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ON A RING PROPERTY UNIFYING REVERSIBLE AND RIGHT DUO RINGS

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Abstract. The concepts of reversible, right duo, and Armendariz rings are known to play important roles in ring theory and they are independent of one another. In this note we focus on a concept that can unify them, calling it a *right Armendarizlike* ring in the process. We first find a simple way to construct a right Armendarizlike ring but not Armendariz (reversible, or right duo). We show the difference between right Armendarizlike rings and strongly right McCoy rings by examining the structure of right annihilators. For a regular ring R, it is proved that R is right Armendarizlike if and only if R is strongly right McCoy if and only if R is Abelian (entailing that right Armendarizlike, Armendariz, reversible, right duo, and IFP properties are equivalent for regular rings). It is shown that a ring R is right Armendarizlike, if and only if so is the polynomial ring over R, if and only if so is the classical right quotient ring (if any). In the process necessary (counter)examples are found or constructed.

1. Right Armendarizlike rings

Throughout this note every ring is associative with identity unless otherwise stated. Given a ring R we use $R[x]$ to denote the polynomial ring with x an indeterminate over R. Let $\mathrm{Mat}_n(R)$ be the n by n full matrix ring over R, and denote by e_{ij} the matrix with (i,j)-entry 1 and elsewhere zeros. We use $\mathbb{Z}$ and $\mathbb{Z}_n$ to denote the ring of integers and the ring of integers modulo n, respectively. Given a ring R, $\ell_R(-)$ (resp. $r_R(-)$) is used for the left (resp. right) annihilator in R.

We extend the McCoy's study of constant zero divisors of polynomial rings ([25, 26]) onto a kind of ring that is near-related to *reversible* rings, *right duo* rings, and *Armendariz* rings. These three properties play important roles in noncommutative ring theory. The near-related concept will be scheduled to unify them, and we will call it a *right Armendarizlike* ring. Another property that unifies them is what Hong et al. in [14] called *strongly right McCoy* rings. In this section we examine the relation between right Armendarizlike rings

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and strongly right McCoy rings. A ring is called *reduced* if it has no nonzero nilpotent elements. Due to Cohn [7], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Reduced rings are clearly reversible. Anderson and Camillo [2], observing the rings whose zero products commute, used the term ZC_2 for what is called reversible. It is obvious that both commutative rings and reduced rings are reversible. The study of reversible rings was continued in [14, 18, 21] and [22]. Especially, Hong et al. obtained the following interesting result for zero-dividing polynomials over reversible rings.

Lemma 1.1 ([14, Theorem 1.6]). *Let R be a reversible ring and $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^n$ be nonzero polynomials over R with $f(x)g(x) = 0$. Then there exists $r = a_0^{l_0} a_1^{l_1} \cdots a_t^{l_t} \in R$ ($t \leq m$ and $l_k \geq 0$ for $k = 0, \ldots, t$) with $g(x)r \neq 0$ and $a_i b_j r = 0$ for all i and j.*

Due to Feller [9], a ring is called *right duo* if every right ideal is two-sided. Various kinds of examples and results can be found in [8, 23, 24] and [31]. Especially, Hong et al. obtained the following interesting result for zero-dividing polynomials over right duo rings.

Lemma 1.2 ([14, Theorem 1.11]). *Let R be a right duo ring and $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ be nonzero polynomials over R with $f(x)g(x) = 0$. Then there exists $r \in R$ with $g(x)r \neq 0$ and $a_i b_j r = 0$ for all i and j.*

For a reduced ring R, Armendariz [4, Lemma 1] proved that

$$(*) \qquad a_i b_j = 0 \text{ for all } i, j \text{ whenever } f(x)g(x) = 0,$$

where $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$ are in $R[x]$. From this result, Rege-Chhawchharia [29] called a ring (not necessarily reduced) *Armendariz* if it satisfies $(*)$. So reduced rings are clearly Armendariz. Essential properties of Armendariz rings are developed in [1, 3, 13, 16, 17, 19, 20] and [29]. Based on Lemmas 1.1 and 1.2, we define the following concept.

Definition 1.3. A ring R is called *right Armendarizlike* provided that if $f(x)g(x) = 0$, then there exists $r \in R$ such that

$$g(x)r \neq 0 \text{ and } a_i b_j r = 0 \text{ for all } i \text{ and } j,$$

where $f(x) = \sum_{i=0}^{m} a_i x^i$ and $0 \neq g(x) = \sum_{j=0}^{n} b_j x^j$ are polynomials over R. *Left Armendarizlike* rings are defined symmetrically. If a ring is both left and right Armendarizlike, then the ring is called *Armendarizlike*.

It is clear that Armendariz rings are Armendarizlike. Both reversible rings and right duo rings are right Armendarizlike by Lemmas 1.1 and 1.2. So the concept of Armendarizlike rings unifies reversible rings, right duo rings, and Armendariz rings. But these implications will be shown to be irreversible by Example 1.4(1) and Example 2.2(1) to follow.

Example 1.4. (1) We expand the argument in [14, Example 1.5]. Let $S = \mathbb{Z}_3[s, t]$ be the polynomial ring with indeterminates s, t over $\mathbb{Z}_3$. Let I be the ideal of S generated by s^3, s^2t^2, t^3 and $R = S/I$. Identify $h(s,t) + I$ with $h(s,t)$ for simplicity. Take $f(x) = s + tx$, $g(x) = s^2 + 2stx + t^2x^2$ in $R[x]$. Then $f(x)g(x) = 0$ but $ts^2 \neq 0$, whence R is not Armendariz. But R is Armendarizlike by Lemma 1.1 since R is commutative. In fact, we have $g(x)t \neq 0$ and $t^2f(x) \neq 0$ so that $abt = 0$ and $t^2ab = 0$, where a and b are coefficients of $f(x)$ and $g(x)$, respectively.

(2) Armendariz (reversible, right duo) rings need not be one-sided Armendarizlike when the rings do not have identities. Let $R = \left(\begin{smallmatrix} 0 & A \\ 0 & 0 \end{smallmatrix}\right)$ be a subring of the 2 by 2 full matrix ring over any ring A. Simple computation shows that R is both reversible and right duo. Next for two polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j$ over R, $f(x)g(x) = 0$ and $a_ib_j = 0$ for all i, j, concluding that R is Armendariz as a ring without identity. But $rf(x) = 0$ and $g(x)r = 0$ for all $r \in R$ and so R is neither left nor right Armendarizlike.

We next find a simple way to construct a right Armendarizlike ring but not Armendariz, from given any right Armendariz(like) ring. Given a ring R and an integer $n \geq 2$, first consider the following subrings of $\mathrm{Mat}_n(R)$:

$$D_n(R) = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

and

$$V_n(R) = \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & \cdots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & \cdots & 0 & a_1 \end{pmatrix} \mid a_1, \ldots, a_n \in R \right\}.$$

Then $D_3(R)$ is Armendariz over a reduced ring R by [17, Proposition 2], but $D_4(R)$ is not Armendariz for any ring R by [17, Example 3]. Also, by [17, Example 5], there exists an Armendariz ring A such that $D_2(A)$ is not Armendariz. So if $D_2(A)$ is proved to be Armendarizlike, then we can always construct an Armendarizlike ring but not Armendariz from any Armendariz ring.

