x, x_2 \le y)$, for any fixed $x$ and $y$.

As in Lin and Wilmott (1999), Tsai and Sun (2004) and Tsai (2005) we define the distribution functions $\Gamma_{i,j}(y) = 1 - \overline{\Gamma}_{i,j}(y)$, $i = 1, 2$, $j = 1, 2, \cdots, 3n-1$, as follows

$$\Gamma_{1,j}(y) = \frac{\int_0^y T_{\rho_j} f_X(t)dt}{E_{1,j}},\ \Gamma_{2,j}(y) = \frac{\int_0^y T_{\rho_j} h(t)dt}{E_{2,j}}, \tag{58}$$

where

$$E_{1,j} = \int_0^\infty T_{\rho_j} f_X(t)dt,\ \ E_{2,j} = \int_0^\infty T_{\rho_j} h(t)dt.$$

Since $\overline{F}(x)$ is a survival function, from Eq. (2.19) in Lin and Willmot (1999) (see also Eq. (1) in Tsai (2005)) it follows that

$$\overline{\Gamma}_{1,j}(y) = \frac{\int_y^\infty e^{-\rho_j(x-y)}\overline{F}_X(x)dx}{E_{1,j}},\ j = 1, 2, \cdots, 3n-1.$$

Now, let $\overline{H}(y) = \int_y^\infty h(t)dt$. Then

$$\begin{aligned}
\overline{\Gamma}_{2,j}(y) &= \frac{\overline{H}(y) - e^{\rho_j y}\int_y^\infty e^{-\rho_j x}h(x)dx}{\rho_j E_{2,j}} \\
&= \frac{e^{\rho_j y}\lim_{x\to+\infty} e^{-\rho_j x}\overline{H}(x) + \rho_j \int_y^\infty e^{-\rho_j(x-y)}\overline{H}(x)dx}{\rho_j E_{2,j}} \\
&= \frac{\int_y^\infty e^{-\rho_j(x-y)}\overline{H}(x)dx}{E_{2,j}},\ j = 1, 2, \cdots, 3n-1,
\end{aligned}$$

since $\overline{H}(x) = -F_X(x)\overline{F}_X(x)$ and thus $\lim_{x\to\infty} e^{-\rho_j x}\overline{H}(x) = 0$ for $j = 1, 2, \cdots, 3n-1$. Furthermore, from Eqs (39) and (58) we have that

$$\begin{aligned}
\overline{\Theta}_\delta(y) &= \int_y^\infty \theta_\delta(t)dt = (1+\kappa_\delta)\int_y^\infty f_{2,\delta}(t|0)dt \\
&= (1+\kappa_\delta)\Big[\sum_{j=1}^{3n-1} b_{1,j}\int_y^\infty T_{\rho_j} f_X(t)dt + \sum_{j=1}^{3n-1} b_{2,j}\int_y^\infty T_{\rho_j} h(t)dt\Big] \\
&= (1+\kappa_\delta)\sum_{i=1}^{2}\sum_{j=1}^{3n-1} b_{i,j}E_{i,j}\overline{\Gamma}_{i,j}(y) \\
&= \sum_{i=1}^{2}\sum_{j=1}^{3n-1} w_{i,j}\overline{\Gamma}_{i,j}(y) \qquad (59)
\end{aligned}$$

where

$$w_{i,j} = (1+\kappa_\delta)b_{i,j}E_{i,j} \ i = 1, 2 \ \ and \ \ j = 1, \cdots, 3n-1.$$

Note that $\sum_{i=1}^{2}\sum_{j=1}^{3n-1} w_{i,j} = (1+\kappa_\delta)\int_0^\infty f_{2,\delta}(t|0)dt = (1+\kappa_\delta)m_\tau(0) = 1$, and thus $\Theta_\delta(y)$ is a weighted distribution function.

Now, let $w(x_1, x_2) = I(x_1 \le x, x_2 \le y)$, for any fixed $x$ and $y$. Then

$$w(x_1, x_2 - x_1) = \begin{cases} 1, \text{ if } x_1 \le x, x_2 \le x_1 + y \\ 0, \text{ otherwise,} \end{cases}$$

and

$$\gamma_1(x_1) = \int_{x_1}^\infty w(x_1, x_2 - x_1) f_X(x_2)dx_2 = \begin{cases} \int_{x_1}^{x_1+y} f_X(x_2)dx_2, \text{ if } x_1 \le x \\ 0, \text{ if } x_1 > x. \end{cases}$$

Since, $T_{\rho_j}\gamma_1(u) = \int_u^\infty e^{-\rho_j(x_1-u)}\gamma_1(x_1)dx_1$ it follows that $T_{\rho_j}\gamma_1(u) = 0$ if $0 < x \le u$ for $j = 1, \cdots, 3n-1$, and if $0 \le u < x$, then for $j = 1, \cdots, 3n-1$ we have

