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# ON A REPRESENTATION OF A STRONGLY HARMONIC RING BY SHEAVES

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**A ring $R$ is strongly harmonic provided that if $M_1$, $M_2$ are a pair of distinct maximal modular ideals of $R$, then there exist ideals $\mathscr{A}$ and $\mathscr{B}$ such that $\mathscr{A} \nsubseteq M_1$, $\mathscr{B} \nsubseteq M_2$ and $\mathscr{A}\mathscr{B} = 0$. Let $\mathscr{M}(R)$ be the maximal modular ideal space of $R$. If $M \in \mathscr{M}(R)$, let $O(M) = \{r \in R \mid$ for some $y \notin M$, $rxy = 0$ for every $x \in R\}$. Define $\mathscr{R}(R) = \bigcup \{R/O(M) \mid M \in \mathscr{M}(R)\}$. If $R$ is a strongly harmonic ring with 1, then $R$ is isomorphic to the ring of global sections of the sheaf of local rings $\mathscr{R}(R)$ over $\mathscr{M}(R)$. Let $\Gamma(\mathscr{M}(R), \mathscr{R}(R))$ be the ring of global sections of $\mathscr{R}(R)$ over $\mathscr{M}(R)$. For every unitary (right) $R$-module $A$, let $A_M = \{a \in A \mid aRx = 0$ for some $x \notin M)$ and let $\tilde{A} = \bigcup\{A/A_M \mid M \in \mathscr{M}(R)\}$. Define $\hat{a}(M) = a + A_M$ and $\hat{r}(M) = r + O(M)$ for every $a \in A$, $r \in R$ and $m \in \mathscr{M}(R)$. Then the mapping $\xi_A$: $a \mapsto \hat{a}$ is a semi-linear isomorphism of $A$ onto $\Gamma(\mathscr{M}(R)), \mathscr{R}(R))$—module $\Gamma(\mathscr{M}(R), \tilde{A})$ in the sense that $\xi_A$ is a group isomorphism satisfying $\xi_A(ar) = \hat{a}\hat{r}$ for every $a \in A$ and $r \in R$.**

1. If $R$ is a ring with 1, $R$ is called *harmonic* (or regular) if the maximal modular ideal space, say $\mathscr{M}(R)$, with the hull-kernel topology, is a Hausdorff space (refer [5]). A ring $R$ is *strongly harmonic* provided that for any pair of distinct maximal modular ideals $M_1$, $M_2$ there exist ideals $\mathscr{A}$, $\mathscr{B}$ in $R$ such that $\mathscr{A} \nsubseteq M_1$, $\mathscr{B} \nsubseteq M_2$ and $\mathscr{A}\mathscr{B} = 0$. For any nonempty subset $S$ of a ring $R$ define $(S)^\perp = \{r \in R \mid sr = 0$ for every $s \in S\}$ and if $a \in R$ let $aR_1$ be the principal right ideal generated by $a$. If $M$ is a prime ideal of a ring $R$ let $O(M) = \{r \in R \mid (rR_1)^\perp \nsubseteq M\}$. An ideal $\mathscr{A}$ of a ring $R$ is called *$M$-primary* for some maximal modular ideal $M$ of $R$ provided that $M/\mathscr{A}$ is the unique maximal modular ideal of $R/\mathscr{A}$ and if $\mathscr{A}'$ is an ideal of $R$ such that $\mathscr{A}' \subseteq \mathscr{A}$ and $\mathscr{A}' \neq \mathscr{A}$ then $R/\mathscr{A}'$ is no longer a local ring (here by a local ring we mean a ring with the unique maximal modular ideal). The principal results in this paper are as follows: Let $R$ be a ring such that if $R/S$ is a local ring for some ideal $S$ of $R$ then $R/S$ has a unit. Then $R$ is strongly harmonic if and only if $O(M)$ is $M$-primary for every maximal modular ideal $M$ of $R$. If $R$ is a strongly harmonic ring with 1 then $R$ is isomorphic to $\Gamma(\mathscr{M}(R), \mathscr{R}(R))$ the ring of global sections of the sheaf of local rings $\mathscr{R}(R) = \bigcup \{R/O(M) \mid M \in \mathscr{M}(R)\}$ over $\mathscr{M}(R)$ and if $A$ is a unitary right $R$-module then the mapping $\xi_A$: $a \mapsto \hat{a}$ is a semi-linear isomorphism of $A$ onto $\Gamma(\mathscr{M}(R), \mathscr{R}(R))$—