Theorem 1.5. *For a ring R and $n \geq 3$, the following conditions are equivalent*:

(1) *R is right* (*resp. left*) *Armendarizlike*;

(2) *$D_2(R)$ is right* (*resp. left*) *Armendarizlike*;

(3) *$V_n(R)$ is right* (*resp. left*) *Armendarizlike.*

Proof. (1)⇒(2): Suppose that R is right Armendarizlike. Note that $D_n(R)[x] \cong D_n(R[x])$ for any $n \geq 2$. Let

$$A(x) = \sum_{i=0}^{m} \begin{pmatrix} a_{1i} & b_{1i} \\ 0 & a_{1i} \end{pmatrix} x^i = \begin{pmatrix} f_1(x) & g_1(x) \\ 0 & f_1(x) \end{pmatrix}$$

and

$$0 \neq B(x) = \sum_{j=0}^{n} \begin{pmatrix} a_{2j} & b_{2j} \\ 0 & a_{2j} \end{pmatrix} x^j = \begin{pmatrix} f_2(x) & g_2(x) \\ 0 & f_2(x) \end{pmatrix}$$

in $D_2(R)[x]$ such that $A(x)B(x) = 0$, where $f_1(x) = \sum_{i=0}^{m} a_{1i}x^i$, $g_1(x) = \sum_{i=0}^{m} b_{1i}x^i$, $f_2(x) = \sum_{j=0}^{n} a_{2j}x^j$, $g_2(x) = \sum_{j=0}^{n} b_{2j}x^j$.

Case 1. $f_2(x) \neq 0$.

Note $f_1(x)f_2(x) = 0$. Then since R is right Armendarizlike, there exists $\alpha \in R$ such that $f_2(x)\alpha \neq 0$ and $a_{1i}a_{2j}\alpha = 0$ for all i, j. So $B(x)\alpha e_{12} \neq 0$ and

$$\begin{pmatrix} a_{1i} & b_{1i} \\ 0 & a_{1i} \end{pmatrix} \begin{pmatrix} a_{2j} & b_{2j} \\ 0 & a_{2j} \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} = 0 \text{ for all } i, j.$$

Case 2. $f_2(x) = 0$ (then $g_2(x) \neq 0$ since $B(x) \neq 0$).

Note $f_1(x)g_2(x) = 0$. Then since R is right Armendarizlike, there exists $\beta \in R$ such that $g_2(x)\beta \neq 0$ and $a_{1i}b_{2j}\beta = 0$ for all i, j. So $B(x)\beta(e_{11}+e_{22}) \neq 0$ and

$$\begin{pmatrix} a_{1i} & b_{1i} \\ 0 & a_{1i} \end{pmatrix} \begin{pmatrix} 0 & b_{2j} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} = 0 \text{ for all } i, j.$$

By Cases 1 and 2, $D_2(R)$ is right Armendarizlike.

(2)⇒(1): Suppose that $D_2(R)$ is right Armendarizlike and let

$$f(x) = \sum_{i=0}^{m} a_i x^i, \ 0 \neq g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$$

with $f(x)g(x) = 0$. Then, letting

$$A(x) = \sum_{i=0}^{m} \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} x^i \text{ and } B(x) = \sum_{j=0}^{n} \begin{pmatrix} b_j & 0 \\ 0 & b_j \end{pmatrix} x^j,$$

we have $A(x) = \left(\begin{smallmatrix} f(x) & 0 \\ 0 & f(x) \end{smallmatrix}\right)$ and $B(x) = \left(\begin{smallmatrix} g(x) & 0 \\ 0 & g(x) \end{smallmatrix}\right)$ with $A(x)B(x) = 0$. Since $D_2(R)$ is right Armendarizlike, there exists $C \in D_2(R)$ such that $B(x)C \neq 0$, say $C = \left(\begin{smallmatrix} s & t \\ 0 & s \end{smallmatrix}\right)$, and

$$\begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} \begin{pmatrix} b_j & 0 \\ 0 & b_j \end{pmatrix} \begin{pmatrix} s & t \\ 0 & s \end{pmatrix} = 0 \text{ for all } i, j.$$

So we have "$g(x)s \neq 0$ and $a_ib_js = 0$" or "$g(x)t \neq 0$ and $a_ib_jt = 0$" for all i, j, concluding that R is right Armendarizlike.

(1)⇒(3): Suppose that R is right Armendarizlike. Note that $V_n(R)[x] \cong V_n(R[x])$ for any $n \geq 3$. Let

$$A(x) = \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_{n-1}(x) & f_n(x) \\ 0 & f_1(x) & \cdots & f_{n-2}(x) & f_{n-1}(x) \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & f_1(x) & f_2(x) \\ 0 & 0 & \cdots & 0 & f_1(x) \end{pmatrix}$$

and

$$B(x) = \begin{pmatrix} g_1(x) & g_2(x) & \cdots & g_{n-1}(x) & g_n(x) \\ 0 & g_1(x) & \cdots & g_{n-2}(x) & g_{n-1}(x) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & g_1(x) & g_2(x) \\ 0 & 0 & \cdots & 0 & g_1(x) \end{pmatrix}$$

in $V_n(R)[x]$ such that $A(x)B(x) = 0$ and $B(x) \neq 0$.

Suppose $g_1(x) \neq 0$. Then $f_1(x)g_1(x) = 0$. Since R is right Armendarizlike, there exists $\alpha \in R$ such that $g_1(x)\alpha \neq 0$ and $a_i b_j \alpha = 0$ for all i, j where a_i and b_j are coefficients of $f_1(x)$ and $g_1(x)$, respectively. So $B(x)\alpha e_{1n} \neq 0$ and $A_i B_j \alpha e_{1n} = 0$ for all i, j, where A_i and B_j are coefficients of $A(x)$ and $B(x)$ respectively.

Suppose that $g_1(x) = 0$ and $g_2(x) \neq 0$. Then $f_1(x)g_2(x) = 0$. Since R is right Armendarizlike, there exists $\beta \in R$ such that $g_2(x)\beta \neq 0$ and $a_i c_j \beta = 0$ for all i, j, where a_i and c_j are coefficients of $f_1(x)$ and $g_2(x)$, respectively. So $B(x)\beta(e_{1(n-1)} + e_{2n}) \neq 0$ and $A_i B_j \beta(e_{1(n-1)} + e_{2n}) = 0$ for all i, j.

We can proceed inductively for the remaining computation, using the matrix $e_{1(n-k+1)} + \cdots + e_{kn}$ when $g_i(x) = 0$ for $i \in \{1, \ldots, k-1\}$ and $g_k(x) \neq 0$ with $2 \leq k \leq n$ (here $f_1(x)g_k(x) = 0$), obtaining that $V_n(R)$ is right Armendarizlike.

(3)⇒(1): Similar to the proof of (2)⇒(1). □

Corollary 1.6. *If R is an Armendariz ring (a reversible ring or a right duo ring), then $D_2(R)$ is an Armendarizlike ring.*

Based on Theorem 1.5, one may ask whether $D_n(R)$ is also right Armendarizlike over a right Armendarizlike ring R when $n \geq 3$. But the following answers negatively.

Example 1.7. We use the examples and arguments in [14, Remark below Theorem 2.2].

(1) There is a mistake about the computation of [14, Remark(1) below Theorem 2.2], and we provide a correction here. Let T be a reduced ring and let $S = D_3(T)$. Then, by [17, Proposition 2], S is an Armendariz ring and hence a right Armendarizlike ring. Next consider $R = D_3(S)$. Take two polynomials

$$A(x) = (e_{12} + e_{45} + e_{78}) + (e_{14} + e_{25} + e_{36})x \text{ and } B(x) = e_{49} - e_{29}x.$$

Then $A(x)B(x) = 0$. Note that the right ideal I of $R = D_3(S)$ generated by the coefficients of $B(x)$ is $e_{29}R + e_{49}R = e_{29}T + e_{49}T$. Let r be any element in $I = e_{29}T + e_{49}T$, say $r = e_{29}a + e_{49}b$. But $A(x)r = 0$ implies $a = b = 0$, entailing $r = 0$. Thus there does not exist nonzero $r \in I$ such that $A(x)r = 0$ and so $R = D_3(S)$ is not right Armendarizlike.