$$\begin{aligned}
T_{\rho_j}\gamma_1(u) &= \int_u^x e^{-\rho_j(x_1-u)}\gamma_1(x_1)dx_1 = \int_u^x e^{-\rho_j(x_1-u)}\int_{x_1}^{x_1+y} f_X(x_2)dx_2dx_1 \\
&= \int_u^x e^{-\rho_j(x_1-u)}\left[\overline{F}_X(x_1) - \overline{F}_X(x_1+y)\right]dx_1 \\
&= \int_u^x e^{-\rho_j(x_1-u)}\overline{F}_X(x_1)dx_1 - \int_{u+y}^{x+y} e^{-\rho_j(z-u-y)}\overline{F}_X(z)dz \\
&= \int_u^\infty e^{-\rho_j(x_1-u)}\overline{F}_X(x_1)dx_1 - \int_x^\infty e^{-\rho_j(x_1-u)}\overline{F}_X(x_1)dx_1 \\
&- \int_{u+y}^\infty e^{-\rho_j(z-u-y)}\overline{F}_X(z)dz + \int_{x+y}^\infty e^{-\rho_j(z-u-y)}\overline{F}_X(z)dz \\
&= E_{1,j}\overline{\Gamma}_{1,j}(u) - e^{-\rho_j(x-u)}\int_x^\infty e^{-\rho_j(x_1-x)}\overline{F}_X(x_1)dx_1 \\
&- E_{1,j}\overline{\Gamma}_{1,j}(u+y) + e^{-\rho_j(x-u)}\int_{x+y}^\infty e^{-\rho_j(z-x-y)}\overline{F}_X(z)dz \\
&= E_{1,j}\overline{\Gamma}_{1,j}(u) - e^{-\rho_j(x-u)}E_{1,j}\overline{\Gamma}_{1,j}(x) - E_{1,j}\overline{\Gamma}_{1,j}(u+y) + e^{-\rho_j(x-u)}E_{1,j}\overline{\Gamma}_{1,j}(x+y) \\
&= E_{1,j}\left[\overline{\Gamma}_{1,j}(u) - \overline{\Gamma}_{1,j}(u+y)\right] - E_{1,j}e^{-\rho_j(x-u)}\left[\overline{\Gamma}_{1,j}(x) - \overline{\Gamma}_{1,j}(x+y)\right],
\end{aligned}$$

and similarly for $j=1,\cdots,3n-1$ we obtain that $T_{\rho_j}\gamma_2(u)=0$ if $0<x\le u$ and

$$T_{\rho_j}\gamma_2(u)=E_{2,j}\left[\overline{\Gamma}_{2,j}(u)-\overline{\Gamma}_{2,j}(u+y)\right]-E_{2,j}e^{-\rho_j(x-u)}\left[\overline{\Gamma}_{2,j}(x)-\overline{\Gamma}_{2,j}(x+y)\right],$$

for $j=1,\cdots,3n-1$ if $0\le u<x$.

Therefore from Eqs (51) and (55) it follows that

$$\begin{aligned}\Lambda_\delta(u) &= (1+\kappa_\delta)G_\delta(u)=(1+\kappa_\delta)\sum_{i=1}^{2}\sum_{j=1}^{3n-1}b_{i,j}T_{\rho_j}\gamma_i(u)\\ &= \sum_{i=1}^{2}\sum_{j=1}^{3n-1}w_{ij}\left\{\overline{\Gamma}_{i,j}(u)-\overline{\Gamma}_{i,j}(u+y)-e^{-\rho_j(x-u)}\left[\overline{\Gamma}_{i,j}(x)-\overline{\Gamma}_{i,j}(x+y)\right]\right\},\end{aligned}$$

and thus using Eq. (59) we obtain

$$\Lambda_\delta(u)=\overline{\Theta}_\delta(u)-\overline{\Theta}_\delta(u+y)-\sum_{i=1}^{2}\sum_{j=1}^{3n-1}w_{i,j}e^{-\rho_j(x-u)}\left[\overline{\Gamma}_{i,j}(x)-\overline{\Gamma}_{i,j}(x+y)\right], \tag{60}$$

and for $u>0$,

$$d\Lambda_\delta(u)=d\Theta_\delta(u+y)-d\Theta_\delta(u)-\sum_{i=1}^{2}\sum_{j=1}^{3n-1}w_{i,j}\rho_je^{-\rho_j(x-u)}\left[\overline{\Gamma}_{i,j}(x)-\overline{\Gamma}_{i,j}(x+y)\right]du. \tag{61}$$

Since for $w(x_1,x_2)=I(x_1\le x,x_2\le y)$, Eq. (57) becomes

$$F_\delta(x,y|u)=\begin{cases}-\frac{1}{\kappa_\delta}\int_0^u m_\tau(u-t)d\Lambda_\delta(t)+\frac{1}{\kappa_\delta}\Lambda_\delta(u)-\frac{1}{\kappa_\delta}\Lambda_\delta(0)m_\tau(u),\ 0\le u<x\\ \\ -\frac{1}{\kappa_\delta}\int_0^x m_\tau(u-t)d\Lambda_\delta(t)-\frac{1}{\kappa_\delta}\Lambda_\delta(0)m_\tau(u),\ 0<x\le u,\end{cases}$$

by replacing $\Lambda_\delta(u)$ and $d\Lambda_\delta(u)$ by Eqs (60) and (61) respectively, and using Eq. (56) we easily get the following Proposition giving the discounted joint distribution function of $U(\tau^-)$ and $|U(\tau)|$ (from which by setting $u=0$ we get $F_\delta(x,y|0)$ using Eq. (56) and $m_\tau(0)=1/(1+\kappa_\delta)$).