module $\Gamma(\mathscr{M}(R), \tilde{A})$ in the sense that $\xi_A$ is a group isomorphism satisfying $\xi_A(ar) = \hat{a}\cdot\hat{r}$ for $a \in A$, $r \in R$ where $\hat{a}(M) = a + A_M$, $\hat{r}(M) = r + O(M)$ for $M \in \mathscr{M}(R)$ and $\hat{A} = \bigcup \{A/A_M \mid M \in \mathscr{M}(R)\}$, the disjoint union of the family of right $R$-modules $A/A_M$) indexed by $\mathscr{M}(R)$, and $A_M = \{a \in A \mid (aR)^{\perp} \nsubseteq M\}$. If $R$ is a ring with 1 such that it contains no nonzero nilpotent elements then $R$ is *biregular* (see [2: p. 104] for definition) if and only if every prime ideal of $R$ is a maximal ideal. Our results here generalize S. Teleman's result that in case $1 \in R$, a strongly semi-simple harmonic ring or a von Neumann algebra can be represented as a ring of global sections of the sheaf of local algebras over its maximal modular ideal space (refer [5], [6] and [7]). The author wishes to express his gratitude to Professors K. H. Hofmann and S. Teleman for their many invaluable suggestions for the preparation of this paper.

2. Let $R$ be a ring and $A$ be a right $R$-module. For each prime ideal $M$ of $R$, define $A_M = \{a \in A \mid (aR_1)^{\perp} \nsubseteq M\}$ where $aR_1$ is the submodule of $A$ which is generated by the element $a$ and $(aR_1)^{\perp} = \{r \in R \mid aR_1 r = 0\}$.

PROPOSITION 2.1. *$A_M$ is a submodule of $A$.*

*Proof.* Let $a, b \in A_M$. Then $(a-b)R_1 \subseteq aR_1 + bR_1$ and $((a-b)R_1)^{\perp} \supseteq (aR_1 + bR_1)^{\perp} = (aR_1)^{\perp} \cap (bR_1)^{\perp} \supseteq (aR_1)^{\perp}(bR_1)^{\perp}$. Hence if $a - b \notin A_M$ then $(aR_1)^{\perp}(bR_1)^{\perp} \subseteq M$ and either $(aR_1)^{\perp} \subseteq M$ or $(bR_1)^{\perp} \subseteq M$ since $M$ is a prime ideal of $R$. Hence either $a \notin A_M$ or $b \notin A_M$. This is impossible. Thus $a - b \in A_M$. Now if $r \in R$ and $a \in A_M$ then $arR_1 \subseteq aR_1$ and $(arR_1)^{\perp} \supseteq (aR_1)^{\perp}$. Since $(aR_1)^{\perp} \nsubseteq M$, $(arR_1)^{\perp} \nsubseteq M$ and $ar \in A_M$.

COROLLARY 2.2. *If $A$ is $R$, whose module multiplication is given by the ring multiplication, then $A_M$ is an ideal of $R$ which is contained in $M$ for any prime ideal $M$ of $R$. In this case, we denote $A_M$ by $O(M)$.*

*Proof.* $O(M)$ is already a right ideal of $R$ by 2.2. Let $r \in R$ and $a \in O(M)$. Then $(raR_1)^{\perp} \supseteq (aR_1)^{\perp}$. Since $(aR_1)^{\perp} \nsubseteq M$, $(raR_1)^{\perp} \nsubseteq M$ and $ra \in O(M)$.

PROPOSITION 2.3. *If $A$ is a right $R$-module for some ring $R$ then $AO(M) \subseteq A_M$ for any prime ideal $M$ of $R$.*

*Proof.* Since $A_M$ is a submodule of $A$, it suffices to show that if $a \in A$ and $x \in O(M)$ then $ax \in A_M$. But this is immediate since $(axR_1)^{\perp} \supseteq (xR_1)^{\perp}$ and $(xR_1)^{\perp} \nsubseteq M$.

THEOREM 2.4. *Let $R$ be a ring such that if $\mathscr{P}$ is a proper ideal of $R$ then there is a maximal modular ideal $M$ in $R$ such that $\mathscr{P} \subseteqq M$. Let $A$ be a right $R$-module such that if $aR = 0$ for some $a \in A$ then $a = 0$. Then $\bigcap \{A_M \mid M$ is a maximal modular ideal of $R\}$ is zero.*

*Proof.* Let $a \in \bigcap \{A_M \mid M$ is a maximal modular ideal of $R\}$ such that $a \neq 0$. Then $(aR_1)^\perp \neq R$, for if $(aR_1)^\perp = R$ then $aR = 0$ and $a = 0$. Since $(aR_1)^\perp \neq R$, $(aR_1)^\perp$ is a proper ideal of $R$. Hence there is a maximal modular ideal $M$ in $R$ such that $(aR_1)^\perp \subseteqq M$. This means that $a \notin A_M$ and $a \notin \bigcap \{A_M \mid M$ is a maximal modular ideal of $R\}$. This is a contradiction.