(2) Let R be any ring and consider the case of $n \geq 4$. Take two polynomials

$$f(x) = e_{12} + e_{13}x \text{ and } g(x) = -e_{3n} + e_{2n}x.$$

Then $f(x)g(x) = 0$. But the right ideal of $D_n(R)$ generated by the coefficients of $g(x)$ is $J = e_{2n}R + e_{3n}R$, there does not exist nonzero $s \in J$ such that $f(x)s = 0$ and so $D_n(R)$ $(n \geq 4)$ is not right Armendarizlike.

Now it is the step to argue the left-right symmetry of the Armendarizlikeness. However there exist right Armendarizlike rings but not left Armendarizlike.

Example 1.8. Let K be a field and $S = K\langle a_0, a_1, a_2, a_3\rangle$ be the free algebra with noncommuting indeterminates a_0, a_1, a_2, a_3 over K. Let I be the ideal of S generated by the following relations

$$a_0a_2, a_1a_2 + a_0a_3, a_1a_3, a_ia_0 (0 \leq i \leq 3), a_ia_1 (0 \leq i \leq 3), a_ia_ja_k (0 \leq i, j, k \leq 3).$$

Set $R = S/I$ and we coincide $\{a_0, a_1, a_2, a_3\}$ with their images in R for simplicity. Put $s(x) = a_0 + a_1x$ and $t(x) = a_2 + a_3x$. Then $s(x)t(x) = 0$. Since $\ell_R(a_0) = \ell_R(a_1) = R - K$, we can't find $r \in R$ such that $rs(x) \neq 0$ and $ra_ia_j = 0$ for all $i = 0, 1, j = 2, 3$, hence R is not left Armendarizlike. Now we claim that R is right Armendarizlike. Take $f(x) = \sum_{i=0}^{m} \alpha_ix^i, 0 \neq g(x) = \sum_{j=0}^{n} \beta_jx^j \in R[x]$ with $f(x)g(x) = 0$. Fix α' to be a monomial of a_i's of smallest degree in the support of α_i's.

If $0 \neq k \in K$ occurs in the support of some β_j, then $\alpha'k$ remains in the support of $\sum \alpha_i\beta_j = 0$, a contradiction. Thus k does not occur in the support of β_j's, and similarly k does not occur in the support of α_i's.

Let H_n be the set of all linear combinations of monomials of degree n over a field K. Then all coefficients α_i, β_j are in H_1 or H_2. If $\beta_k \in H_2$ for all $0 \leq k \leq n$, then $\alpha_i\beta_j = 0$ for all i, j. If $\beta_\ell \in H_1$ for some ℓ, then we can find a nonzero polynomial $g_1(x)$ from $g(x)$ such that $g(x) = g_1(x) + g_2(x)$ and $f(x)g_1(x) = 0$ where $g_1(x) \in H_1[x]$ and $g_2(x) \in H_2[x]$. Clearly, we can easily find $r \in H_1$ such that $g_1(x)r \neq 0$ and $\alpha_i\beta_jr = 0$ for all i, j. So R is a right Armendarizlike ring.

According to Hong et al. [14], a ring R is *strongly right McCoy* provided that $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero r in the right ideal of R generated by the coefficients of $g(x)$, where $f(x)$ and $0 \neq g(x)$ are polynomials in $R[x]$. The strongly left McCoy ring can be defined symmetrically. Right Armendarizlike rings are clearly strongly right McCoy, but the converse need not hold by the following.

Example 1.9. Let K be a field and $A = K\langle a_0, b_0, a_1, b_1, a_2, b_2\rangle$ be the free algebra generated by the noncommuting indeterminates $a_0, b_0, a_1, b_1, a_2, b_2$ over K. Let I be the ideal of A generated by

$$a_0b_0,\ a_0b_1 + a_1b_0,\ a_0b_2 + a_1b_1 + a_2b_0,\ a_1b_2 + a_2b_1,$$
$$a_2b_2,\ a_1b_0b_2,\ a_2b_0b_2,\ b_{j_1}b_{j_2}b_{j_3},\ a_{i_1}a_{i_2},\ b_ja_i,$$

where $i, j, i_1, i_2, j_1, j_2, j_3 \in \{0, 1, 2\}$. Define $R = A/I$ and identify a_i's and b_j's with their images in R for simplicity. We first get an equality $a_0b_1b_2 = 0$ from

$0 = (a_0b_1 + a_1b_0)b_2 = a_0b_1b_2$. Every element $r \in R$ can be rewritten by

$$\begin{aligned} r = u &+ \sum_{s=0}^{2} v_{i_s}a_s + \sum_{t=0}^{2} v_{j_t}b_t + \sum_{l_1,l_2\in\{0,1,2\}} v_{1_l}b_{l_1}b_{l_2} \\ &+ v_2a_0b_1 + v_3a_0b_2 + v_4a_1b_1 + v_5a_1b_2 \\ &+ v_6a_0b_1b_0 + v_7a_0b_1b_1 + v_8a_0b_2b_0 + v_9a_0b_2b_1 + v_{10}a_0b_2b_2 + v_{11}a_1b_1b_0 \\ &+ v_{12}a_1b_1b_1 + v_{13}a_1b_1b_2 + v_{14}a_1b_2b_0 + v_{15}a_1b_2b_1 + v_{16}a_1b_2b_2, \end{aligned}$$

where $u, v_m, v_{i_s}, v_{j_t}, v_{1_l} \in K$ for all $m, i_s, j_t, 1_l$.

Let $a(x) = a_0+a_1x+a_2x^2$ and $b(x) = b_0+b_1x+b_2x^2 \in R[x]$. Then $a(x)b(x) = 0$ and $b(x) \neq 0$. Assume that $a_ib_jr = 0$ and $b(x)r \neq 0$ for some $r \in R$. We use the preceding expression for r. Note $b(x)r = b(x)(u + v_{j_0}b_0 + v_{j_1}b_1 + v_{j_2}b_2)$, so we can let $r = u + v_{j_0}b_0 + v_{j_1}b_1 + v_{j_2}b_2$. From $a_ib_jr = 0$, we have $a_0b_1u = a_0b_1v_{j_0}b_0 = a_0b_1v_{j_1}b_1 = a_0b_2v_{j_2}b_2 = 0$. This yields $u = v_{j_0} = v_{j_1} = v_{j_2} = 0$ and $r = 0$, a contradiction. So R is not right Armendarizlike.

Next we will show that R is strongly right McCoy. Note that every polynomial over R can be expressed by

$$(*) \qquad h_0(x) + \sum_{s=0}^{2} h_{1_s}(x)b_s + \sum_{t=1}^{9} h_{2_t}(x)k_t,$$

where $h_0(x), h_{1_s}(x), h_{2_t}(x) \in K\langle a_0, a_1, a_2\rangle[x]$ and $k_1 = b_0b_0, k_2 = b_0b_1, k_3 = b_0b_2, k_4 = b_1b_0, k_5 = b_1b_1, k_6 = b_1b_2, k_7 = b_2b_0, k_8 = b_2b_1, k_9 = b_2b_2$. Note that each coefficient of $h_0(x)$, $h_{1_s}(x)$'s, and $h_{2_t}(x)$'s is of the form $z + \sum_{i=0}^{2} z_ia_i$ with $z, z_i \in K$.