**Proposition 6.** *Let $\Psi_j(u)=m_\tau(u)+\rho_j\int_0^u e^{\rho_jt}m_\tau(u-t)dt$, $j=1,2,\cdots,3n-1$. Then the discounted joint distribution function of $U(\tau^-)$ and $|U(\tau)|$ is given by*

$$F_\delta(x,y)|u)=\begin{cases}\frac{1+\kappa_\delta}{\kappa_\delta}\left[m_\tau(u)-m_\tau(u+y)\right]-\frac{1}{\kappa_\delta}\Theta_\delta(y)m_\tau(u)+\frac{1}{\kappa_\delta}\int_0^y m_\tau(u+y-t)d\Theta_\delta(t)\\ +\frac{1}{\kappa_\delta}\sum_{i=1}^2\sum_{j=1}^{3n-1}w_{i,j}e^{-\rho_jx}\left[\overline{\Gamma}_{i,j}(x)-\overline{\Gamma}_{i,j}(x+y)\right]\\ \times\left[\Psi_j(u)-e^{-\rho_ju}\right],\ 0\le u<x,\\ \\ \frac{1}{\kappa_\delta}\int_0^x m_\tau(u-t)[d\Theta_\delta(t)-d\Theta_\delta(y+t)]-\frac{1}{\kappa_\delta}\Theta_\delta(y)m_\tau(u)\\ +\frac{1}{\kappa_\delta}\sum_{i=1}^2\sum_{j=1}^{3n-1}w_{i,j}e^{-\rho_jx}\left[\overline{\Gamma}_{i,j}(x)-\overline{\Gamma}_{i,j}(x+y)\right]\\ \times\left\{\Psi_j(u)+e^{\rho_jx}\left[m_\tau(u-x)-\Psi_j(u-x)\right]\right\},\ 0<x\le u,\end{cases}$$

*with*

$$F_\delta(x,y|0)=\frac{1}{1+\kappa_\delta}\left\{\Theta_\delta(y)-\sum_{i=1}^{2}\sum_{j=1}^{3n-1}w_{i,j}e^{-\rho_jx}\left[\overline{\Gamma}_{i,j}(x)-\overline{\Gamma}_{i,j}(x+y)\right]\right\}.$$

The marginal discounted distribution functions $F_{1,\delta}(x|u)$ of $U(\tau^-)$ and $F_{2,\delta}(y|u)$ of $|U(\tau)|$ given $U(0) = u$, can be derived immediately by letting $y \to \infty$ and $x \to \infty$ respectively.

The discounted joint probability density functions $f_\delta(x, y|u)$ of $U(\tau^-)$ and $|U(\tau)|$ given $U(0) = u$, can be obtained directly from

$$f_\delta(x, y)|u) = \frac{\partial^2 F_\delta(x, y|u)}{\partial x \partial y},$$

and is given by the following

**Proposition 7.** *The discounted joint probability density function $f_\delta(x, y|u)$ of $U(\tau^-)$ and $|U(\tau)|$ can be written as*

$$f_\delta(x,y|u) = \begin{cases} \frac{1+\kappa_\delta}{\kappa_\delta} \sum_{j=1}^{3n-1} \left[b_{1,j} f_X(x+y) + b_{2,j} h(x+y)\right] e^{-\rho_j x} \left[e^{\rho_j u} - \Psi_j(u)\right] \ , 0 \le u < x \\ \frac{1+\kappa_\delta}{\kappa_\delta} \sum_{j=1}^{3n-1} \left[b_{1,j} f_X(x+y) + b_{2,j} h(x+y)\right] \left[\Psi_j(u-x) - e^{-\rho_j x} \Psi_j(u)\right] \ , 0 < x \le u, \end{cases}$$

*with*

$$f_\delta(x, y|0) = \sum_{j=1}^{3n-1} \left[b_{1,j} f(x+y) + b_{2,j} h(x+y)\right] e^{-\rho_j x}.$$

The marginal discounted probability density function $f_{1,\delta}(x|u)$ of $U(\tau^-)$ given $U(0) = u$ is obtained immediately from $f_{1,\delta}(x|u) = \int_0^\infty f_\delta(x, y|u)dy$ and is given by the following

**Proposition 8.** *The discounted probability density function $f_\delta(x|u)$ of $U(\tau^-)$ can be written as*

$$f_{1,\delta}(x|u) = \begin{cases} \frac{1+\kappa_\delta}{\kappa_\delta} \sum_{j=1}^{3n-1} \left[b_{1,j} - b_{2,j} F_X(x)\right] \overline{F}_X(x) e^{-\rho_j x} \left[e^{\rho_j u} - \Psi_j(u)\right], \ 0 \le u < x \\ \frac{1+\kappa_\delta}{\kappa_\delta} \sum_{j=1}^{3n-1} \left[b_{1,j} - b_{2,j} F_X(x)\right] \overline{F}_X(x) \left[\Psi_j(u-x) - e^{-\rho_j x} \Psi_j(u)\right], \ 0 < x \le u, \end{cases}$$

*with*

$$f_{1,\delta}(x|0) = \sum_{j=1}^{3n-1} \left[b_{1,j} - b_{2,j} F_X(x)\right] \overline{F}_X(x) e^{-\rho_j x}.$$

The marginal discounted probability density function $f_{2,\delta}(y|u)$ of $|U(\tau)|$ is given by $f_{2,\delta}(y|u) = \int_0^\infty f_\delta(x, y|u)dx$. When $\delta \to 0$ in all the aforementioned Propositions we get the joint and marginal distribution functions and probability density functions of $U(\tau^-)$ and $|U(\tau)|$.

