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ON A RESULT OF TZITZEICA AND A NEW ASYMPTOTIC TRANSFORMATION OF MINIMAL PROJECTIVE SURFACES

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In the memory of the eminent Geometer E. ČECH

FIRST PART

1. Introduction. Tzitzeica's result with which we are concerned here is not mentioned in his classical book, *Géométrie différentielle projective des réseaux* [3], and was brought to our attention through R. Calapso's lecture at the *Symposium de géométrie et analyse globale, G. Tzitzeica et D. Pompeiu* [4].

It is well known that Tzitzeica discovered a class of surfaces whose curvature K at any point M is proportional to the fourth power of the distance d of a fixed point 0 (the so-called *central point of the surface*) to the tangent plane at M. Most important in this class is the subclass which satisfies the condition

$$(\alpha) \qquad \frac{K}{d^4} = \text{constant} ,$$

and which was characterized as S-surfaces, see [1], p. 128. Tzitzeica showed that the coordinates x, y, z, of a generic point M on an unruled surface S relative to the asymptotics, are three linear independent integrals of a system which can be reduced to the from

$$(1.1) \qquad \frac{\partial^2 x}{\partial u^2} = \frac{h_u}{h}\frac{\partial x}{\partial u} + \frac{1}{h}\frac{\partial x}{\partial v}; \quad \frac{\partial^2 x}{\partial u\,\partial v} = hx\,; \quad \frac{\partial^2 x}{\partial v^2} = \frac{1}{h}\frac{\partial x}{\partial u} + \frac{h_v}{h}\frac{\partial x}{\partial v},$$

h being a solution of the equation

$$(1.2) \qquad \frac{\partial^2 \log h}{\partial u\,\partial v} = h - \frac{1}{h^2}.$$

E. WILCYNSKI [2] studied unruled surfaces with indeterminate directrices. Such a surface is determined up to a homography, by a completely integrable system which can be reduced to

$$\frac{\partial^2 x}{\partial u^2} = -\frac{\beta_u}{\beta} x_u + \beta x_v\,; \quad x_{vv} = \beta x_u - \frac{\beta_v}{\beta} x_v\,, \tag{1.3}$$

β being a solution of the equation

$$\frac{\partial^2 \log \beta}{\partial u\, \partial v} = \beta^2 + \frac{k}{\beta}\,, \tag{1.4}$$

where k is an arbitrary constant, (see [5]). Wilczynski [2] also determined in finite form the surfaces belonging to the case $k = 0$, that is, surfaces with indeterminate directrices and the asymptotic curves belonging to linear complexes. FUBINI and ČECH showed in [5] that the surfaces (1.3) coincide for $k \neq 0$ with Tzitzeica's S-surfaces and, in recognition of Wilcynski's important contribution, called them Tzitzeica-Wilcynski surfaces for $k \neq 0$, and limited Tzitzeica-Wilcynski surfaces for $k = 0$.

Tzitzeica's result. In two other notes [1], p. 188–190, 191–192, Tzitzeica stated (without demonstration) the following results.

If $M(x, y, x)$ is a generical point of an S-surface, that is, a surface whose coordinates satisfy a system of the form (1.1), *then the point $\overline{M}(\bar{x}, \bar{y}, \bar{z})$ such that*

$$\bar{x} = x - \frac{1+c}{h}\frac{\partial \log R}{\partial v}\frac{\partial x}{\partial u} - \frac{1-c}{h}\frac{\partial \log R}{\partial u}\frac{\partial x}{\partial v}\,, \tag{1.5}$$

(a similar formula for $\bar{y}, \bar{z}$) *where R is a solution of the completely integrable system*

$$\frac{\partial^2 R}{\partial u^2} = \frac{h_u}{h}\frac{\partial R}{\partial u} - \frac{1+c}{1-c}\frac{1}{h}\frac{\partial R}{\partial v}\,, \tag{1.6}$$

$$\frac{\partial^2 R}{\partial u\, \partial v} = hR\,,$$

$$\frac{\partial^2 R}{\partial v^2} = -\frac{1-c}{1+c}\frac{1}{h}\frac{\partial R}{\partial u} + \frac{h_v}{h}\frac{\partial R}{\partial v}\,,$$

and c an arbitrary constant, — describes the second focal surface of the congruence generated by the straight line $(M, \overline{M})$. This focal surface is also an S-surface, and the congruence $(M, \overline{M})$ is a W-congruence.

We note here that c cannot be zero, as will be shown later.

Tzitzeica did not explain how he had arrived at the system (1.6) or what relation exists between (1.1) and (1.6).