Take two nonzero polynomials $f(x) = \sum_{v=0}^{m} C_vx^v, g(x) = \sum_{w=0}^{n} D_wx^w \in R[x]$ with $f(x)g(x) = 0$. According to the expression $(*)$, rewrite them by

$$f(x) = f_0(x) + \sum_{s=0}^{2} f_{1_s}(x)b_s + \sum_{t=1}^{9} f_{2_t}(x)k_t \quad \text{and}$$
$$g(x) = g_0(x) + \sum_{s=0}^{2} g_{1_s}(x)b_s + \sum_{t=1}^{9} g_{2_t}(x)k_t.$$

Since $f(x)g(x) = 0$, we have $f_0(x)g_0(x) = 0$ and

$$\begin{aligned} &f_0(x)(\sum_{s=0}^{2} g_{1_s}(x)b_s + \sum_{t=1}^{9} g_{2_t}(x)k_t) + (\sum_{s=0}^{2} f_{1_s}(x)b_s + \sum_{t=1}^{9} f_{2_t}(x)k_t)g_0(x) \\ &+ (\sum_{s=0}^{2} f_{1_s}(x)b_s + \sum_{t=1}^{9} f_{2_t}(x)k_t)(\sum_{s=0}^{2} g_{1_s}(x)b_s + \sum_{t=1}^{9} g_{2_t}(x)k_t) = 0. \end{aligned}$$

So every coefficient of $f_0(x)$ and $g_0(x)$ must be of the form $\sum_{i=0}^{2} z_ia_i$ with $z_i \in K$ when $f_0(x) \neq 0$ or $g_0(x) \neq 0$.

Suppose $g_0(x) \neq 0$. Then $g(x)b_1^2 = g_0(x)b_1^2 \neq 0$ and moreover $C_vD_wb_1^2 = 0$ for all v, w. So it suffices to compute the case of $g_0(x) = 0$. Based on this conclusion, we now rewrite $g(x)$ (with $g_0(x) = 0$) by

$$(\dagger) \qquad g(x) = \sum_{p=0}^{2} \alpha_p b_p + \sum_{s=0}^{2} g'_{1_s}(x) b_s + \sum_{q=1}^{9} \beta_q k_q + \sum_{t=1}^{9} g'_{2_t}(x) k_t,$$

where $\alpha_p, \beta_q \in K[x]$ and every coefficient of $g'_{1_s}(x)$'s and $g'_{2_t}(x)$'s is of the form $\sum_{i=0}^{2} z_i a_i$ $(z_i \in K)$. Thus, for any nonzero $r \in R$, $g(x)r$ is of the form

$$g(x)r = \sum_{p,h=0}^{2} \alpha_p b_p b_h + \sum_{s,h=0}^{2} g'_{1_s}(x) b_s b_h.$$

`Case 1.` $(f_0(x) \neq 0,\ g_0(x) = 0)$.

If each coefficient of $g_{1_s}(x)$'s and $g_{2_t}(x)$'s is of the form $\sum_{i=0}^{2} z_i a_i$ $(z_i \in K)$, then $C_vD_w = 0$ for all v, w. So assume that not every coefficient of $g_{1_s}(x)$'s and $g_{2_t}(x)$'s is of the form $\sum_{i=0}^{2} z_i a_i$ $(z_i \in K)$. From $f(x)g(x) = 0$ we have

$$\sum_{s=0}^{2} f_0(x) g_{1_s}(x) b_s + \sum_{t=1}^{9} f_0(x) g_{2_t}(x) k_t + \sum_{u,v=0}^{2} f_{1_u}(x) b_u g_{1_v}(x) b_v = 0.$$

Then by the construction of I, we have

$$\sum_{s=0}^{2} f_0(x) g_{1_s}(x) b_s = 0 \text{ and } \sum_{t=1}^{9} f_0(x) g_{2_t}(x) k_t + \sum_{u,v=0}^{2} f_{1_u}(x) b_u g_{1_v}(x) b_v = 0.$$

If we use the expression $(\dagger)$, then we also obtain

$$(**) \qquad \sum_{p=0}^{2} \alpha_p f_0(x) b_p = 0 \ \text{ and } \ \sum_{q=1}^{9} \beta_q f_0(x) k_q + \sum_{u,p=0}^{2} \alpha_p f_{1_u}(x) b_u b_p = 0$$

since $\sum_{s=0}^{2} f_0(x) g'_{1_s}(x) b_s = 0$, $\sum_{t=1}^{9} f_0(x) g'_{2_t}(x) k_t = 0$, and

$$\sum_{u,v=0}^{2} f_{1_u}(x) b_u g'_{1_v}(x) b_v = 0.$$

We will use these equalities in $(**)$ freely.

(`Subcase 1-1`) $f_0(x) \in K\langle a_0 \rangle [x]$.

In this case we have the following two cases:

(i) Assume that $\sum_{p=0}^{2} \alpha_p b_p \neq 0$. Then $\sum_{p=0}^{2} \alpha_p f_0(x) b_p = 0$ yields

$$\sum_{p=0}^{2} \alpha_p b_p = \alpha_0 b_0.$$

Whence we get

$$g(x)b_2 = \alpha_0 b_0 b_2 + \sum_{s=0}^{2} g'_{1_s}(x) b_s b_2 \neq 0$$

and this yields that $C_v D_w b_2 = 0$ for all v, w.

(ii) Assume that $\sum_{p=0}^{2} \alpha_p b_p = 0$. Then $\sum_{q=1}^{9} \beta_q f_0(x) k_q = 0$. Here if $\sum_{q=1}^{9} \beta_q k_q = 0$, then $g(x) = \sum_{s=0}^{2} g'_{1_s}(x) b_s + \sum_{t=1}^{9} g'_{2_t}(x) k_t$ and so $C_v D_w = 0$ for all v, w. If $\sum_{q=1}^{9} \beta_q k_q \neq 0$, then $\sum_{q=1}^{9} \beta_q k_q$ consists of $\beta_q b_0 b_j$ or $\beta_q b_1 b_2$. We also have that $C_v D_w = 0$ for all v, w.

For the cases of $f_0(x) \in \langle a_1 \rangle [x]$ and $f_0(x) \in \langle a_2 \rangle [x]$, the computations are similar to the preceding case.

(`Subcase 1-2`) Suppose that both a_0 and a_2 occur in the coefficients of $f_0(x)$.