## 8 Explicit Results for exponentially distributed claims

In this section, we assume that the r.v. $X$ representing the individual claim amount, follows an exponential distribution with parameter $\alpha > 0$, i.e. $f_X(x) = \alpha e^{-\alpha x}$, $x > 0$, with $\widehat{f}_X(s) = \frac{\alpha}{\alpha+s}$. From the Propositions in Section 7, it is clear that the discounted joint and marginal distributions of $U(\tau^-)$ and $|U(\tau)|$ can be evaluated explicitly whenever the function $m_\tau(u)$ is known. Therefore, at first we will find an explicit expression for $m_\tau(u)$ for exponentially distributed claims. Taking Laplace transforms in both sides of the first equation in Proposition 4, we get that

$$\widehat{m}_\tau(s) = \frac{m_\tau(0) - \widehat{f}_{2,\delta}(s)}{s[1 - \widehat{f}_{2,\delta}(s)]} = \frac{1 - \widehat{f}_{2,\delta}(s) - [1 - m_\tau(0)]}{s[1 - \widehat{f}_{2,\delta}(s)]} \tag{62}$$

From Eqs (49) and (50) we have that

$$\widehat{h}_{1,\delta}(s)-\widehat{h}_{2,\delta}(s)=[1-\widehat{f}_{2,\delta}(s)]\prod_{i=1}^{3n-1}(\rho_i-s)$$

and thus Eq. (62) becomes

$$\widehat{m}_\tau(s)=\frac{\widehat{h}_{1,\delta}(s)-\widehat{h}_{2,\delta}(s)-[1-m_\tau(0)]\prod_{i=1}^{3n-1}(\rho_i-s)}{s[\widehat{h}_{1,\delta}(s)-\widehat{h}_{2,\delta}(s)]}. \tag{63}$$

Now, for $f_X(x)=\alpha e^{-\alpha x}$, from Eqs (25), (26) we easily obtain that

$$\widehat{h}_{1,\delta}(s)-\widehat{h}_{2,\delta}(s)=\frac{Q_{3n+1,\delta}(s)}{c^{3n-1}(\alpha+s)(2\alpha+s)}, \tag{64}$$

where

$$\begin{aligned}Q_{3n+1,\delta}(s) &= (\alpha+s)(2\alpha+s)(\delta+\lambda-cs)^n(\delta+2\lambda-cs)^{2n-1}-\alpha\lambda^n(2\alpha+s)(\delta+2\lambda-cs)^{2n-1}\\ &- \theta\alpha\lambda^n s\Bigg[2(\delta+\lambda-cs)^n\sum_{i=0}^{n-1}\binom{n+i-1}{i}\lambda^i(\delta+2\lambda-cs)^{n-i-1}\\ &- (\delta+2\lambda-cs)^{2n-1}\Bigg].\end{aligned}$$

Note that $Q_{3n+1,\delta}(s)$ is a polynomial of degree $3n+1$ with leading coefficient $(-c)^{3n-1}$ and therefore the equation $Q_{3n+1,\delta}(s)=0$ has $3n+1$ roots in the complex plane. Since $\widehat{h}_{1,\delta}(s)-\widehat{h}_{2,\delta}(s)=0$ is Lundberg's generalized equation, it follows from Proposition 1 and Eq. (64) that the equation $Q_{3n+1,\delta}(s)=0$ has $3n-1$ roots $\rho_1,\cdots,\rho_{3n-1}$ with positive real part and two roots say $-R_i=-R_i(\delta)$, with $Re(R_i)>0$, $i=1,2$. Therefore we can rewrite $Q_{3n+1,\delta}(s)$ as

$$\begin{aligned}Q_{3n+1,\delta}(s) &= (-c)^{3n-1}(s+R_1)(s+R_2)\prod_{i=1}^{3n-1}(s-\rho_i)\\ &= c^{3n-1}(s+R_1)(s+R_2)\prod_{i=1}^{3n-1}(\rho_i-s).\end{aligned} \tag{65}$$

So, from Eqs (65), and (64), Eq. (63) yields

$$\widehat{m}_\tau(s)=\frac{\prod_{j=1}^{2}(s+R_j)-[1-m_\tau(0)](\alpha+s)(2\alpha+s)}{s\prod_{j=1}^{2}(s+R_j)} \tag{66}$$

Since $\widehat{m}_\tau(s)<\infty$ for $s\geq 0$, the numerator in Eq. (66) must be zero for $s=0$, i.e.

$$1-m_\tau(0)=\frac{R_1R_2}{2\alpha^2}$$

and hence Eq. (66) becomes

$$\widehat{m}_\tau(s)=\frac{\left(1-\frac{R_1R_2}{2\alpha^2}\right)s+R_1+R_2-\frac{3R_1R_2}{2\alpha}}{(s+R_1)(s+R_2)}.$$

Assuming that $R_1$, $R_2$ are distinct, using partial fractions yields

$$\widehat{m}_\tau(s) = \sum_{j=1}^{2} \frac{\zeta_{j,\delta}}{s+R_j},$$

where

$$\begin{aligned}
\zeta_{1,\delta} &= \frac{R_2}{R_2-R_1}\left(1-\frac{3R_1}{2\alpha}+\frac{{R_1}^2}{2\alpha^2}\right)\\
\zeta_{2,\delta} &= \frac{R_1}{R_2-R_1}\left(1-\frac{3R_2}{2\alpha}+\frac{{R_2}^2}{2\alpha^2}\right).
\end{aligned}$$

Inverting the above Laplace transform gives

$$m_\tau(u) = \zeta_{1,\delta}e^{-R_1 u} + \zeta_{2,\delta}e^{-R_2 u},\ u \geq 0, \tag{67}$$

and the ruin probability $\psi(u)$ is easily obtained by letting $\delta \to 0$.