H. IONAS, the well-known geometer, who achieved great progress in metric study of W-congruences, gave in [6] a proof of existence of an asymptotic transformation

Π_n (n a constant) which transforms a surface characterized by

$$\left\{ \begin{matrix} 1\ 1 \\ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 2\ 2 \\ 1 \end{matrix} \right\}, \tag{1.7}$$

$\left(\left\{ \begin{matrix} i,\ y \\ k \end{matrix} \right\} \text{ being Christoffel's second set of symbols} \right)$ into another surface of the same class. (This is the transformation by W-congruence of isothermic-surfaces which Fubini, [5] p. 283–287, obtained in his study of asymptotic transformations of surfaces, preserving Darboux's curves).

Next Ionas [6] noted that surfaces with K/d^4 = const. belong to the class characterized by (1.7), particularized his transformation Π_n to S-surfaces and obtained after rather complicated calculations, Tzitzeica's system (1.6) showing at the same time that surfaces generated by the point (1.5) are also S-surfaces. Ionas likewise failed to show the connection between (1.1) and (1.6). To show this connection we shall define the $\mathcal{D}$-correspondence between two unruled surfaces and we shall show that *Tzitzeica's asymptotic transformation is a particular case of a general asymptotic transformation of minimal projective surfaces obtained through $\mathcal{D}$-asymptotic correspondences.*

2. $\mathcal{D}$-correspondence between two unruled minimal projective surfaces. According to G. Thomsen [9], unruled surfaces which satisfy the condition

$$\delta \iint I \sqrt{(A)}\, du\, dv = 0\,, \tag{2.1}$$

where A and I are, respectively, the discriminant of Fubini's first fundamental form and Fubini-Pick's invariant, are minimal projective. If $x(u, v)$ is an unruled surface and u, v its asymptotic parameters, then it is determined by the completely integrable system

$$x_{uu} = \theta_u x_u + \beta x_v + p_{11} x \,.\, x_{vv} = \gamma x_u + \theta_v x_v + p_{22} x\,. \tag{2.2}$$

The integrability conditions are

$$L_v + 2\beta\gamma_u + \gamma\beta_u = 0\,; \quad M_u + 2\gamma\beta_v + \beta\gamma_v = 0\,, \tag{2.3}$$
$$\beta M_v + 2M\beta_v + \beta_{vvv} = \gamma L_u + 2\gamma L_u + \gamma_{uuu}\,,$$

where

$$L = \theta_{uu} - \tfrac{1}{2}\theta_u^2 - \beta_v - \beta\theta_v - 2p_{11}\,; \quad M = \theta_{vv} - \tfrac{1}{2}\theta_v^2 - \gamma_u - \gamma\theta_u - 2p_{22}\,. \tag{2.4}$$

According to Mayer [10], one of the following conditions

$$\beta M_v + 2M\beta_v + \beta_{vvv} = 0\,; \quad \gamma L_u + 2L\gamma_u + \gamma_{uuu} = 0\,, \tag{2.5}$$

characterizes a minimal projective surface. Mayer succeeded in proving the following result: *A minimal projective surface always allows* (exclusive of some other *W*-transformations) *a class of* ∞^5 *W-transformations, which are also minimal projective surfaces.*

In this case we have the relation

$$\mathscr{B} = \sigma \mathscr{A}\,, \tag{2.6}$$

where σ is an arbitrary constant and the expressions $\mathscr{B}$ and $\mathscr{A}$ will be explained in the next section.

According to Čech [8], asymptotic correspondences between two unruled surfaces with contact invariants r and s, see [5′], fall into three categories, the third of which is characterized by

$$r = k_1\,; \quad s = k_2\,, \tag{2.7}$$

k_1, k_2 being constants such that $k_1 \neq k_2 \neq 0 \neq \pm 1$.

Let there be two unruled surfaces in asymptotic correspondence, such that one of the surfaces is determined by the system (2.2). If the correspondence is of the third category, then the second surface is determined by the quantities

$$\frac{\beta^*}{\beta} = k_1\,; \quad \frac{\gamma^*}{\gamma} = k_2\,, \tag{2.8}$$

which should satisfy integrability conditions like (2.3)

We denote by

$$x^*_{uu} = \theta^*_u x^*_u + \beta^* x^*_v + p^*_{11} x^*\,; \quad x^*_{vv} = \gamma^* x^*_u + \theta^*_v x^*_v + p^*_{22} x^*\,, \tag{2.9}$$

the completely integrable system which should determine the second surface.