COROLLARY 2.5. *If $R$ is a ring with 1 and $A$ is a unitary right $R$-module, then $\bigcap \{AO(M) \mid M$ is a maximal ideal of $R\}$ is zero.*

*Proof.* By 2.4, $\bigcap \{A_M \mid M$ is a maximal ideal of $R\} = 0$. Since $AO(M) \subseteqq A_M$ for any prime ideal of $R$ by 2.3, the conclusion now follows.

DEFINITION 2.6. We say that a ring $R$ is *strong harmonic* provided that for any pair of distinct maximal modular ideals $M_1$, $M_2$ there exist ideals $\mathscr{A}$, $\mathscr{B}$ in $R$ such that $\mathscr{A} \not\subseteqq \mathscr{M}_1$, $\mathscr{B} \not\subseteqq \mathscr{M}_2$ and $\mathscr{A}\mathscr{B} = 0$.

PROPOSITION 2.7. *If $R$ is strongly harmonic, then $\mathscr{M}(R)$ is Hausdorff.*

*Proof.* If $M_1$, $M_2$ are distinct maximal modular ideals of $R$, then, by definition, there exist ideals $\mathscr{A}$ and $\mathscr{B}$ such that $\mathscr{A} \not\subseteqq M_1$, $\mathscr{B} \not\subseteqq M_2$ and $\mathscr{A}\mathscr{B} = 0$. Therefore, two open sets $\{M \in \mathscr{M}(R) \mid \mathscr{A} \not\subseteqq M\}$ and $\{M \in \mathscr{M}(R) \mid \mathscr{B} \not\subseteqq M\}$ are disjoint.

EXAMPLE 2.8. Let $R$ be a strongly semi-simple ring, that is a ring in which the intersection of maximal modular ideals is zero. If the maximal modular ideal space, $\mathscr{M}(R)$ with the hull-kernel topology, is a Hausdorff space, then $R$ is strongly harmonic.

EXAMPLE 2.9. If $R$ is a ring with 1 such that it is strongly harmonic then it is harmonic. However, if $1 \notin R$ then a strongly harmonic ring may not be harmonic. For example, let $R$ be the algebra of sequences $(a_n)_{n \geq 0}$ of $2 \times 2$-matrices over the field of complex numbers $C$, such that $a_n \to \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$ for $n \to \infty$ for some $\lambda \in C$. Then the intersection of the maximal modular ideals of $R$ is zero and $\mathscr{M}(R)$ is Hausdorff. Hence $R$ is strongly harmonic; however, it is not harmonic.

EXAMPLE 2.10. Let $R$ be a von Neumann algebra. Then for any distinct pair of maximal ideals $M_1$, $M_2$ there exist central idempotents $e_1$, $e_2$ in $R$ such that $e_1 \notin M_1$, $e_2 \notin M_2$ and such that $e_1 \cdot e_2 = 0$. Hence $R$ is strongly harmonic.

EXAMPLE 2.11. Let $Q$ be the field of rational numbers and let $p_1, p_2, \cdots, p_l$ be a finite number of distinct prime numbers. Let $R = \{m/n \in Q \mid n$ is not divisible by any $p_i, 1 \leqq i \leqq l\}$. Then $\mathscr{M}(R)$ consist of $l$ points and it is a Hausdorff space. However, since $R$ is an integral domain, $R$ is not strongly harmonic if $l > 1$.

DEFINITION 2.12. Let $R$ be a ring and $M$ be a maximal modular ideal of $R$. An ideal $\mathscr{O}$ in $R$ is said to be $M$-primary, for some maximal modular ideal $M$ of $R$, provided that $\mathscr{O} \subseteqq M$, $R/\mathscr{O}$ is a ring with a unique maximal modular ideal $M/\mathscr{O}$, and if $P$ is an ideal of $R$ such that $P \subseteqq \mathscr{O}$ and $P \neq \mathscr{O}$, then $R/P$ is not a local ring. Here, by a *local ring* we mean a ring with a unique maximal modular ideal.

PROPOSITION 2.13. *Let $R$ be a ring and $M$ be a maximal modular ideal of $R$. If an $M$-primary ideal, say $\mathscr{O}$, exists, then it is unique.*

*Proof.* Let $\mathscr{P}$ be a $M$-primary ideal of $R$. If either $\mathscr{P} \subseteqq \mathscr{O}$ or $\mathscr{O} \subseteqq \mathscr{P}$ then, by definition, $\mathscr{P} = \mathscr{O}$. So assume $\mathscr{O} \cap \mathscr{P}$ is properly contained in $\mathscr{O}$ or $\mathscr{P}$. Then the ideal $\mathscr{O}\mathscr{P}$ is properly contained in $\mathscr{O}$ and $R/\mathscr{O}\mathscr{P}$ is not a local ring. Hence there is a maximal modular ideal $N$ in $R$ such that $N \neq M$ and $\mathscr{O}\mathscr{P} \subseteqq N$. Since $N$ is a prime ideal, this means that either $\mathscr{O} \subseteqq N$ or $\mathscr{P} \subseteqq N$. In either case, this means that $\mathscr{O}$ or $\mathscr{P}$ is not $M$-primary. This is a contradiction.