In order to find the discounted joint and marginal distributions of $U(\tau^-)$ and $|U(\tau)|$ from Propositions of the previous section, we need to find $\Psi_j(u)$ as defined in Proposition 6. From Eq. (67), these functions can be written for $j = 1, 2, \cdots, 3n-1$, as

$$\Psi_j(u) = \frac{\zeta_{1,\delta}R_1}{R_1+\rho_j}e^{-R_1 u} + \frac{\zeta_{2,\delta}R_2}{R_2+\rho_j}e^{-R_2 u} + \left(\frac{\zeta_{1,\delta}}{R_1+\rho_j}+\frac{\zeta_{2,\delta}}{R_2+\rho_j}\right)\rho_j e^{\rho_j u}.$$

For example, by Proposition 7, the discounted joint probability density function of $U(\tau^-)$ and $|U(\tau)|$ is given by

$$f_\delta(x,y)|u) = \begin{cases}
\frac{2\alpha^3}{R_1R_2}e^{-\alpha(x+y)}\sum_{j=1}^{3n-1}\left[b_{1,j}-b_{2,j}(1-2e^{-\alpha(x+y)})\right] \\
\times\left[\frac{(\alpha+\rho_j)(2\alpha+\rho_j)R_1R_2}{2\alpha^2}e^{-\rho_j(x-u)} - \sum_{i=1}^{2}\frac{\zeta_{i,\delta}R_i}{R_i+\rho_j}e^{(-R_i u+\rho_j x)}\right], 0 \leq u < x \\
\frac{2\alpha^3}{R_1R_2}e^{-\alpha(x+y)}\sum_{j=1}^{3n-1}\left[b_{1,j}-b_{2,j}(1-2e^{-\alpha(x+y)})\right] \\
\times\sum_{i=1}^{2}\frac{\zeta_{i,\delta}R_i}{R_i+\rho_j}e^{-R_i u}\left[e^{R_i x}-e^{-\rho_j x}\right], 0 < x \leq u
\end{cases}$$

Similarly, since we can easily obtain the other discounted (and non - discounted by letting $\delta \to 0$) joint and marginal distributions of $U(\tau^-)$ and $|U(\tau)|$, the details are omitted.

## 8.1 Numerical illustration

In this subsection, using a particular model, we illustrate why we consider the extension from the Poisson arrival process to Erlang(n) interarrival claim times as well as we indicate the impact of the dependence parameter $\theta$ on the ruin probability and the Laplace transform of the ruin time.

### 8.1.1 Why Erlang(n) arrivals?

The answer to this question for the independent case, i.e. for $\theta = 0$, was explained in details by De Vylder and Goovaerts (1998) in the discussions of the paper of Dickson (1998). The authors explained why Erlang risk models are justified in practice by showing that the ruin probabilities calculated in an Erlangian risk model are significantly different (and especially much smaller) from those calculated in a "corresponding" classical compound Poisson risk model.

Here, we adopt the same approach in order to compare the ruin probabilities calculated in an Erlang risk model under the FGM copula with those calculated by the "corresponding" classical compound Poisson risk model also under the FGM copula for $\theta \neq 0$. The two compared "corresponding" models are risk models with the same claim-size distribution, the same expected number of claims in any time interval $[0, t]$, the same security loading, and the same initial surplus. As stated by De Vylder and Goovaerts (1998), this definition of the "corresponding" models can only be adopted asymptotically for $t \to \infty$.

We assume for the claim amount r.v. that $X \sim Exp(1)$ for both risk models. For the Erlang risk model under the FGM copula, we assume that the interclaim r.v. $W \sim Erl(2, 2)$, i.e., $k(t) = 4te^{-2t}$ and thus the expected number of claims in $[0, t]$ is $E[N(t)] = t - \frac{1}{4}(1 - e^{-4t})$. For the "corresponding" classical compound Poisson risk model under the FGM copula with Poisson parameter $\mu > 0$ we denote by $\psi_P(u)$ the ruin probability. Since, the expected number of claims in $[0, t]$ for the "corresponding" classical compound Poisson risk model is equal to $\mu t$, we take $\mu = 1$, because the expected number of claims in time interval $[0, t]$ must asymptotically be the same in both models as $t \to \infty$. Also, in the Erlangian risk model we take the premium rate $c = 1.5$ and then we also take $c = 1.5$ in the "corresponding" classical model because the security loadings in $[0, t]$ must asymptotically be the same. The above choice of the parameters implies that the security loading is $50\%$ for both models.

For the Erlang(2,2) risk model, using $\delta = 0$ from Eq. (67), we provide the analytic expressions for the ruin probability $\psi(u)$ (derived with Maple) as function of the initial surplus $u \geq 0$ and for different values of the dependence parameter $\theta$, while the ruin probabilities $\psi_P(u)$ for the "corresponding" classical compound Poisson risk model are taken from Example 8.1 in Cossette et al. (2010), see also in Figure 1:

- with $\theta = -1$

$$\begin{aligned} \psi(u) &= 0.6416701672e^{-0.3487732254\ u} - 0.0169012248e^{-2.1517194000\ u} \\ \psi_P(u) &= 0.7201508967e^{-0.2687389645\ u} - 0.01854637723e^{-2.220708719\ u} \end{aligned}$$

- with $\theta = -0.5$

$$\begin{aligned} \psi(u) &= 0.6111640019e^{-0.3833132642\ u} - 0.0096651749e^{-2.0792454120\ u} \\ \psi_P(u) &= 0.6957948813e^{-0.2976043940\ u} - 0.01047590296e^{-2.114760590\ u} \end{aligned}$$