We define a $\mathscr{D}$*-correspondence as an asymptotic correspondence of the third category between two surfaces or between their systems, which conserves Fubini's first and third differential forms.*

Consequently a $\mathscr{D}$*-correspondence exists iff*:

$$k_1 k_2 = 1 \quad \text{and} \quad L^* = L; \quad M^* = M\,. \tag{2.10}$$

We want to show that such a correspondence may hold *only between minimal projective surfaces.*

Indeed, let us suppose that surfaces $x(u, v)$ and $x^*(u, v)$ are in a $\mathscr{D}$-correspondence. Then condition (2.10) holds. From the third integrability condition of (2.3) and the similar condition for system (2.9), results if we denote by $k_1 = \alpha$,

$$(\alpha^2 - 1)\,(\beta M_v + 2M\beta_v + \beta_{vvv}) = 0\,. \tag{2.11}$$

Taking into account that $\alpha \neq \pm 1$ it follows that

$$\beta M_v + 2M\beta_v + \beta_{vvv} = 0\,, \tag{2.12}$$

and according to Mayer's above-mentioned condition, *it follows that* $x(u, v)$ *is a minimal projective surface.*

Therefore we have the following result.

A $\mathcal{D}$-asymptotic correspondence can exist only between minimal projective surfaces.

In a paper in the Archivum Mathematicum, Brno [7] we obtained some results which can be formulated, using the above definition of $\mathcal{D}$-correspondence, in the following manner.

1. *Every surface in $\mathcal{D}$-asymptotic correspondence with an unruled minimal projective surface is also minimal projective.*

2. *An unruled minimal projective surface* (Γ) *yields through a $\mathcal{D}$-asymptotic correspondence a class C of surfaces of the same category.*

3. *These last surfaces do not belong to the class of* ∞^5 *W-transformation of Mayer.*

Let us now recall the surfaces considered by Tzitzeica. The projective invariants of system (1.1) are

$$\beta = \gamma = \frac{1}{h}\,; \quad L = \frac{h_{uu}}{h} - \frac{3h_u^2}{2h^2}\,; \quad M = \frac{h_{vv}}{h} - \frac{3h_v^2}{2h^2}\,, \tag{2.13}$$

and we see immediately that Mayer's condition (2.5) is satisfied, that is (see also [10]) *Tzitzieca's surfaces are minimal projective.*

We can interpret system (1.6) used by Tzitzeica for his asymptotic transformation of surfaces (1.1) as a system which determines for different $c \neq 0$ a class of surfaces in $\mathcal{D}$-asymptotic correspondence with surfaces (1.1) which are also minimal projective surfaces.

Denoting by

$$\alpha = \frac{a}{b}\,, \quad a = 1 + c\,; \quad b = c - 1\,; \tag{2.14}$$

then in (2.8) we have

$$\beta^* = -\frac{1 + c}{1 - c}\,\beta\,; \quad \gamma^* = -\frac{1 - c}{1 + c}\,\gamma\,. \tag{2.8'}$$

Consequently, with a $\mathcal{D}$-asymptotic correspondence we obtain precisely system (1.6).

We thus have the following result:

The system (1.6) *used by Tzitzeica for asymptotic transformation of his surfaces is none other than the one obtained through $\mathcal{D}$-correspondence of system* (1.1).

This is the connection between system (1.6) and (1.1). Regarding the asymptotic transformation discovered by Tzitzeica, we shall show that it is a particular case of general asymptotic transformation of minimal projective surfaces, including also asymptotic transformation of the limiting Tzitzeica-Wilczynski surfaces.

I am indebted to the results of the late eminent geometer, George Tzitzeica, which appeared in the mentioned two papers, which enabled us to extend our results recently obtained for minimal projective surfaces in Archivum Mathematicum Tomus 6 Brno.

SECOND PART

3. Recapitulation of Fubini's general method for the study of *W*-congruences with a given focal surface. Let $x(u, v)$ be an unruled surface referred to the asymptotic parameters u, v the first focal surface of a congruence generated by the tangents to the curves

$$\frac{du}{A} = \frac{dv}{B}. \tag{3.1}$$

Then, according to Fubini [5], the point

$$\bar{x} = \mu x + A x_u + B x_v, \tag{3.2}$$

(similar equations for $\bar{y}$, $\bar{z}$, $\bar{t}$) describes the second focal surface of the congruence.