PROPOSITION 2.14. *Let $R$ be a ring such that if $R/\mathscr{O}$ is a local ring for some ideal $\mathscr{O}$ in $R$, then $R/\mathscr{O}$ has a unit. If $R/O(M)$ is a local ring for some maximal modular ideal $M$ in $R$, then $O(M)$ is $M$-primary.*

*Proof.* Observe that $O(M) \subseteqq M$. Hence $M/O(M)$ is the unique maximal modular ideal of the local ring $R/O(M)$. Let $\mathscr{P}$ be an ideal of $R$ such that $\mathscr{P} \subseteqq O(M)$, $\mathscr{P} \neq O(M)$ and $R/\mathscr{P}$ is a local ring. Let $t \in O(M)$ such that $t \notin \mathscr{P}$. Then $(tR_1)^{\perp} \nsubseteqq M$. If $\mathscr{P} + (tR_1)^{\perp} \neq$

$R$ then there is a maximal modular ideal $N$ in $R$ such that $\mathscr{P} + (tR_1)^\perp \subseteqq N$, since $R/\mathscr{P}$ has a unit. Since $(tR_1)^\perp \nsubseteqq M$, this means that $M \neq N$. This is impossible. Hence $R = \mathscr{P} + (tR_1)^\perp$. Let $e + \mathscr{P}$ be the identity of $R/\mathscr{P}$ for some $e \in R$. Then $e = p + s$ for some $p \in \mathscr{P}$ and $s \in (tR_1)^\perp$. Hence $te = tp$ and $t - te = t - tp \in \mathscr{P}$. This means that $t \in \mathscr{P}$ and this is a contradiction. Thus $O(M)$ must be $M$-primary.

THEOREM 2.15. *Let $R$ be a ring such that if $R/\mathscr{O}$ is a local ring for some ideal $\mathscr{O}$, then it has a unit. Then $R$ is strongly harmonic if, and only if, $O(M)$ is $M$-primary for every maximal modular ideal $M$ in $R$.*

*Proof.* Assume $R$ is strongly harmonic. By 2.14, it suffices to show that $R/O(M)$ is a local ring for each maximal modular ideal $M$ of $R$. If $R/O(M)$ is not a local ring for some maximal modular ideal $M$, then there is a maximal modular ideal $N$ in $R$ such that $N \neq M$ and $O(M) \subseteqq N$. Since $R$ is strongly harmonic, there exist ideals $\mathscr{A}$ and $\mathscr{B}$ such that $\mathscr{A} \nsubseteqq N$, $\mathscr{B} \nsubseteqq M$ and $\mathscr{A}\mathscr{B} = 0$. This means that $\mathscr{A} \subseteqq O(M)$. Since $O(M) \subseteqq N$, $\mathscr{A} \subseteqq N$. This is a contradiction. Conversely, assume $O(M)$ is $M$-primary for each maximal modular ideal $M$ of $R$. Let $M_1$, $M_2$ be two distinct maximal modular ideals or $R$. Then $O(M_1) \nsubseteqq M_2$ and $O(M_2) \nsubseteqq M_1$. Hence there exist $a \in O(M_1)$ such that $a \notin M_2$ and $b \in O(M_2)$ such that $b \notin M_1$. Then $(b)$, the ideal generated by $b$, is not contained in $M$. Let $\mathscr{A} = (b)$ and let $\mathscr{B} = (bR_1)^\perp$. Then $\mathscr{A} \nsubseteqq M_1$, $\mathscr{B} \nsubseteqq M_2$ and $\mathscr{A}\mathscr{B} = 0$.

REMARK 2.16. If $R$ is a strongly semi-simple ring with 1 such that $\mathscr{M}(R)$, the maximal modular ideal space of $R$, is a Hausdorff space, then by [5: Theorem 6.5] and [5: Theorem 6.15], the $M$-primary ideal exists for each maximal modular ideal $M$ in $R$. In this case, the $M$-primary ideal $p(M)$ is given by the set $\{x \in R \mid \overline{\operatorname{supp}(RxR)} \cap \{M\} = \phi\}$, where $\operatorname{supp}(RxR) = \{M \in \mathscr{M}(R) \mid RxR \nsubseteqq M\}$ by [5: Theorem 6.14].