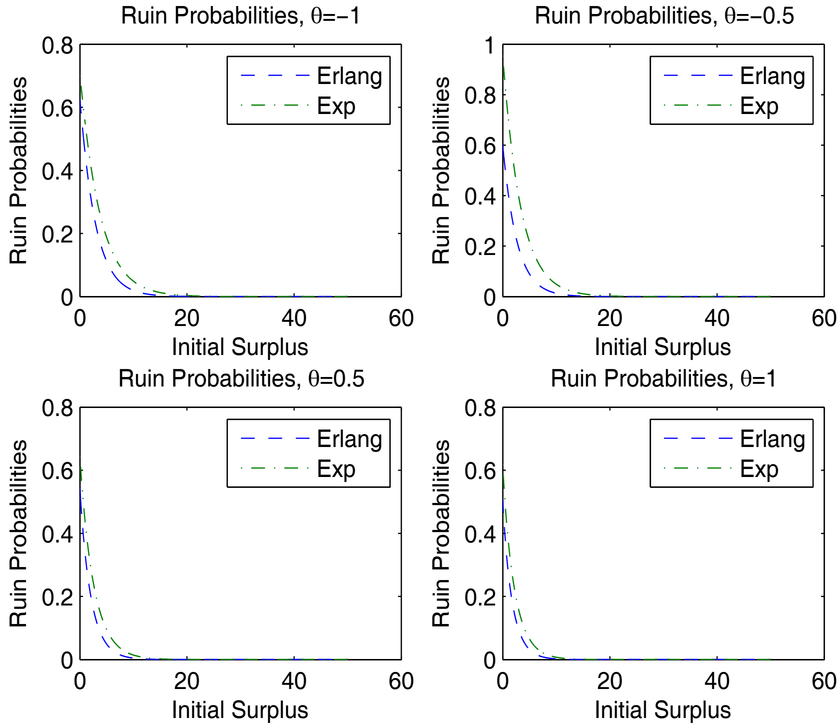
- with $\theta = 0.5$

$$\begin{aligned} \psi(u) &= 0.5314436215e^{-0.4762087115\ u} + 0.01332254042e^{-1.911908905\ u} \\ \psi_P(u) &= 0.6311261756e^{-0.3788264025\ u} + 0.01399640216e^{-1.873562242\ u} \end{aligned}$$

- with $\theta = 1$

$$\begin{aligned} \psi(u) &= 0.4774717870e^{-0.5409429369\ u} + 0.03255482730e^{-1.811552947\ u} \\ \psi_P(u) &= 0.5865437312e^{-0.4391578659\ u} + 0.03347620593e^{-1.730494168\ u} \end{aligned}$$

Figure 1: Ruin Probabilities in Corresponding Risk Models for $\theta \neq 0$




Numerical values of these ruin probabilities corresponding to particular values of the initial surplus $u \geq 0$ are given in Table 1.

Table 1. Ruin Probabilities in Corresponding Risk Models

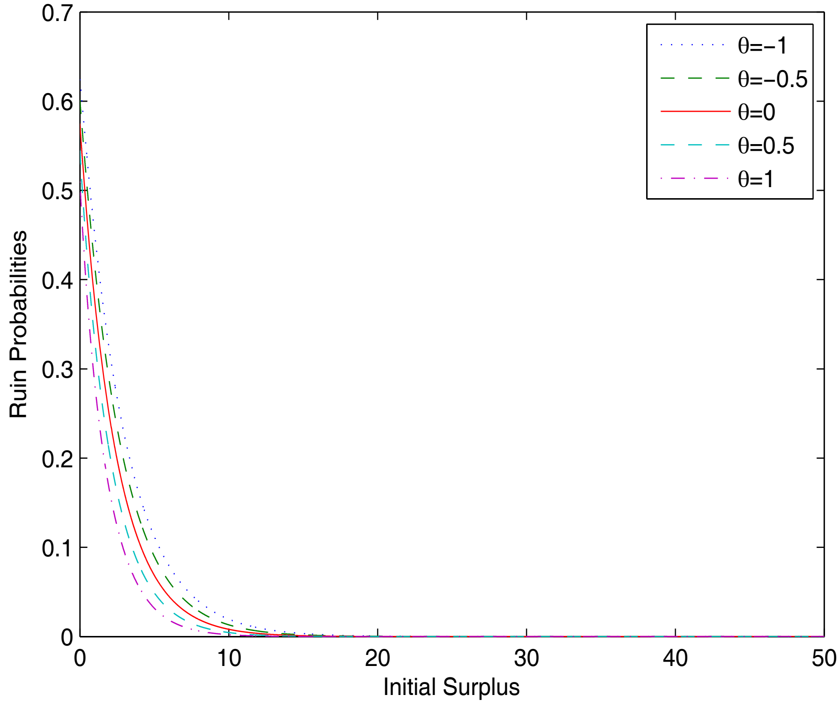
| | $\theta = -1$ | | $\theta = -0.5$ | | $\theta = 0.5$ | | $\theta = 1$ | |
|---|---|---|---|---|---|---|---|---|
| $u$ | $\psi(u)$ | $\psi_P(u)$ | $\psi(u)$ | $\psi_P(u)$ | $\psi(u)$ | $\psi_P(u)$ | $\psi(u)$ | $\psi_P(u)$ |
| 0 | 0.6248 | 0.7016 | 0.6015 | 0.95532 | 0.54477 | 0.64512 | 0.51003 | 0.62002 |
| 5 | 0.1122 | 0.18787 | 0.089909 | 0.21809 | 0.049135 | 0.094953 | 0.031942 | 0.065271 |
| 10 | 0.01962 | 0.049012 | 0.013227 | 0.04925 | 0.0045426 | 0.014285 | 0.0021363 | 0.0072621 |
| 15 | 0.003430 | 0.012786 | 0.0019458 | 0.011122 | 0.00041999 | 0.0021492 | 0.0001429 | 0.00080806 |
| 20 | 0.0005997 | 0.0033357 | 0.00028625 | 0.0025115 | 3.8829e-005 | 0.00032335 | 9.5582e-006 | 8.9913e-005 |
| 25 | 0.0001049 | 0.00087022 | 4.211e-005 | 0.00056713 | 3.5899e-006 | 4.8648e-005 | 6.3934e-007 | 1.0005e-005 |
| 30 | 1.8332e-005 | 0.00022702 | 6.1949e-006 | 0.00012807 | 3.319e-007 | 7.319e-006 | 4.2765e-008 | 1.1132e-006 |
| 35 | 3.2052e-006 | 5.9226e-005 | 9.1134e-007 | 2.8921e-005 | 3.0686e-008 | 1.1011e-006 | 2.8605e-009 | 1.2387e-007 |
| 40 | 5.604e-007 | 1.5451e-005 | 1.3407e-007 | 6.5308e-006 | 2.837e-009 | 1.6566e-007 | 1.9134e-010 | 1.3783e-008 |
| 45 | 9.7983e-008 | 4.0308e-006 | 1.9723e-008 | 1.4748e-006 | 2.623e-010 | 2.4924e-008 | 1.2799e-011 | 1.5337e-009 |
| 50 | 1.7132e-008 | 1.0516e-006 | 2.9015e-009 | 3.3303e-007 | 2.425e-011 | 3.7498e-009 | 8.5609e-013 | 1.7065e-010 |