The congruence is W iff

$$\frac{\partial}{\partial u}\left(\frac{A_v + B\gamma}{A}\right) = \frac{\partial}{\partial v}\left(\frac{B_u + A\beta}{B}\right). \tag{3.3}$$

In this case, one can find a common factor δ so that $\bar{A} = \delta A$ and $\bar{B} = \delta B$ can be a solution (other $\bar{A} = \bar{B} = 0$) of the system

$$\bar{A}_v + \bar{B}\gamma = 0; \quad \bar{B}_u + \bar{A}\beta = 0. \tag{3.4}$$

Let us write again A, B, instead of $\bar{A}$, $\bar{B}$ respectively. Then we have:

$$\mu = -\tfrac{1}{2}(A_u + A\theta_u + B_v + B\theta_v); \quad \lambda = -\tfrac{1}{2}(A_u + A\theta_u - B_v - B\theta_v). \tag{3.5}$$

The following quantities

$$\mathcal{A} = AA_{uu} - \tfrac{1}{2}A_u^2 + A^2 L; \quad \mathcal{B} = BB_{vv} - \tfrac{1}{2}B_v^2 + B^2 M, \tag{3.6}$$

$$N = 2(\mathcal{B} - \mathcal{A}); \tag{3.7}$$

are very important in Fubini's theory.

For the second focal surface $\bar{x}(u, v)$ we have

$$\bar{\theta} = \theta + \log|N|, \tag{3.8}$$

$$\bar{\beta} + \beta = -\frac{B}{A}\frac{N_u}{N}; \quad \bar{\gamma} + \gamma = -\frac{A}{B}\frac{N_v}{N},$$

$$\bar{p}_{11} = p_{11} + \beta_v + \beta\theta_v + \frac{1}{A}\lambda\frac{N_u}{N}; \quad \bar{p}_{22} = p_{22} + \gamma_u + \gamma\theta_u - \frac{\lambda}{B}\frac{N_v}{N}.$$

N is constant for W-transformations obtained by application of a null reciprocity to the first surface, and $N = 0$ for W-congruences with a degenerate second focal surface.

4. A new asymptotic transformation of minimal projective surfaces. Let $x(u, v)$ be an unruled minimal projective surface and $x^*(u, v)$, one of the surfaces in $\mathcal{D}$-correspondence with $x(u, v)$.

For simplicity, we suppose that the coordinates of the two surfaces are non-homogeneous. Then, they are the respective solutions for the systems

$$x_{uu} = \theta_u x_u + \beta x_v \, ; \qquad x_{vv} = \gamma x_u + \theta_v x_v \, , \tag{4.1}$$

$$x^*_{uu} = \theta^*_u x^*_u + \beta^* x^*_v \, ; \quad x^*_{vv} = \gamma^* x^*_u + \theta^*_v x^*_v \, . \tag{4.2}$$

Setting

$$\alpha = \frac{a}{b}\, , \tag{4.3}$$

we obtain from (2.8), (2.10) and (2.4)

$$\beta^* = \frac{a}{b}\beta \, ; \quad \gamma^* = \frac{b}{a}\gamma \, , \tag{4.4}$$

and

$$\theta^*_{uu} - \tfrac{1}{2}\theta^{*2}_u - \frac{a}{b}\beta\theta^*_v - \frac{a}{b}\beta_v = \theta_{uu} - \tfrac{1}{2}\theta^2_u - \beta_v - \beta\theta_v \, , \tag{4.5}$$

$$\theta^*_{vv} - \tfrac{1}{2}\theta^{*2}_v - \frac{b}{a}\gamma\theta^*_u - \frac{b}{a}\gamma_u = \theta_{vv} - \tfrac{1}{2}\theta^2_v - \gamma_u - \gamma\theta_u \, .$$

We note that in our case, if one of the systems (4.1) or (4.2) is completely integrable, so is the other. For example if $x(u, v)$ is a minimal projective surface. Then conditions (2.3) are satisfied. From (4.4) and (2.10) it follows that the conditions (2.2) for system (4.2) are also satisfied, and according to Mayer's condition (2.8), we have

$$\beta^* M^*_v + 2M^*\beta^*_v + \beta^*_{vvv} = \gamma^* L^*_u + 2L^*\gamma^*_u + \gamma^*_{uuu} = 0 \, . \tag{4.6}$$

We first prove the following result.

For every non-constant solution of one of the systems (4.1) *and* (4.2) *the functions*

$$A = -a(\log x^*)_v \, ; \quad B = b(\log x^*)_u \, ; \tag{4.7}$$

or

$$A^* = b(\log x)_v \, ; \quad B^* = -a(\log x)_u \, ; \tag{4.7'}$$

satisfy Fubini's conditions (3.3).