3. If $\mathscr{A}$ is an ideal of a ring $R$, let

$$\operatorname{supp}(\mathscr{A}) = \{M \in \mathscr{M}(R) \mid \mathscr{A} \nsubseteqq M\}, \quad h(A) = \mathscr{M}(R) \backslash \operatorname{supp}(\mathscr{A}),$$
$$k(F) = \bigcap \{M \in \mathscr{M}(R) \mid M \in F\}.$$

THEOREM 3.1. *Let $R$ be a ring and let*

$$\mathscr{R}(R) = \bigcup \{R/O(M) \mid M \in \mathscr{M}(R)\},$$

*the disjoint union of a family of rings $\{R/O(M) \mid M \in \mathscr{M}(R)\}$. For* *each* $r \in R$ *define* $\hat{r}$ *to be the function from* $\mathscr{M}(R)$ *into* $\mathscr{R}(R)$ *such that* $\hat{r}(M) = r + O(M)$ *for each* $M \in \mathscr{M}(R)$. *Let* $\tau = \{\hat{r}(U) \mid r \in R$ *and* $U$ *is an open set in* $\mathscr{M}(R)\}$. *Let* $\rho$ *be a family of sets consisting of arbitrary unions of the members of* $\tau$. *Then* $(\mathscr{R}(R), \rho)$ *is a topological space and each point* $\hat{r}(M)$ *of* $\mathscr{R}(R)$, $r \in R$ *and* $M \in \mathscr{M}(R)$, *is contained in an open set which is homeomorphic to an open set of* $\mathscr{M}(R)$ *under the canonical projection*: $\hat{r}(M) \mid\rightarrow M$, *that is,* $\mathscr{R}(R)$ *is a sheaf of rings over* $\mathscr{M}(R)$.

*Proof.* In $\eta \in \hat{r}_1(U) \cap \hat{r}_2(V)$ for some $r_1, r_2 \in R$ and some open sets $U$, $V$ in $\mathscr{M}(R)$ then there is $M \in U \cap V$ such that $r_1 - r_2 \in O(M)$. Hence $((r_1 - r_2)R_1)^{\perp} \nsubseteq M$. Let $W = U \cap V \cap \operatorname{supp}((r_1 - r_2)R_1)^{\perp})$. Then $M \in W$ and $\eta \in \hat{r}_1(W) \subseteqq \hat{r}_1(U) \cap \hat{r}_2(V)$. Since $W$ is an open set of $\mathscr{M}(R)$, $\hat{r}_1(W) \in \tau$ and hence $(\mathscr{R}(R), \rho)$ is a topological space. In view of [1: 2.2 p. 151], it suffices to show that if $\hat{r}(M) = 0$ for some $r \in R$ and $M \in \mathscr{M}(R)$ then there exists an open set $U$ of $M$ such that $\hat{r}(U) = 0$. But this is immediate since if $\hat{r}(M) = 0$ then $r \in O(M)$ and $(rR_1)^{\perp} \nsubseteq M$. Therefore, if we let $U = \operatorname{supp}((rR_1)^{\perp})$ then $\hat{r}(U) = 0$ since $r \in \bigcap \{O(M) \mid M \in U\}$.

THEOREM 3.2. *Let* $R$ *be a strongly harmonic ring. If* $F$ *is a compact subset of* $\mathscr{M}(R)$ *and* $M_0 \notin F$ *for some* $M_0 \in \mathscr{M}(R)$ *then there exist ideals* $\mathscr{A}$ *and* $\mathscr{B}$ *such that* $\mathscr{A}\mathscr{B} = O$, $M_0 \in \operatorname{supp}(\mathscr{A})$ *and* $F \subseteqq \operatorname{supp}(\mathscr{B})$.

*Proof.* Since $R$ is strongly harmonic, for any $M \in F$ there exist ideals $\mathscr{A}'$, $\mathscr{B}'$ in $R$ such that $M_0 \in \operatorname{supp}(\mathscr{A}')$, $M \in \operatorname{supp}(\mathscr{B}')$ and $\mathscr{A}'\mathscr{B}' = 0$. Since $F$ is compact, there exist a finite number of ideals, say $\mathscr{A}_1, \mathscr{A}_2, \cdots, \mathscr{A}_n, \mathscr{B}_1, \mathscr{B}_2, \cdots, \mathscr{B}_n$ such that

$$M_0 \in \bigcap_{i=1}^{n} \operatorname{supp}(\mathscr{A}_i) = \operatorname{supp}(\mathscr{A}_1\mathscr{A}_2 \cdots \mathscr{A}_n)$$

and $F \subseteqq \bigcup_{i=1}^{n} \operatorname{supp}(\mathscr{B}_i) = \operatorname{supp} \sum_{i=1}^{n} \mathscr{B}_i$ such that $\mathscr{A}_i\mathscr{B}_i = 0$ for all $i = 1, 2, \cdots, n$, and $(\mathscr{A}_1\mathscr{A}_2 \cdots \mathscr{A}_n)(\sum_{i=1}^{n} \mathscr{B}_i) = 0$.