In this case, we can express $f_0(x)$ by

$$f_0(x) = f'_0(x) + f''_0(x) \text{ with } f'_0(x) \in K\langle a_0 \rangle [x] \text{ and } f''_0(x) \in K\langle a_2 \rangle [x].$$

Then

$$\begin{aligned} 0 = \sum_{p=0}^{2} \alpha_p f_0(x) b_p &= \sum_{p=1}^{2} \alpha_p f'_0(x) b_p + \sum_{p=0}^{1} \alpha_p f''_0(x) b_p \\ \Leftrightarrow \sum_{p=1}^{2} \alpha_p f'_0(x) b_p &= 0 = \sum_{p=0}^{1} \alpha_p f''_0(x) b_p. \end{aligned}$$

Assume that $\sum_{p=0}^{2} \alpha_p b_p \neq 0$. Then $\sum_{p=1}^{2} \alpha_p f'_0(x) b_p + \sum_{p=0}^{1} \alpha_p f''_0(x) b_p \neq 0$, a contradiction. Thus $\sum_{p=0}^{2} \alpha_p b_p = 0$ and $g(x) = \sum_{s=0}^{2} g'_{1_s}(x) b_s + \sum_{q=1}^{9} \beta_q k_q + \sum_{t=1}^{9} g'_{2_t}(x) k_t$, entailing

$$\begin{aligned} 0 = f(x)g(x) = \sum_{q=1}^{9} \beta_q f_0(x) k_q &= \sum_{q=1}^{9} \beta_q f'_0(x) k_q + \sum_{q=1}^{9} \beta_q f''_0(x) k_q \\ &= \beta_4 f'_0(x) k_4 + \beta_5 f'_0(x) k_5 + \beta_7 f'_0(x) k_7 + \beta_8 f'_0(x) k_8 + \beta_9 f'_0(x) k_9 \\ &\quad + \beta_1 f''_0(x) k_1 + \beta_2 f''_0(x) k_2 + \beta_4 f''_0(x) k_4 + \beta_5 f''_0(x) k_5 + \beta_6 f''_0(x) k_6. \end{aligned}$$

Here suppose $\sum_{q=1}^{9} \beta_q k_q \neq 0$. Then $\sum_{q=1}^{9} \beta_q k_q = \beta_3 k_3 = \beta_3 b_0 b_2$ by the preceding argument, entailing $g(x) = \sum_{s=0}^{2} g'_{1_s}(x) b_s + \beta_3 b_0 b_2 + \sum_{t=1}^{9} g'_{2_t}(x) k_t$. Whence $C_v D_w = 0$ for all v, w. Next if $\sum_{q=1}^{9} \beta_q k_q = 0$, then $g(x) = \sum_{s=0}^{2} g'_{1_s}(x) b_s + \sum_{t=1}^{9} g'_{2_t}(x) k_t$ and so $C_v D_w = 0$ for all v, w.

(`Subcase 1-3`) Suppose that both a_0 and a_1 occur in the coefficients of $f_0(x)$.

The computation is similar to one of `Subcase 1-2`.

(`Subcase 1-4`) Suppose that both a_1 and a_2 occur in the coefficients of $f_0(x)$.

The computation is similar to one of `Subcase 1-2`.

(`Subcase 1-5`) Suppose that the coefficients of $f_0(x)$ contain all of a_0, a_1, and a_2.

First note that $f_0(x)$ and the expression (†) of $g(x)$ can be rewritten by

$$f_0(x) = p(x)(a_0 + a_1x + a_2x^2) + f_0'''(x)$$

and

$$g(x) = (b_0 + b_1x + b_2x^2)q(x) + \sum_{p=0}^{2} \alpha_p' b_p + \sum_{s=0}^{2} g_{1_s}'(x)b_s + \sum_{q=1}^{9} \beta_q' k_q + \sum_{t=1}^{9} g_{2_t}'(x)k_t$$

for some $p(x), q(x), \alpha_p', \beta_q' \in K[x]$ and $f_0'''(x) \in K\langle a_0, a_1, a_2\rangle[x]$, where $f_0'''(x)$ (resp. $\sum_{p=0}^{2} \alpha_p' b_p + \sum_{q=1}^{9} \beta_q' k_q$) does not contain polynomials of the form $p(x)(a_0 + a_1x + a_2x^2)$ (resp. $(b_0 + b_1x + b_2x^2)q(x)$) as sum-factors. From $f(x)g(x) = 0$, we have

$$\sum_{p=0}^{2} \alpha_p' p(x)(a_0 + a_1x + a_2x^2)b_p = 0;$$

$$\sum_{p=0}^{2} \alpha_p' f_0'''(x)b_p = 0;$$

$$f_0'''(x)(b_0 + b_1x + b_2x^2)q(x) = 0;$$

and

$$\sum_{q=1}^{9} \beta_q' p(x)(a_0 + a_1x + a_2x^2)k_q = 0,$$

$$\sum_{s=0}^{2} f_{1_s}(x)b_s(b_0 + b_1x + b_2x^2)q(x) + \sum_{q=1}^{9} \beta_q' f_0'''(x)k_q + \sum_{s,p=0}^{2} \alpha_p' f_{1_s}(x)b_u b_p = 0.$$

Here if $p(x)(a_0 + a_1x + a_2x^2) \neq 0$, then $\sum_{p=0}^{2} \alpha_p' b_p = 0$. Moreover since $\sum_{q=1}^{9} \beta_q' p(x)(a_0 + a_1x + a_2x^2)k_q = 0$, we have $\sum_{q=1}^{9} \beta_q' k_q = \beta_3' b_0 b_2$ when $\sum_{q=1}^{9} \beta_q' k_q \neq 0$. Thus we have

$$g(x) = (b_0 + b_1x + b_2x^2)q(x) + \beta_3' b_0 b_2 + \sum_{s=0}^{2} g_{1_s}'(x)b_s + \sum_{t=1}^{9} g_{2_t}'(x)k_t.$$

If $p(x)(a_0 + a_1x + a_2x^2) = 0$, then $f_0'''(x) \neq 0$ (since $f_0(x) \neq 0$) and so $\sum_{p=0}^{2} \alpha_p' b_p = 0 = (b_0 + b_1x + b_2x^2)q(x)$, entailing

$$g(x) = \sum_{s=0}^{2} g_{1_s}'(x)b_s + \sum_{q=1}^{9} \beta_q' k_q + \sum_{t=1}^{9} g_{2_t}'(x)k_t.$$

From
$$\sum_{s=0}^{2} f_{1_s}(x)b_s(b_0+b_1x+b_2x^2)q(x)+\sum_{q=1}^{9}\beta'_q f_0'''(x)k_q+\sum_{s,p=0}^{2}\alpha'_p f_{1_s}(x)b_sb_p=0,$$
we also obtain $\sum_{q=1}^{9}\beta'_q f_0'''(x)k_q=0$ and this yields $\sum_{q=1}^{9}\beta'_q k_q=\beta'_3 b_0b_2$ when $\sum_{q=1}^{9}\beta'_q k_q\neq 0$. Thus we have
$$g(x)=\beta'_3 b_0b_2+\sum_{s=0}^{2} g'_{1_s}(x)b_s+\sum_{t=1}^{9} g'_{2_t}(x)k_t.$$
If $(b_0+b_1x+b_2x^2)q(x)\neq 0$, then $f_0'''(x)=0$ and this yields $f_0(x)=p(x)(a_0+a_1x+a_2x^2)$.

Thus we have the following three cases.

(i) Suppose that $p(x)(a_0+a_1x+a_2x^2)\neq 0$ and $(b_0+b_1x+b_2x^2)q(x)\neq 0$. Then
$$f_0(x)=p(x)(a_0+a_1x+a_2x^2)$$
and
$$g(x)=(b_0+b_1x+b_2x^2)q(x)+\beta'_3 b_0b_2+\sum_{s=0}^{2} g'_{1_s}(x)b_s+\sum_{t=1}^{9} g'_{2_t}(x)k_t.$$
So some coefficients of $g(x)$ are $D=z'b_0+z''b_0b_2+\sum_{s=0}^{2}\delta_s b_s+\sum_{t=1}^{9}\gamma_t k_t$, where $0\neq z',z''\in K$ and nonzero elements δ_s,γ_t are of the form $\sum_{i=0}^{2} z_ia_i$. It then follows that
$$Db_2=zb_0b_2+\sum_{s=0}^{2}\delta_s b_sb_2\neq 0$$
and $f(x)Db_2=0$.