Indeed, let for example x^* be a non-constant solution of system (4.2). Bearing in mind (4.2), one verifies that

$$(4.8)\qquad \frac{A_v + B\gamma}{A} = \theta_v^* - (\log x^*)_v\,; \quad \frac{B_u + A\beta}{B} = \theta_u^* - (\log x^*)_u\,;$$

consequently

$$(4.9)\qquad \frac{\partial}{\partial u}\frac{A_v + B\gamma}{A} = \frac{\partial}{\partial v}\frac{B_u + A\beta}{B} = \theta_{uv}^* - (\log x^*)_{uv}\,.$$

It is seen that the straight line $(x, \bar{x})$ *where*

$$(4.10)\qquad \bar{x} = \mu x + A x_u + B x_v\,,$$

generates a W-congruence.

Similarly we can verify that the point

$$(4.10')\qquad \bar{x}^* = \mu^* x^* + A^* x_u^* + B^* x_v^*\,,$$

generates the second focal surface of a W-congruence if A^* *and* B^* *are given by* (4.7′). This is further proof of the connection between a minimal projective unruled surface and one of his $\mathscr{D}$-correspondence surface.

From (4.8) it follows that we may multiply the functions A and B by the factor

$$(4.11)\qquad \sigma = x^* e^{-\theta^*}, \quad \theta^* \text{ not constant}\,,$$

or A^*, B^* by

$$(4.11')\qquad \sigma^* = x e^{-\theta}, \quad \theta \text{ not constant}\,,$$

to yield

$$(4.12)\qquad A_v + B\gamma = 0\,; \quad B_u + A\beta = 0\,;$$

or

$$(4.12')\qquad A_v^* + B^*\gamma^* = 0\,; \quad B_u^* + A^*\beta^* = 0\,;$$

in which case we have

$$(4.13)\qquad A = -a e^{-\theta^*} x_v^*\,; \quad B = b e^{-\theta^*} x_u^*\,;$$

or

$$(4.13')\qquad A^* = b e^{-\theta} x_v\,; \quad B^* = -a e^{-\theta} x_u\,.$$

One may restate the result just proved as follows:

Theorem. *With every non-constant solution of one of the systems which determines a minimal projective surface and another of a* $\mathscr{D}$*-asymptotic correspondence, can* *be associated a rectilinear W-congruence whose first focal surface is that determined by the other system.*

This connection could not have been noticed by Tzitzeica or by Ionas. Using Fubini's method, we can simplify Tzitzeica's equation (1.5) in the following form

$$\bar{x} = x - \frac{1+c}{h} R_v x_u - \frac{1-c}{h} R_u x_v ,$$

which was overlooked by H. Ionas [6].

We now prove the following theorem.

Theorem. *The second focal surfaces defined by points* (4.10) *or* (4.10′) *are also minimal projective surfaces.*

To prove this we have first to evaluate the expressions (3.6) and (3.7). From (4.13) we obtain

$$(4.14) \qquad A_u = a\theta^*_u e^{-\theta^*} x^*_v - a e^{-\theta^*} x^*_{uv} ; \quad B_v = -b\theta^*_v e^{-\theta^*} x^*_u + b e^{-\theta^*} x^*_{uv} ;$$

$$A_{uu} = -a e^{-\theta^*} x^*_u (\theta^*_{uv} + \beta^* \gamma^*) + a e^{-\theta^*} x^*_v (\theta^*_{uu} - \theta^{*2}_u - \beta^*_v - \beta^* \theta^*_v) + a e^{-\theta^*} x^*_{uv} ;$$

hence

$$AA_{uu} = a^2 e^{-2\theta^*} x^*_u x^*_v (\theta^*_{uv} + \beta^* \gamma^*) -$$

$$- a^2 e^{-2\theta^*} x^{*2}_v (\theta^*_{uu} - \theta^{*2}_u - \beta^*_v - \beta^* \theta^*_v) - a^2 e^{-2\theta^*} x^*_v x^*_{uv} ,$$

$$(4.15) \qquad -\tfrac{1}{2} A^2_u = -\tfrac{1}{2} a^2 e^{-2\theta^*} (\theta^{*2}_u x^{*2}_{uv} - 2\theta^*_u x^*_v x^*_{uv}) ,$$

$$A^2 L = a^2 e^{-2\theta^*} x^{*2}_v (\theta^*_{uu} - \tfrac{1}{2}\theta^{*2}_u - \beta^*_v - \beta^* \theta^*_v) ,$$

because

$$L^* = L .$$

Therefore we have

$$(4.16) \qquad \mathscr{A} = AA_{uu} - \tfrac{1}{2} A^2_u + A^2 L = a^2 e^{-2\theta^*} [(\theta^*_{uv} + \beta^* \gamma^*) x^*_u x^*_v - \tfrac{1}{2} x^{*2}_{uv}] .$$