Figure 1 and Table 1 show that the ruin probabilities $\psi(u)$ for the Erlang risk model under the FGM copula are significantly different and especially much smaller than the "corresponding" classical risk model under the FGM copula for the practical values of the initial surplus $u$ and for all $\theta \in [-1, 0) \cup (0, 1]$, indicating why it is worthwhile to consider Erlangian risk models under the FGM copula.

### 8.1.2 Impact of the dependence parameter $\theta$

In Figure 2 we plot the values $\psi(u)$ calculated in the previous subsection. As we can see from Figure 2, the dependence parameter $\theta$ has a clear impact on the ruin probabilities. It is clear that the higher the dependence parameter the lower the ruin probability is. This happens because when for example the dependence relation is positive, the probability of having an important claim increases as the time elapsed since the last claim increases. Thus the ruin probability will be lower since the probability that the insurance company will have enough premium income to pay the claim will be higher. Similar results for the case of the exponential distribution were obtained from Cossette et al. (2010).

Figure 2: Ruin Probabilities




Furthermore using $\delta = 0.05$, we provide the analytic expressions for the Laplace transform of the time of ruin $m_\tau(u)$ (derived with Maple) as function of the initial surplus $u$, ($u \geq 0$) and for different values of the dependence parameter $\theta$,

- with $\theta = -1$

$$m_\tau(u) = 0.588107070542046e^{-0.4015607208\ u} - 0.0198616515195528e^{-2.150382538\ u}$$

- with $\theta = -0.5$

$$m_\tau(u) = 0.558265539590616e^{-0.4358563215\ u} - 0.0112379309905072e^{-2.078539964\ u}$$

- with $\theta = 0$

$$m_\tau(u) = 0.5230305556e^{-0.4769694444\ u}$$

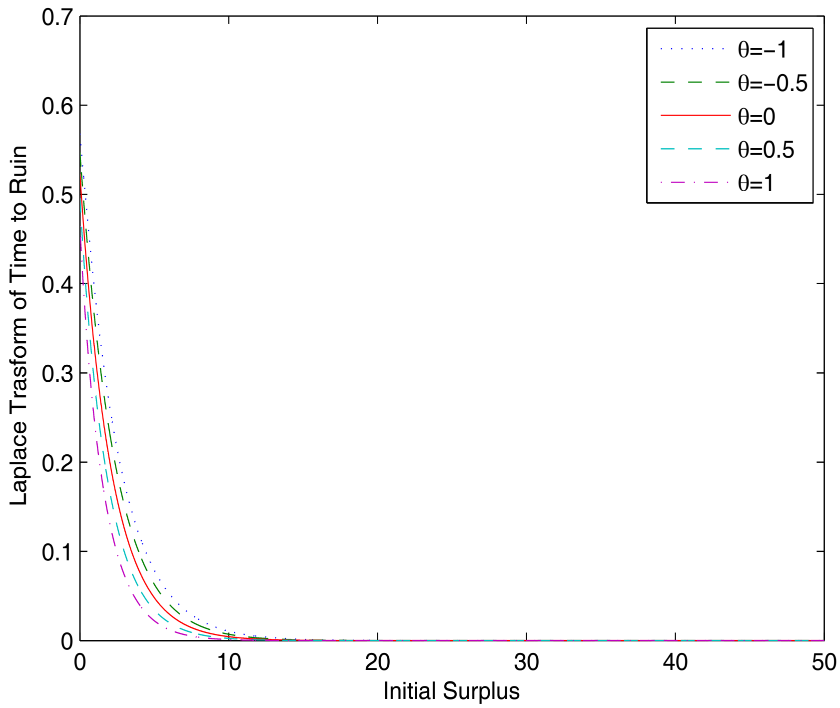
- with $\theta = 0.5$

$$m_\tau(u) = 0.480589531459186e^{-0.5272636613\ u} + 0.0151619535823271e^{-1.912699668\ u}$$

- with $\theta = 1$

$$m_\tau(u) = 0.427916113486677e^{-0.5905527687\ u} + 0.0366819441278372e^{-1.813223037\ u}$$

Figure 3: Laplace Transform of Time to Ruin




As we can see from Figure 3, the dependence parameter $\theta$ has a clear impact on the values of the LT of time to ruin. It is clear that the higher the dependence parameter the lower the value of the LT of time to ruin is.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# 9 References


1. H. Gerber, E. Shiu, On the time value of ruin, North American Actuarial Journal 2 (1998) 48-78.
2. X.S. Lin, G.E. Willmot, Analysis of a defective renewal equation arising in ruin theory, Insurance: Mathematics and Economics 25 (1999) 63-84.