(ii) Suppose that $p(x)(a_0+a_1x+a_2x^2)\neq 0$ and $(b_0+b_1x+b_2x^2)q(x)=0$. Then
$$f_0(x)=p(x)(a_0+a_1x+a_2x^2)+f_0'''(x)$$
and
$$g(x)=\sum_{s=0}^{2} g'_{1_s}(x)b_s+\sum_{q=1}^{9}\beta'_q k_q+\sum_{t=1}^{9} g'_{2_t}(x)k_t.$$
Here if $\sum_{q=1}^{9}\beta'_q k_q\neq 0$, then we get
$$g(x)=\beta'_3 b_0b_2+\sum_{s=0}^{2} g'_{1_s}(x)b_s+\sum_{t=1}^{9} g'_{2_t}(x)k_t \text{ (with } \beta'_3\neq 0)$$
by the arguments above. Whence $C_vD_w=0$ for all v,w.

Next if $\sum_{q=1}^{9}\beta'_q k_q=0$, then we also have that $C_vD_w=0$ for all v,w.

(iii) Suppose that $p(x)(a_0+a_1x+a_2x^2)=0$ and $f_0'''(x)\neq 0$. Then if $\sum_{q=1}^{9}\beta'_q k_q\neq 0$, then
$$f_0(x)=f_0'''(x)$$

and

$$g(x) = \beta_3' b_0 b_2 + \sum_{s=0}^{2} g'_{1_s}(x) b_s + \sum_{t=1}^{9} g'_{2_t}(x) k_t \text{ (with } \beta_3' \neq 0)$$

by the arguments above. Whence $C_v D_w = 0$ for all v, w.

Next if $\sum_{q=1}^{9} \beta_q' k_q = 0$, then we also have that $C_v D_w = 0$ for all v, w.

Case 2. ($f_0(x) = 0$, $g_0(x) = 0$).

From $f(x)$ and the expression (†) of $g(x)$, $f(x)g(x) = 0$ implies

$$\sum_{s,p=0}^{2} \alpha_p f_{1_s}(x) b_s b_p = 0.$$

If $\sum_{p=0}^{2} \alpha_p b_p = 0$, then $C_v D_w = 0$ for all v, w. So assume $\sum_{p=0}^{2} \alpha_p b_p \neq 0$. Then some coefficients of $g(x)$ are of the form $\sum_{j=0}^{2} z_j' b_j + \sum_{s=0}^{2} \delta_s b_s + \sum_{t=1}^{9} z_t'' k_t + \sum_{t=1}^{9} \gamma_t k_t$, where $z_j', z_t'' \in K$, $\sum_{j=0}^{2} z_j' b_j \neq 0$, and nonzero elements δ_s, γ_t are of the form $\sum_{i=0}^{2} z_i a_i$. It then follows that $g(x) b_j \neq 0$ for all j and $C_v D_w b_j = 0$ for all v, w.

By Cases 1 and 2, R is strongly right McCoy.

Note. (1) Consider the strongly right McCoy ring R in Example 1.9. We will show that R is not strongly left McCoy. Recall $a(x)b(x) = (a_0 + a_1 x + a_2 x^2)(b_0 + b_1 x + b_2 x^2) = 0$. Consider the left ideal L of R generated by a_0, a_1, a_2. Then $L = Ka_0 + Ka_1 + Ka_2$. Assume that $sb(x) = 0$ for some $0 \neq s \in L$, $s = \gamma_0 a_0 + \gamma_1 a_1 + \gamma_2 a_2$ say. From $sb_0 = 0$, we have

$$0 = sb_0 = (\gamma_0 a_0 + \gamma_1 a_1 + \gamma_2 a_2) b_0 = \gamma_1 a_1 b_0 + \gamma_2 a_2 b_0.$$

This yields $\gamma_1 = \gamma_2 = 0$, entailing $s = \gamma_0 a_0$. From $0 = sb_1 = \gamma_0 a_0 b_1$, we get $\gamma_0 = 0$ and so $s = 0$, a contradiction. So R is not strongly left McCoy.

(2) Let K be a field and $A = K\langle a_0, b_0, a_1, b_1, a_2, b_2 \rangle$ be the free algebra generated by the noncommuting indeterminates $a_0, b_0, a_1, b_1, a_2, b_2$ over K.

Let J be the ideal of A generated by

$$\begin{gathered} a_0 b_0,\ a_0 b_1 + a_1 b_0,\ a_0 b_2 + a_1 b_1 + a_2 b_0,\ a_1 b_2 + a_2 b_1, \\ a_2 b_2,\ a_0 a_0 b_1,\ a_0 a_0 b_2,\ a_{i_1} a_{i_2} a_{i_3},\ b_{j_1} b_{j_2},\ b_j a_i, \end{gathered}$$

where $i, j, i_1, i_2, i_3, j_1, j_2 \in \{0, 1, 2\}$. Then one can show that A/J is strongly left McCoy, but neither strongly right McCoy nor left Armendarizlike, through similar computations to Example 1.9 and (1) above.

But right (left) Armendarizlike property and strong right (left) McCoy property are equivalent for regular rings by Theorem 2.5, to follow.

2. Properties of right Armendarizlike rings

In this section we examine basic properties of right Armendarizlike rings, and relations between right Armendarizlike rings and related concepts. Especially it is proved in Theorem 2.5 to follow that the properties of right Armendarizlike, Armendariz, reversible, right duo, and strong right McCoy are equivalent for regular rings. First we obtain a similar result to [1, Proposition 1].

Proposition 2.1. *Let R be a right Armendarizlike ring and suppose that $f_1(x), \ldots, f_n(x) \in R[x]$ are such that $f_1(x) \cdots f_n(x) = 0$ and $f_i(x) \neq 0$ for all $i \in \{2, \ldots, n\}$. Then for any choice $a_1, \ldots, a_n$ with a_i a coefficient of $f_i(x)$ there exists $r \in R$ such that $f_n(x)r \neq 0$ and $a_1 \cdots a_n r = 0$.*

Proof. If $f_2(x) \cdots f_n(x) \neq 0$, then there exists $r_1 \in R$ such that

$$(f_2(x) \cdots f_n(x))r_1 \neq 0 \text{ and } a_1(f_2(x) \cdots f_n(x))r_1 = 0$$

for any coefficient a_1 of $f_1(x)$. From $(f_2(x) \cdots f_n(x))r_1 \neq 0$, we have

$$f_3(x) \cdots f_n(x)r_1 \neq 0$$

and hence there exists $r_2 \in R$ such that

$$(f_3(x) \cdots f_n(x)r_1)r_2 \neq 0 \text{ and } (a_1a_2)(f_3(x) \cdots f_n(x)r_1)r_2 = 0$$

for any coefficient a_1a_2 of $a_1f_2(x)$. We inductively obtain $r_{n-1} \in R$ such that

$$(f_n(x)r_1 \cdots r_{n-2})r_{n-1} \neq 0 \text{ and } (a_1 \cdots a_{n-1})f_n(x)(r_1 \cdots r_{n-2})r_{n-1} = 0$$

for any coefficient $a_1 \cdots a_{n-2}a_{n-1}$ of $(a_1 \cdots a_{n-2})f_{n-1}(x)$.

Letting $r = r_1 \cdots r_{n-2}r_{n-1}$, we are done.

