

ON A RESULT OF CASSELS

R. T. WORLEY

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Let α be an irrational algebraic number of degree k over the rationals. Let K denote the field generated by α over the rationals and let \mathfrak{a} denote the ideal denominator of α . Then Cassels [3] has shown that for sufficiently large integral $N > 0$ distinctly more than half the integers n ,

$$N < n \leq N + [10^{-6}N]$$

are such that $(n+\alpha)\mathfrak{a}$ is divisible by a prime ideal \mathfrak{p}_n which does not divide $(m+\alpha)\mathfrak{a}$ for any integer $m \neq n$ satisfying $N < m \leq N + [10^{-6}N]$. The purpose of this note is to point out that minor modification of Cassel's proof enables the extension of the interval for n from $N < n \leq N + [10^{-6}N]$ to $0 \leq n \leq N$, and to derive results on the proportion of values n , $0 \leq n \leq N$ for which the values $f(n)$ of a given integral polynomial in n are divisible by a prime $p > N$.

THEOREM. *The proportion $\rho(\alpha, N)$ of integers n , $0 \leq n \leq N$, for which $(n+\alpha)\mathfrak{a}$ is divisible by a prime ideal \mathfrak{p}_n not dividing $(m+\alpha)\mathfrak{a}$ for integral $m \neq n$, $0 \leq m \leq N$, has the properties*

- (i) $\liminf_{N \rightarrow \infty} \rho(\alpha, N) > \frac{1}{2}$.
- (ii) $\liminf_{N \rightarrow \infty} \rho(\alpha, N) \geq 1 - \frac{1}{2k} + O(k^{-\frac{3}{2}})$ as $k \rightarrow \infty$.

PROOF. The proof is basically that of Cassels with $N \log N$ for M instead of $[10^{-6}N]$ and with the estimate cn^k as a lower bound for norm $\{(n+\alpha)\mathfrak{a}\}$ instead of cn^2 . For simplicity we prove (i) and (ii) for $\sigma(\alpha, N)$ which denotes the proportion of integers n , $N < n \leq N \log N$, for which $(n+\alpha)\mathfrak{a}$ is divisible by a prime ideal \mathfrak{p}_n not dividing $(m+\alpha)\mathfrak{a}$ for integral $m \neq n$, $N < m \leq N \log N$. Plainly

$$\liminf_{N \rightarrow \infty} \rho(\alpha, N) = \liminf_{N \rightarrow \infty} \sigma(\alpha, N).$$

As in Cassels' proof \mathfrak{p} shall denote a prime ideal of K which is of the first degree and unambiguous i.e. \mathfrak{p}^2 does not divide $\mathfrak{p} = \text{norm } \mathfrak{p}$ and if

$p|(n+\alpha)a$ for any integer n then $(n+\alpha)a$ is not divisible by any $p' \neq p$ with norm $p' = p$. For any integer n ,

$$(1) \quad (n+\alpha)a = b \prod_p p^{u(p)}$$

where the $u(p)$ are integers and b contains all the factors of $(n+\alpha)a$ which are not p 's. Norm b is bounded and norm $\{(n+\alpha)a\} > cn^k$ where $c > 0$ is independent of n provided n is large enough. Hence on taking logarithms

$$(2) \quad \sum u(p) \log p \geq k \log n - C$$

where C is independent of n (n large enough).

Denote by $\mathfrak{S} = \mathfrak{S}_N$ the set of all n in the range $N < n \leq N \log N$ with the property that

$$(3) \quad p^{u(p)} < N \log N$$

for all p in the factorization (1). An upper bound for the number $S = \rho N \log N$ of elements of \mathfrak{S} is found as follows.

For any p and integral $v > 0$ write

$$\phi(p^v, n) = \begin{cases} \log p & \text{if } p^v|(n+\alpha)a \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sigma(n) = \sum \phi(p^v, n)$$

where the summation is over all p and v with $p^v < N \log N$. Take $\epsilon > 0$ arbitrarily — then $\log N > (1-\epsilon) \log(N \log N)$ for sufficiently large N . From (2) and (3) we therefore obtain

$$\sigma(n) \geq (k-\epsilon) \log(N \log N)$$

for all $n \in \mathfrak{S}$ and

$$(4) \quad \sum_{n \in \mathfrak{S}} \sigma(n) \geq (k-\epsilon)S \log(N \log N)$$

for N sufficiently large.