Likewise

$$(4.17) \qquad \mathscr{B} = BB_{vv} - \tfrac{1}{2} B^2_v + B^2 M = b^2 e^{-2\theta^*} [(\theta^*_{uv} + \beta^* \gamma^*) x^*_u x^*_v - \tfrac{1}{2} x^{*2}_{uv}] .$$

Consequently from (3.7) we obtain

$$(4.18) \qquad N = 2(\mathscr{B} - \mathscr{A}) = (b^2 - a^2) e^{-2\theta^*} [2(\theta^*_{uv} + \beta^* \gamma^*) x^*_u x^*_v - x^{*2}_{uv}] .$$

Setting

$$(4.19) \qquad b^2 = \delta a^2 ,$$

we have by (4.16) and (4.17)

$$\text{(j)} \qquad \mathscr{B} = \delta \mathscr{A} \, .$$

According to Mayer's above-mentioned result, it follows that the surfaces $\bar{x}(u, v)$ or $\bar{x}^*(u, v)$ are minimal projective surfaces.

5. The particular case of Tzizeica-Wilczyinski surfaces. Suppose that

$$\text{(5.1)} \qquad \theta^* = \theta \, .$$

Then from (4.5) we have

$$\text{(5.2)} \qquad \left(1 - \frac{a}{b}\right)(\beta_v + \beta\theta_v) = 0 \, ; \quad \left(1 - \frac{b}{a}\right)(\gamma_u + \gamma\theta_u) = 0 \, .$$

But by hypothesis $\alpha = a/b \neq 1$, i.e. the surfaces (4.1) and (4.2) are not projectively applicable. Thus

$$\text{(5.3)} \qquad \beta_v + \beta\theta_v = 0 \, ; \quad \gamma_u + \gamma\theta_u = 0 \, ,$$

and we have by (5.3)

$$\text{(5.4)} \qquad \frac{\partial^2 \log \beta : \gamma}{\partial u \, \partial v} = 0 \, ,$$

i.e., $x(u, v)$ is isothermic asymptotic surface of Fubini. By a change of parameters u, v, we can reduce systems (4.1) and (4.2) to the form

$$\text{(5.5)} \qquad x_{uu} = -\frac{\beta_u}{\beta} x_u + \beta x_v \, ; \quad x_{vv} = \beta x_u - \frac{\beta_v}{\beta} x_v \, ,$$

and

$$\text{(5.5')} \qquad x^*_{uu} = -\frac{\beta_u}{\beta} x^*_u + \frac{a}{b} \beta x^*_v \, ; \quad x^*_{vv} = \frac{b}{a} \beta x^*_u - \frac{\beta_v}{\beta} x^*_v \, ,$$

which are Tzitzeica-Wilczynski surfaces.

If β is a solution of the equation

$$\text{(5.6)} \qquad \frac{\partial^2 \log \beta}{\partial u \, \partial v} = \beta^2 + \frac{k}{\beta} \, ,$$

then the two systems are completely integrable. If $k \neq 0$, we may associate with the systems (5.5) and (5.5') respectively the third equation

$$\text{(5.5}_1\text{)} \qquad x_{uv} = \frac{k}{\beta} x \, ,$$

$$\text{(5.5}'_1\text{)} \qquad x^*_{uv} = \frac{k}{\beta} x^* \, ,$$

(for x, y, z, x^*, y^*, z^*, but not for the fourth coordinate t, and t^*), and the systems are completely integrable throughout.

Setting

$$w = x\,; \quad R = x^*\,, \quad h = \frac{1}{\beta}\,, \quad a = 1 + c\,; \quad b = c - 1\,, \tag{5.7}$$

the systems (5.5) and (5.5_1), (5.5′) and ($5.5'_1$) yield Tzitzeica's systems (1.1) and (1.6) for $k = -1$.

From (5.3) and (5.5′) results

$$\theta_{uv} + \beta\gamma = \theta^*_{uv} + \beta^*\gamma^* = -\frac{\partial^2 \log \beta}{\partial u\, \partial v} + \beta^2 = \frac{k}{\beta} \quad (k \neq 0) \tag{5.8}$$

and taking into account (4.18), (5.6) and

$$\beta = \mathrm{e}^{-\theta} = \mathrm{e}^{-\theta^*}\,, \tag{5.9}$$

we have

$$N = (b^2 - a^2)\,\beta^2 \left(2\frac{k}{\beta}\; x^*_u x^*_v - x^{*2}_{uv}\right). \tag{5.10}$$

It remains to note that the constant c in Tzitzeica's system (1.6) (see [1]), where it is asserted that it is arbitrarily constant, cannot be zero because if $c = 0$, then $a = b$ and by (4.18), $N = 0$. According to Fubini [5] however, $N = 0$ signifies that the surfaces generated by the point $\bar{x}$ given by (1.5) degenerates in a straight line.