If $f_2(x) \cdots f_n(x) = 0$, then take $k \geq 3$ such that k is smallest with respect to the property

$$f_k(x)f_{k+1}(x) \cdots f_n(x) \neq 0$$

(this k exists since $f_n(x) \neq 0$). We then get

$$(a_1 \cdots a_{k-1})(f_k(x) \cdots f_n(x)) = 0 \text{ for any choice } a_1, \ldots, a_{k-1}$$

with a_i a coefficient of $f_i(x)$ for $i \in \{1, \ldots, k-1\}$. Applying the manner of the preceding case, we can also obtain $r \in R$ such that $f_n(x)r \neq 0$ and $a_1 \cdots a_n r = 0$, starting from the product

$$(a_1 \cdots a_{k-1}f_k(x))(f_{k+1}(x) \cdots f_n(x)) = 0$$

with $f_{k+1}(x) \cdots f_n(x) \neq 0$. $\square$

A ring is usually called *Abelian* if every idempotent is central. Armendariz rings are Abelian by the proof of [1, Theorem 6] or [16, Corollary 8]. Due to Bell [5], a ring R is called *IFP* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. IFP rings are clearly Abelian and it is also trivial to check that reversible rings and right duo rings are IFP. The study of IFP rings was developed by many authors containing [6, 12, 13, 16, 18, 22, 30, 27] and [28].

The concepts of Armendariz rings and IFP rings are independent of each other. Rege-Chhawchharia showed that commutative (hence IFP) rings need not be Armendariz in [29, Example 3.2]; and by [16, Example 14] or [3, Example 4.8], there exist Armendariz rings that is not IFP. It then follows that Armendariz rings (and so Armendarizlike rings) need not be reversible or right duo.

Recall that if a ring R is right duo or reversible, then R is right Armendarizlike. Also recall that the class of Abelian rings contains Armendariz rings, right duo rings, and reversible rings. So it is natural to check the implications between right Armendarizlike rings and Abelian rings. But they are independent of each other as follows.

Example 2.2. (1) There exists an Armendarizlike ring but not Abelian. We use the ring in [6, Theorem 7.1]. Let K be a field and $K\langle e, x, y, z\rangle$ be the free algebra with noncommuting indeterminates e, x, y, z over K. Set R be the factor ring of $K\langle e, x, y, z\rangle$ with the relations

$$e^2 = e,\ ex = x,\ xe = 0,\ ey = ye = 0,\ ez = ze = z,$$
$$x^2 = y^2 = z^2 = xy = xz = yx = yz = zx = zy = 0.$$

We coincide $\{e, x, y, z\}$ with their images in R for simplicity. Note that R is non-Abelian since e is an idempotent that does not commute with x. By the computation of [14, Example 1.10], we can show that R is an Armendarizlike ring.

(2) By help of Theorem 1.5, we can find an Abelian ring but not right Armendarizlike. Let S be a reduced ring and let $R = D_4(S)$. Then R is an Abelian ring by [15, Lemma 2], but not right Armendarizlike by Example 1.7.

The class of IFP rings contains right duo rings and reversible rings. So one may conjecture that IFP rings are right Armendarizlike. However there exists an IFP ring that is not right Armendarizlike by help of [28, Section 3]. Nielsen [28] and Rege-Chhawchharia [29] called a ring R (possibly without identity) *right McCoy* when the equation $f(x)g(x) = 0$ implies $f(x)r = 0$ for some nonzero $r \in R$, where $f(x), 0 \neq g(x)$ are polynomials in $R[x]$. Left McCoy rings are defined similarly. Strongly right (resp. left) McCoy rings are clearly right (resp. left) McCoy. Following the literature, a ring R is called *directly finite* if $uv = 1$ implies $vu = 1$ for $u, v \in R$.

Proposition 2.3. (1) *Left or right McCoy rings are directly finite.*
(2) *Strongly left or right McCoy rings are directly finite.*
(3) *Left or right Armendarizlike rings are directly finite.*

Proof. It suffices to prove (1). Right McCoy rings are directly finite by [6, Theorem 5.2].

Let R be a left McCoy ring. Suppose that $uv = 1$ but $vu \neq 1$. Consider two polynomials $f(x) = (vu - 1) + (vu - 1)ux$ and $g(x) = v + (vu - 1)x$

over R. Then $f(x)g(x) = 0$ with $f(x) \neq 0$. Since R is left McCoy, there exists $0 \neq r \in R$ such that $rg(x) = 0$. So $rv = 0$ and $r(vu-1) = 0$; but $0 = r(vu-1) = rvu - r = -r \neq 0$, a contradiction. □

It is easily checked that Abelian rings are directly finite. So by Proposition 2.3, the class of directly finite rings contains Abelian rings and one-sided Armendarizlike rings. However neither implication is reversible by the following.

Example 2.4. Let R be the 2 by 2 upper triangular matrix ring over any reduced ring. Consider the two polynomials $f(x) = e_{11} - e_{12}x$ and $g(x) = e_{22} + e_{12}x$ over R. Then $f(x)g(x) = 0$ but for any nonzero matrix $r \in R$ cannot annihilate $f(x)$ on the right. So R is not right Armendarizlike. Similar computation implies that R is also not left Armendarizlike. Note that R is non-Abelian but directly finite. Note that every n by n ($n \geq 2$) full (upper triangular) matrix ring is neither left nor right Armendarizlike by similar computations to above.

The concepts of right Armendarizlike rings and Abelian rings are independent of each other by Example 2.2. But for regular rings they are equivalent as follows. A ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$. It is well-known that the ring of all column finite infinite matrices over a field is regular but not directly finite; hence regular rings need not be right Armendarizlike by Proposition 2.3.

Theorem 2.5. *For a regular ring R, the following conditions are equivalent*:
(1) *R is right* (*left*) *Armendarizlike*;
(2) *R is Abelian*;
(3) *R is Armendariz*;
(4) *R is reversible*;
(5) *R is right* (*left*) *duo*;
(6) *R is strongly right* (*left*) *McCoy.*

Proof. The conditions (2) and (5) are equivalent by [10, Theorem 3.2]. If R is Abelian, then R is reduced (hence reversible) by [10, Theorem 3.2] and so Armendariz by [4, Lemma 1], obtaining (2)⇒(3) and (2)⇒(4). (3)⇒(1) and (1)⇒(6) are obvious. (4)⇒(1) is obtained from Lemma 1.1. So it suffices to prove (6)⇒(2). Let R be strongly right McCoy and assume on the contrary that there exist $e^2 = e, r \in R$ such that $er(1-e) \neq 0$. Since R is regular, there exists $y \in R$ with $er(1-e) = er(1-e)yer(1-e)$. Here we can put $y = (1-e)ye$.

Next consider two nonzero polynomials

$$f(x) = er(1-e) - ex \text{ and } g(x) = er(1-e) + yer(1-e)x.$$

Then $f(x)g(x) = 0$. Since R is strongly right McCoy, there exist $s, t \in R$ such that $\alpha = er(1-e)s + yer(1-e)t \neq 0$ and $f(x)\alpha = 0$. But $er(1-e)\alpha = 0$ yields $0 = er(1-e)yer(1-e)t = er(1-e)t$ (hence $yer(1-e)t = 0$) and $e\alpha = 0$ yields $0 = eer(1-e)s = er(1-e)s$, entailing $\alpha = 0$. This induces a contradiction. The left cases can be proved similarly. □

Since IFP rings are Abelian, the IFP can be an equivalent condition in Theorem 2.5.

One may hope directly finite regular rings to be right Armendarizlike (hence Abelian). But by [10, Example 5.10], there exists a directly finite regular ring but not Abelain (hence not right Armendarizlike by Theorem 2.5).

A ring R is called *π-regular* if for each $a \in R$ there exist a positive integer $n = n(a)$, depending on a, and $x \in R$ such that $a^n = a^n x a^n$. Regular rings are obviously π-regular. One may also ask whether Abelian π-regular rings are right Armendarizlike. But the answer is also negative by the ring $R = D_4(S)$ over a division ring S. This ring R is clearly π-regular and by [15, Lemma 2] R is Abelian, but R is not right Armendarizlike by Example 1.7(2).