3. D.C.M. Dickson, On a class of renewal risk processes, North American Actuarial Journal 2 (1998) 60-68.

4. D.C.M. Dickson, C. Hipp, Ruin probabilities for Erlang(2) risk processes, Insurance: Mathematics and Economics 22 (1998) 251-262.

5. D.C.M. Dickson, C. Hipp, On the time to ruin for Erlang(2) risk processes, Insurance: Mathematics and Economics 29 (2001) 333-344.

6. Y. Cheng, Q. Tang, Moments of the surplus before ruin and the deficit of ruin in the Erlang (2) risk process, North American Actuarial Journal 7 (2003) 1-12.

7. C. Tsai, L. Sun, On the discounted distribution functions for the Erlang(2) risk process, Insurance: Mathematics and Economics 35 (2004) 5-19.

8. S. Li, J. Garrido, On ruin for the Erlang(n) risk process, Insurance: Mathematics and Economics 34 (2004) 391-408.

9. L. Sun, The expected discounted penalty at ruin in the Erlang(2) risk process, Statistics and Probability Letters 72 (2005) 205-217.

10. H. Gerber, E. Shiu, The time value of ruin in a Sparre Andersen model, North American Actuarial Journal 9 (2005) 49-84.

11. M. Boudreault, Modeling and pricing earthquake risk, First prize in the Canadian actuarial research competition, SCOR Canada, (2003).

12. A. Nikoloulopoulos, D. Karlis, Fitting copulas to bivariate earthquake data: the seismic gap hypothesis revisited, Environmetrics 19 (2008) 251-269.

13. H. Albrecher, O. Boxma, A ruin model with dependence between claim sizes and claim intervals, Insurance: Mathematics and Economics 35 (2004) 245-254.

14. H. Albrecher, O. Boxma, On the discounted penalty function in a Markov - dependent risk model, Insurance: Mathematics and Economics 37 (2005) 650-672.

15. H. Albrecher, J. Teugels, Exponential behavior in the presence of dependence in risk theory, Journal of Applied Probability 43 (2006) 265-285.

16. M. Boudreault, H. Cossette, D. Landriault and E. Marceau, On a risk model with dependence between interclaim arrivals and claim sizes, Scandinavian Actuarial Journal (2006) 265-285.

17. Q. Meng, X. Zhang, J. Guo , On a risk model with dependence between claim sizes and claim intervals, Statistics and Probability Letters 78 (2008) 1727-1734.

18. H. Cossette, E. Marceau, F. Marri, On the compound Poisson risk model with dependence based on a generalized Farlie Gumbel Morgenstern copula, Insurance: Mathematics and Economics 43 (2008) 444-455.

19. H. Cossette, E. Marceau, F. Marri, Analysis of ruin measures for the classical compound Poisson risk model with dependence, Scandinavian Actuarial Journal (2010) 221-245.

20. A. Badescu, E. Cheung, D. Landriault, Dependent risk models with bivariate phase-type distributions, Journal of Applied Probability 46 (2009) 113-131.

21. R.S. Ambagaspitiya, Ultimate ruin probability in the Sparre Andersen model with dependent claim sizes and claim occurrence times, Insurance: Mathematics and Economics 44 (2009) 464-472.

22. E. Cheung, D. Landriault, G.E. Willmot, Structural properties of Gerber - Shiu functions in dependent Sparre Andersen models, Insurance: Mathematics and Economics 46 (2010) 117-126.

23. H. Albrecher, C. Constantinescu, S. Loisel, Explicit ruin formulas for models with dependence among risks, Insurance: Mathematics and Economics 48 (2011) 265-270.

24. Z. Zhang, H. Yang, Gerber - Shiu analysis in a perturbed risk model with dependence between claim sizes and interclaim times, Journal of Computational and Applied Mathematics 235 (2011) 1189-1204.

25. Z. Zhang, H. Yang, H. Yang, On a Sparre Andersen risk model with time - dependent claim sizes and jump - diffusion perturbation, Methodology and Computing in Applied Probability (2011) 1-23.

26. J.K. Woo, Some remarks on delayed renewal risk models, Astin Bulletin 40 (2011) 199-219.

27. R.B. Nelsen, An introduction to copulas, 2nd ed., Springer-Verlag, New York, 2006.

28. V. Klimenok, On the modification of Rouché's theorem for the queuing theory problems, Queuing Systems 38 (2001) 431-434.

29. S. Li, J. Garrido, On a general class of renewal risk process: analysis of the Gerber - Shiu function, Advances in Applied Probability 37 (2005) 836-856.

30. C. Tsai, Discussion of H. Gerber, E. Shiu, The time value of ruin in a Sparre Andersen risk model, North American Actuarial Journal 9 (2005) 74-77.

31. F. De Vylder, M. Goovaerts, Discussion of D.C.M. Dickson, On a class of renewal risk processes, North American Actuarial Journal 2 (1998) 68-70.