6. The case of the limiting Tzitzeica-Wilcynski surfaces. Let us suppose that $k = 0$. In such a case the equations

$$x_{uv} = \frac{k}{\beta}\, x\,; \qquad x^*_{uv} = \frac{k}{\beta}\, x^*\,, \tag{6.1}$$

cannot be associated with the systems (5.5) and (5.5′). Observing that we now have

$$\theta_{uv} + \beta\gamma = \theta^*_{uv} + \beta^*\gamma^* = 0\,, \tag{6.2}$$

then it follows from (5.10) that

$$N = (a^2 - b^2)\,\beta^2 x^{*2}_{uv}\,; \tag{6.3}$$

since the limiting Tzitzeica-Wilczynski surfaces are known in finite form (see [2]), we can calculate (6.3).

Indeed, according to Wilczynski or Terracini (see [11]), these surfaces are generated by the points

(A) $$x_1 = 4(U + V) - 2(u - v)(U' - V')\,; \quad x_2 = U' - V'\,;$$

$$x_3 = u + v\,; \quad x_4 = 1\,;$$

where u, v, are asymptotic parameters, U is an arbitrary function of u alone and V of v alone, whose derivatives of third order do not vanish.

The coordinates (A) must satisfy the system (see [2]),

$$(6.4) \qquad \frac{\partial^2 x}{\partial u^2} + \frac{2\Phi_u}{\Phi}\frac{\partial x}{\partial u} + 2\Phi\frac{\partial x}{\partial v} = 0\,; \quad \frac{\partial^2 x}{\partial v^2} + 2\Phi\frac{\partial x}{\partial u} + \frac{\Phi_v}{\Phi}\frac{\partial x}{\partial v}$$

with

$$(6.5) \qquad \frac{\partial^2 \log \Phi}{\partial u\, \partial v} = 4\Phi^2\,.$$

Consequently $U(u)$ and $V(v)$ must be third order polynomials of u alone and v alone. We can therefore take, according to O. Borůvka (see [11]), for the surface (5.5) the coordinates

$$(\mathrm{B}) \qquad x_1 = -u^3 + 3u^2v + 3uv^2 - v^3\,; \quad x_2 = u^2 - v^2\,;$$
$$x_3 = -u - v\,; \quad x_4 = 1$$

and then

$$(6.6) \qquad \beta = \gamma = -\frac{1}{u+v}\,; \quad \theta_u = \theta_v = \frac{1}{u+v}\,,$$

and for the surfaces (5.5′)

$$(\mathrm{C}) \qquad x_1^* = -a^{*2}u^3 + 3u^2v + 3uv^2 - \frac{v^3}{a^{*2}}\,; \quad x_2^* = a^*u^2 - \frac{v^2}{a^*}\,;$$
$$x_3^* = -a^*u - \frac{v}{a^*}\,; \quad x_4 = 1\,,$$

with

$$(6.7) \qquad \alpha = \frac{a}{b} = -a^{*2}\,; \quad \beta^* = \frac{a}{b}\beta = -\frac{a}{b}\frac{1}{u+v}\,; \quad \gamma^* = -\frac{b}{a}\frac{1}{u+v}\,;$$
$$\theta_u^* = \theta_v^* = \frac{1}{u+v}\,.$$

And now let us calculate the function N from (6.3). We see immediately that for the coordinates x_3^* and x_2^* follows

$$(6.8) \qquad N = 0\,,$$

and for $x_1^* = -a^{*2}u^3 + 3u^2v + 3uv^2 - v^3/a^{*2}$, we have

$$(6.9) \qquad N = (a^2 - b^2)\,\beta^2 x_{uv}^{*2} = 36(a^2 - b^2) = \text{non zero constant.}$$

In the first case it follows that the second focal surface of the congruence generated by $(x, \bar{x})$ degenerated into a straight line, and in the other case the second focal surface generated by the point

$$\bar{x} = \mu x + A x_u + B x_v$$

with

$$A = -a\beta x_v^* ; \quad B = b\beta x_u^* ,$$

for $x^* = x_1^*$ is correlative to the first focal surface (x).

In [11] we proved the following theorem.

Theorem. *The surfaces* (A) *of Wilczynski or Terracini's surfaces of third class, allow a group G_2 of collineations into themselves iff $U(u)$ and $V(v)$ are polynomials of the third degree.*

Consequently we have the following result.

With every non-constant solution of one of the systems defining a Tzitzeica-Wilczynski limiting surface, which allows a group G_2 of collineation into themselves and one of the sytems in $\mathscr{D}$-correspondence can also be associated with a rectiliniar W-congruence, whose first focal surface is the one determined by the other system

The congruences belong to special complexes, or the second focal surfaces are correlative to the first ones.