THEOREM 3.3. *Let* $R$ *be a strongly harmonic ring. If* $F$ *is a compact subset of* $\mathscr{M}(R)$ *then* $F = h(\bigcap \{O(M) \mid M \in F\})$.

*Proof.* Since $\bigcap_{M \in F} O(M) \subseteqq k(F)$, $F \subseteqq h(\bigcap_{M \in F} O(M))$. Suppose there is $M_0 \in h(\bigcap_{M \in F} O(M))$ such that $M_0 \notin F$. Then by 3.2 there exist ideals $\mathscr{A}$, $\mathscr{B}$ in $R$ such that $M_0 \in \operatorname{supp}(\mathscr{A})$, $F \subseteqq \operatorname{supp}(\mathscr{B})$ and $\mathscr{A}\mathscr{B} = 0$. Hence if $M \in F$ then $\mathscr{B} \nsubseteq M$ and $\mathscr{A} \subseteqq O(M)$. Thus $A \subseteqq \bigcap_{M \in F} O(M)$. Since $M_0 \in h(\bigcap_{M \in F} O(M))$, this means that $\mathscr{A} \subseteqq M_0$ and this is a contradiction.

THEOREM 3.4. *Let $R$ be a strongly harmonic ring with* 1 *and let $\mathscr{R}(R)$ be the sheaf of local rings over $\mathscr{M}(R)$, which is described in* 3.1. *If $F_0$ is a compact subset of $\mathscr{M}(R)$ and $\sigma$ is a section from $F_0$ into $\mathscr{R}(R)$, then there is $r \in R$ such that $\hat{r}|_{F_0} = \sigma$.*

*Proof.* If $M_0 \in F_0$ then there exists an open set $U$ in $\mathscr{M}(R)$ which contains $M_0$ and $r \in R$ such that if $M \in U \cap F_0$ then $\sigma(M) = \hat{r}(M)$. Let $U_0 = \mathscr{M}(R)\backslash F_0$. Since $\mathscr{M}(R)$ is Hausdorff by 2.7, $F_0$ is a closed set. Hence $U_0$ is an open subset of $\mathscr{M}(R)$. There exist a finite number of points $M_1, M_2, \cdots, M_n$ in $F_0$, open sets $U_1, U_2, \cdots, U_n$ such that $M_i \in U_i$, $i = 1, 2, \cdots, n$, and $r_1, r_2, \cdots, r_n$ in $R$ such that $\sigma(M) = \hat{r}_i(M)$ for every $M \in U_i \cap F_0$ for every $i = 1, 2, \cdots, n$. Furthermore, $F_0 \subseteqq \bigcup_{i=1}^n U_i$ and $\mathscr{M}(R) = \bigcup_{i=0}^n U_i$. Let $F_i = \mathscr{M}(R)\backslash U_i$ and let $I_i = \bigcap_{M \in F} O(M)$ for each $i = 0, 1, 2, \cdots, n$. Since $F_i$ is a closed subset of a compact space, it is compact. Hence $F_i = h(I_i)$ for each $i = 0, 1, 2, \cdots, n$ by 3.3. Since $\phi = \bigcap_{i=0}^n F_i = \bigcap_{i=0}^n h(I_i) = h(\sum_{i=0}^n I_i)$, $R = \sum_{i=0}^n I_i$ and $1 = \sum_{i=1}^n e_i$ for some $e_i \in I_i$, $i = 0, 1, 2, \cdots, n$. If $M \in F_i \cap F_0$, then $\hat{r}_i(M)\hat{e}_i(M) = O(M) = \sigma(M)\hat{e}_i(M)$. If $M \in U_i \cap F_0$, then $\hat{r}_i(M)\hat{e}_i(M) = \sigma(M)\hat{e}_i(M)$. Hence, for every $M \in F_0$, $\hat{r}_i(M)\hat{e}_i(M) = \sigma(M)\hat{e}_i(M)$. Thus if we let $r = e_0 + \sum_{i=1}^n r_i e_i$, then for every

$$\begin{aligned} M \in F_0\ \hat{r}(M) &= \hat{e}_0(M) + \sum_{i=1}^n \hat{r}_i(M)\hat{e}_i(M) \\ &= \sigma(M)\hat{e}_0(M) + \sum_{i=1}^n \sigma(M)\hat{e}_i(M) \\ &= \sigma(M)(\sum_{i=0}^n \hat{e}_i(M)) = \sigma(M) . \end{aligned}$$

COROLLARY 3.5. *If $R$ is a strongly harmonic ring with* 1 *then* $R \cong \Gamma(\mathscr{M}(R), \mathscr{R}(R))$.