Writing $\sigma(n) = \sigma_1(n) + \sigma_2(n) + \sigma_3(n)$ where $\sigma_1, \sigma_2, \sigma_3$ are the sums of $\phi(p^v, n)$ with p^v in the sets

$$\begin{aligned} \sigma_1 : v > 1, p^v < N \log N \\ \sigma_2 : v = 1, (N \log N)^{\frac{1}{2}} \leq p < N \log N \\ \sigma_3 : v = 1, p < (N \log N)^{\frac{1}{2}} \end{aligned}$$

we obtain, as in Cassels' proof, that

$$(5) \quad \sum_{n \in \mathfrak{S}} \sigma_1(n) = O(M \log M),$$

$$(6) \quad \sum_{n \in \mathfrak{G}} \sigma_2(n) \leq (\frac{1}{2} + o(1)) M \log M$$

and

$$(7) \quad \sum_{n \in \mathfrak{G}} (\sigma_3(n))^2 \leq (\frac{3}{8} + o(1)) M \log^2 M$$

where we have used M to denote $N \log N$ for simplicity and where $o(1)$ refers to the limit as $N \rightarrow \infty$.

Combining (4), (5) and (6) yields

$$(8) \quad \sum_{n \in \mathfrak{G}} \sigma_3(n) \geq ((k - \epsilon)\rho - \frac{1}{2} + o(1)) M \log M.$$

Then either $(k - \epsilon)\rho < \frac{1}{2} + o(1)$ or

$$((k - \epsilon)\rho - \frac{1}{2} + o(1))^2 M^2 \log^2 M \leq (\rho M)(\frac{3}{8} + o(1)) M \log^2 M,$$

from which it follows that for sufficiently large N ,

$$(9) \quad \rho \leq \frac{k - \epsilon + \frac{3}{8}}{2(k - \epsilon)^2} + \frac{1}{2(k - \epsilon)^2} (\frac{9}{64} + \frac{3}{4}(k - \epsilon) + o(1)k^2)^{\frac{1}{2}}$$

where $o(1)$ refers to the limit as $N \rightarrow \infty$. But, as in Cassels' proof, $\sigma(\alpha, N) \geq 1 - \rho$, so the desired results follow from (9).

COROLLARY. *Let $f(n)$ denote a polynomial in n with integral coefficients and leading coefficients 1, irreducible over the rationals, and let $\rho(f, N)$ denote the number of integers n , $0 \leq n \leq N$, for which $f(n)$ is divisible by a prime $p > N$. Then*

$$(i) \quad \liminf_{N \rightarrow \infty} \rho(f, N) > \frac{1}{2}.$$

(ii) $\liminf_{N \rightarrow \infty} \rho(f, N) \geq 1 - (1/2k) + O(k^{-\frac{1}{2}})$ as $k \rightarrow \infty$ where k denotes the degree of f .

PROOF. Apply the theorem to $-\alpha$, where α is a root of f , noting that $f(n) = \text{norm}(n - \alpha)$.

It should be pointed out the bound on $\rho(f, N)$ seems to be much weaker than the probable bound. The argument, however implausible it may sound, that the numbers $f(0), f(1), \dots, f(N)$ are evenly distributed amongst the numbers $1, 2, \dots, N^k$ with respect to divisibility by primes greater than N leads to the conclusion that $\lim_{N \rightarrow \infty} \rho(f, N) = 1 - \rho(k)$ where $\rho(k)$ is defined as follows. Let $\psi(x, y)$ be the number of positive integers $\leq x$ free of prime divisors $> y$ — then de Bruijn [1], Buchstab [2] Chowla and Vijayaraghavan [4] and Ramaswami [5] have shown that

$$\lim_{y \rightarrow \infty} y^{-k} \psi(y^k, y) = \rho(k)$$

where $\rho(k)$ can be calculated in an inductive manner and $\rho(k) = o(k^{-n})$ for

all positive integers n . A limited amount of computing indicated that $1-\rho(2)$ is the correct limit for $\rho(f, N)$ for $f(n) = n^2+n+1$ and that $1-\rho(3)$ was a possible limit for $\rho(f, N)$ for $f(n) = n^3+n^2+n+2$.

References

- [1] N. G. de Bruijn, 'On the number of positive integers $\leq x$ and free of prime divisors $> y$ ', *Proc. K. Nederl. Akad. Wetensch.* 54 (1951) 50—60.
- [2] A. A. Buchstab, 'On those numbers in an arithmetic progression all prime factors of which are small in magnitude', *Doklady Akad. Nauk SSSR* 67 (1949) 5—8.
- [3] J. W. S. Cassels, 'Footnote to a note of Davenport and Heilbronn', *J. Lond. Math. Soc.* 36 (1961) 177—184.
- [4] S. Chowla and T. Vijayaraghavan, 'On the largest prime divisors of numbers', *J. Ind. Math. Soc. (N.S.)* 11 (1947) 31—37.
- [5] V. Ramaswami, 'The number of positive integers $< x$ and free of prime divisors $> x^\epsilon$, and a problem of S. Pillai', *Duke Math. J.* 16 (1949) 99—109.

Monash University
Clayton, Vic. 3168