Some remarks. 1. If we change the asumptotic parameters in Tzitzeica's system (1,1), letting

$$R = x ; \quad \bar{u} = -\frac{u}{3\sqrt[3]{\alpha}} ; \quad \bar{v} = \sqrt[3]{\alpha v} ; \quad \alpha = -\frac{1+c}{1-c} ,$$

we obtain

$$R_{\bar{u}\bar{u}} = \frac{h_{\bar{u}}}{h} R_{\bar{u}} + \frac{1}{h} \frac{1+c}{1-c} R_{\bar{v}} ; \quad R_{\bar{v}\bar{v}} = \frac{1-c}{1+c} \frac{1}{h} R_{\bar{u}} + \frac{h_{\bar{v}}}{h} R_{\bar{v}} , \quad R_{\bar{u}\bar{v}} = hR ,$$

which is the adjoint of Tzitzeica's system (1.6).

2. It is easy to prove that only for coincidence minimal projective surfaces, the pair A, B given by (4.13) and the pair A^*, B^*, given by (4.13′) may satisfy the systems (4.2) and (4.1), respectively.

3. If the coordinates of two minimal projective surfaces $x(u, v)$ and $x^*(u, v)$ in a $\mathscr{D}$-asymptotic correspondence are homogeneous, then the particular case of Tzitzeica-Wilczynski surfaces is obtained by (5.1) and $p_{11}^* = p_{11}$; $p_{22}^* = p_{22}$.

4. If the ratio a/b also varies, one obtain ∞^5 W-congruences on all minimal projective surfaces.

One can pass from systems (4.1) and (4.2) to the systems

(4.1′) $$x'_{uu} = \beta x'_v + p'_{11} x' ; \quad x'_{vv} = \gamma x'_u + p'_{22} x' ,$$

respectively

$$(4.2') \qquad x^{*\prime}_{uu} = \beta^* x^{*\prime}_v + p^{*\prime}_{11} \; ; \quad x^{*\prime}_{vv} = \gamma^* x^{*\prime}_u + p^{*\prime}_{22} x^{*\prime}$$

by means of the transformation

$$x = \lambda x' \; ; \quad x^* = \sigma x^{*\prime} \, ,$$

where

$$\lambda^2 = e^{\theta} \; ; \quad \sigma^2 = e^{\theta^*} \, .$$

From (4.1), (4.2) and (4.1′) and (4.2′) it follows

$$p'_{11} = 2\left(\frac{\lambda_u}{\lambda}\right)^2 - \frac{\lambda_{uu}}{\lambda} + \beta \, \frac{\lambda_v}{\lambda} \; ; \quad p'_{22} = 2\left(\frac{\lambda_v}{\lambda}\right)^2 - \frac{\lambda_{vv}}{\lambda} + \gamma \, \frac{\lambda_u}{\lambda} \, ,$$

$$p^{*\prime}_{11} = 2\left(\frac{\sigma_u}{\sigma}\right)^2 - \frac{\sigma_{uu}}{\sigma} + \beta^* \frac{\sigma_v}{\sigma} \; ; \quad p^{*\prime}_{22} = 2\left(\frac{\sigma_u}{\sigma}\right)^2 - \frac{\sigma_{uu}}{\sigma} + \gamma^* \frac{\sigma_u}{\sigma} \, .$$

Now it is easy to see that the expressions

$$A' = \frac{a}{\sigma^2}(\sigma x^{*\prime})_v \; ; \quad B' = \frac{b}{\sigma^2}(\sigma x^{*\prime})_u$$

and

$$A^{*\prime} = \frac{b}{\lambda^2}(\lambda x')_v \; ; \quad B^{*\prime} = \frac{-a}{\lambda^2}(\lambda x'_u) \, ,$$

verify the conditions

$$A'_v + B'\gamma = 0 \; ; \quad B'_u + A'\beta = 0 \, ,$$

and

$$A^{*\prime}_v + B^{*\prime}\gamma^* = 0 \; ; \quad B^{*\prime}_u + A^{*\prime}\beta^* = 0 \, .$$

Consequently the straight ines $(x'\bar{x}')$ and $(x^{*\prime}, \bar{x}^{*\prime})$ generate W congruences, where

$$\bar{x}' = \mu' x' + A' x'_u + B' x'_v$$

and

$$\bar{x}^{*\prime} = \mu^{*\prime} x^{*\prime} + A^{*\prime} x^{*\prime}_u + B^{*\prime} x^{*v}_{\prime} \, .$$

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