*Proof.* By 2.5, $r \mapsto \hat{r}$ is a monomorphism from $R$ into $\Gamma(\mathscr{M}(R), \mathscr{R}(R))$. Since $\mathscr{M}(R)$ is a compact space, by 3.4 if $\sigma \in \Gamma(\mathscr{M}(R), \mathscr{R}(R))$ then there is $r \in R$ such that $\sigma = \hat{r}$. Thus $r \mapsto \hat{r}$ is an isomorphism of $R$ onto $\Gamma(\mathscr{M}(R), \mathscr{R}(R))$.

DEFINITION 3.6. We say that a sheaf $\mathscr{R}$ over the space $X$ is soft provided that if $F$ is a compact subset of $X$ and $\sigma \in \Gamma(F, \mathscr{R})$ then there is $\bar{\sigma} \in \Gamma(X, \mathscr{R})$ such that $\bar{\sigma}|_F = \sigma$.

THEOREM 3.7.[1] *Let $R$ be a strongly harmonic ring with* 1. *Then the sheaf $\mathscr{R}(R)$ of local rings which is constructed in* 3.1 *is soft. Conversely, if $\mathscr{R}$ is a soft sheaf of local rings over a Hausdorff compact space $\mathscr{M}$, then $\Gamma(\mathscr{M}, \mathscr{R})$ is a strongly harmonic ring.*

[1] The author is indebted to Professor S. Teleman for this theorem.

*Proof.* By 3.4, $\mathscr{R}(R)$ is soft if $R$ is a strongly harmonic ring with 1. Suppose now that $\mathscr{R}$ is a soft sheaf of local rings over a Hausdorff compact space $\mathscr{M}$. Let $R = \Gamma(\mathscr{M}, \mathscr{R})$. By Theorem 11 of [6: p. 712], $\mathscr{M}$ is homeomorphic to $\mathscr{M}(R)$. Hence we may take $R = \Gamma(\mathscr{M}(R), \mathscr{R})$. Since $\mathscr{M}$ is Hausdorff, if $M_1, M_2 \in \mathscr{M}(R)$ such that $M_1 \neq M_2$ then there exist open sets $U_i$, $i = 1, 2$, in $\mathscr{M}(R)$ such that $M_1 \in U_1$, $M_2 \in U_2$ and $U_1 \cap U_2 = \phi$. If $\sigma \in R$, define

$$|\sigma| = \{M \in \mathscr{M}(R) \mid \sigma(M) \neq 0\} .$$

Let $A_i = \{\sigma \in R \mid |\sigma| \subsetneqq U_i\}$, $i = 1, 2$. Clearly, $A_1$, $A_2$ are ideals of $R$ and $A_1A_2 = 0 = A_2A_1$ since $U_1 \cap U_2 = \phi$. There exists compact sets $K_1$, $K_2$ such that $M_i \in K_i$ and $K_i \subsetneqq U_i$, $i = 1, 2$. Let $F_i = \mathscr{M}(R)\backslash U_i$. Since $\mathscr{R}$ is soft there exist $\sigma_i$ in $\Gamma(\mathscr{M}(R), \mathscr{R})$ such that $\sigma_i(K_i) = 1$ and $\sigma_i(F_i) = 0$, $i = 1, 2$. Hence $A_i \not\subset M_i$ for $i = 1, 2$. Thus $R$ is strongly harmonic.

REMARK 3.8. Let $R$ be a ring and $A$ be a right $R$-module. We will associate with $A$ a sheaf if $\mathscr{R}(R)$-modules over $\mathscr{M}(R)$ (refer [4] for definition). For $M \in \mathscr{M}(R)$, denote $\tilde{A} = \bigcup \{A/A_M \mid M \in \mathscr{M}(R)\}$, the disjoint union of a family of $R$-modules $A/A_M$ indexed by $\mathscr{M}(R)$. Let $\pi: \tilde{A} \mapsto \mathscr{M}(R)$ be given by $\pi^{-1}(M) = A/A_M$. For $a \in A$ and $M \in \mathscr{M}(R)$, let $t_a(M)$ be the image of $a$, under the natural homomorphism of $A$ onto $A/A_M$. Topologize $\tilde{A}$ by taking all sets $t_a(U)$, with $a \in A$, $U$ is an open set in $\mathscr{M}(R)$, as a basis for the open sets. Then $\tilde{A}$ becomes a sheaf of $\mathscr{R}(R)$-modules over $\mathscr{M}(R)$. The justification of this statement and proof of this result require only slight modifications of 3.1.

THEOREM 3.9. *Let $R$ be a strongly harmonic ring with 1 and let $A$ be a unitary right $R$-module. Then the mapping $\xi_A: a \mapsto t_a$ is a semi-linear isomorphism of $A$ onto the $\Gamma(\mathscr{M}(R), \mathscr{R}(R))$-module $\Gamma(\mathscr{M}(R), \tilde{A})$ in the sense that $\xi_A$ is a group isomorphism satisfying $\xi_A(ar) = t_a \cdot \hat{r}$ for $a \in A$, $r \in R$ where $t_a(M) = a + A_M$ for all $m \in \mathscr{M}(R)$.*

*Proof.* We omit the proof because it is only a variant of the proof of 3.4. However, it is worth noting that the full strength of 2.4 is needed here to prove that $\xi_A$ is an injection.

4. A ring is called *biregular* if every principal ideal of the ring is generated by a central idempotent. In [2], Dauns and Hofmann proved that if $R$ is a ring with 1 then $R$ is biregular if and only if $R$ is isomorphic to the ring of all global sections of a sheaf of simple rings over a Boolean space. By applying this theorem, we

will show that if $R$ is a ring with 1 such that it contains no nonzero nilpotent elements then $R$ is biregular if, and only if, every prime ideal of $R$ is a maximal ideal of $R$.

PROPOSTION 4.1. *If $R$ is a biregular ring then every prime ideal $M$ of $R$ is a maximal ideal of $R$.*

*Proof.* If $R$ is biregular then so is the ring $R/M$ for any ideal $M$ of $R$. Hence if $M$ is a prime ideal then $R/M$ is a prime biregular ring. Therefore, $R/M$ contains no proper principal ideal for if $R/M$ contains a proper principal ideal, then $R/M$ would have two nonzero ideals whose product is zero. Thus $R/M$ is a simple ring and $M$ is a maximal ideal of $R$.

PROPOSITION 4.2. *Let $R$ be a ring and $M$ be a prime ideal of $R$. Define $O_M = \{x \in R \mid xy = 0$ for some $y \notin M\}$. If $R$ contains no nonzero nilpotent elements then $O_M = O(M)$.*

*Proof.* Clearly $O(M) \subseteqq O_M$. If $x, y$ are elements of $R$ such that $xy = 0$ then $yx$ is zero since $yxyx = 0$ and $R$ contains no nonzero nilpotent elements. Furthermore, if $r \in R$, $xry = 0$ since $xry\,xry = 0$. Thus $O(M) = O_M$.

PROPOSITION 4.3. *Let $R$ be a ring without nilpotent elements. If every prime ideal of $R$ is maximal, then $M = O(M)$ for every prime ideal $M$ of $R$.*

*Proof.* If every prime ideal of $R$ is maximal, then every prime ideal is a maximal prime ideal. Hence by [3: 2.4], $M = O_M$ for each prime ideal $M$ of $R$. Thus by 4.2 $M = O(M)$.

PROPOSITION 4.4. *If $R$ is a ring with 1 such that $R$ contains no nonzero nilpotent elements and if every prime ideal of $R$ is maximal, then $\mathscr{M}(R)$ is a Boolean space.*

*Proof.* This is a direct consequence of [3: 2.5].

THEOREM 4.5. *Let $R$ be a ring with 1 such that it contains no nonzero nilpotent elements. Then $R$ is biregular if, every prime ideal of $R$ is maximal.*

*Proof.* If $R$ is biregular then by 4.1, every prime ideal is maximal. Conversely, suppose that every prime ideal of $R$ is maximal. Since $R$ is a ring without nilpotent elements, the intersection of prime ideals of $R$ is zero. Since $\mathscr{M}(R)$ is a Hausdorff space by 4.4, if $M_1$, $M_2$ are two distinct elements in $\mathscr{M}(R)$, then there exist ideals $\mathscr{A}$ and $\mathscr{B}$ such that $\mathscr{A} \nsubseteq M_1$, $\mathscr{B} \nsubseteq M_2$ and $\mathscr{A}\mathscr{B} = 0$. Hence $O(M)$ is $M$-primary for every $M \in \mathscr{M}(R)$ by 2.13 and thus $R \cong \Gamma(\mathscr{M}(R), \mathscr{R}(R))$ by 3.5. Since $\mathscr{M}(R)$ is a Boolean space by 4.4 and $M = O(M)$ by 4.3, $R$ is a biregular ring by [2: 2.19, p. 108].

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