3. Examples of right Armendarizlike rings

In this section we examine the interesting properties of the class of Armendarizlike rings, and find various kinds of Armendarizlike rings.

Theorem 3.1. *A ring R is right Armendarizlike if and only if so is $R[x]$.*

Proof. Let R be a right Armendarizlike ring and let $R[x][t]$ denote the polynomial ring with an indeterminate t over $R[x]$. Let $f(t) = \sum_{i=0}^{m} f_i(x)t^i, g(t) = \sum_{j=0}^{n} g_j(x)t^j \in R[x][t]$ be polynomials with $f(t)g(t) = 0$ and $g(t) \neq 0$. Say $f_i(x) = \sum_{h=0}^{n_i} a(i)_h x^h$ and $g_j(x) = \sum_{k=0}^{m_j} b(j)_k x^k$. Set $k = \sum_{i=0}^{m} \deg(f_i(x)) + \sum_{j=0}^{n} \deg(g_j(x))$ where the degree of the zero polynomial is taken to be 0. Letting $F(x) = f_0 + f_1 x^k + \cdots + f_m x^{km}, G(x) = g_0 + g_1 x^k + \cdots + g_n x^{kn}$, then $F(x), G(x) \in R[x]$ and $G(x) \neq 0$. Since the set of coefficients of the f_i's (resp. g_j's) coincides with the set of coefficients of $F(x)$ (resp. $G(x)$), we get $F(x)G(x) = 0$ from $f(t)g(t) = 0$. Now since R is right Armendarizlike, there exists $r \in R$ such that $G(x)r \neq 0$ and $a(i)_h b(j)_k r = 0$ for all i, h, j and k. This implies that $g(t)r \neq 0$ and $f_i(x)g_j(x)r = 0$ for all i, j, concluding that $R[x]$ is right Armendarizlike.

Conversely, suppose that $R[x]$ is right Armendarizlike. Set $R[x][y]$ be the polynomial ring with an indeterminate y over $R[x]$. Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ such that $f(x)g(x) = 0$ with $g(x) \neq 0$. Putting $f(y) = \sum_{i=0}^{m} a_i y^i$ and $g(y) = \sum_{j=0}^{n} b_j y^j \in R[x][y]$, we get $f(y)g(y) = 0$. Clearly $g(y) \neq 0$ from $g(x) \neq 0$. Since $R[x]$ is right Armendarizlike, there exists $c(x) \in R[x]$ such that $g(y)c(x) \neq 0$ and $a_i b_j c(x) = 0$ for all i, j. Since $g(y)c(x) \neq 0$, $g(y)c_k \neq 0$ for some coefficient c_k of $c(x)$. Thus $g(x)c_k \neq 0$ and $a_i b_j c_k = 0$ for all i, j. Therefore R is right Armendarizlike. $\square$

A ring R is called right *Ore* if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. Note that R is a right Ore ring if and only if the classical right quotient ring of R exists. There exist many reduced rings which are not right Ore as can be seen by the free algebra in two indeterminates over a field (this ring is a domain but cannot have its classical right (left) quotient ring).

Theorem 3.2. *Let R be a right Ore ring with the classical right quotient ring $Q_r(R)$. Then R is right Armendarizlike if and only if so is $Q_r(R)$.*

Proof. Let $Q = Q_r(R)$. Suppose $F(x)G(x) = 0$ for $F(x), G(x) \in Q[x]$ with $G(x) \neq 0$. Then we can write $F(x) = a_0u^{-1} + a_1u^{-1}x + \cdots + a_mu^{-1}x^m$ and $G(x) = b_0v^{-1} + b_1v^{-1}x + \cdots + b_nv^{-1}x^n$, where u, v are regular. Since $F(x)G(x) = 0$, $(a_0u^{-1} + a_1u^{-1}x + \cdots + a_mu^{-1}x^m)(b_0 + b_1x + \cdots + b_nx^n) = 0$ and so

$$(\dagger) \qquad a_0u^{-1}b_0 = 0, a_0u^{-1}b_1 + a_1u^{-1}b_0 = 0, \ldots, a_mu^{-1}b_n = 0.$$

Now for $u^{-1}b_0, u^{-1}b_1, \ldots, u^{-1}b_n$, there exist $c_0, c_1, \ldots, c_n, s \in R$ and s regular such that $u^{-1}b_i = c_is^{-1}$ for all i. Then from Eq.($\dagger$), we have $a_0c_0 = 0, a_0c_1 + a_1c_0 = 0, \ldots, a_mc_n = 0$ and so $f(x)g(x) = 0$, where $f(x) = a_0+a_1x+\cdots+a_mx^m$ and $0 \neq g(x) = c_0 + c_1x + \cdots + c_nx^n$ in $R[x]$. Since R is right Armendarizlike, there exists $r \in R$ such that $g(x)r \neq 0$ and $a_ic_jr = 0$ for all i, j. Since $b_kv^{-1}vsr = b_ksr = uc_ks^{-1}sr = uc_kr \neq 0$ for some k, there exists $vsr \in Q$ such that $G(x)vsr \neq 0$ and so $a_iu^{-1}b_jv^{-1}vsr = a_ic_js^{-1}v^{-1}vsr = a_ic_jr = 0$ for all i, j. Therefore Q is right Armendarizlike.

Conversely, let $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $0 \neq g(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x]$ such that $f(x)g(x) = 0$. Then $f(x), g(x) \in Q[x]$. Since Q is right Armendarizlike, there exists $rs^{-1} \in Q$ such that $g(x)rs^{-1} \neq 0$ and $a_ib_jrs^{-1} = 0$ for all i, j. So $r \in R$ such that $g(x)r \neq 0$ and $a_ib_jr = 0$ for all i, j. Therefore R is right Armendarizlike. □

Note. Let R be a right Ore ring with the classical right quotient ring Q. We also note that if R is left Armendarizlike, then so is Q. Letting $F(x) = a_0u^{-1} + a_1u^{-1}x + \cdots + a_mu^{-1}x^m$ and $G(x) = b_0v^{-1} + b_1v^{-1}x + \cdots + b_nv^{-1}x^n$ in $Q[x]$ with $F(x)G(x) = 0$, then $(a_0 + a_1x + \cdots + a_mx^m)u^{-1}(b_0 + b_1x + \cdots + b_nx^n) = 0$. Note that $u^{-1}b_i = b_i'u'^{-1}$ for some $b_i', u' \in R$ with u' regular. Then $(a_0+a_1x+\cdots+a_mx^m)(b_0'+b_1'x+\cdots+b_n'x^n) = 0$. Since R is left Armendarizlike, there exists $r \in R$ such that $r(a_0 + a_1x + \cdots + a_mx^m) \neq 0$ and $ra_ib_j' = 0$ for all i, j. Then $rF(x) \neq 0$ and $0 = ra_ib_j'u'^{-1} = ra_iu^{-1}b_jv^{-1}$, proving that Q is left Armendarizlike.

Proposition 3.3. (1) *The class of right* (*left*) *Armendarizlike rings is closed under direct limits.*

(2) *A direct product of rings $R = \prod_{i\in I} R_i$ is right* (*resp. left*) *Armendarizlike if and only if all rings R_i for all i are right* (*resp. left*) *Armendarizlike rings.*

(3) *A direct sum of rings $R = \sum_{i\in I} R_i$ is right* (*resp. left*) *Armendarizlike if and only if all rings R_i for all i are right* (*resp. left*) *Armendarizlike rings.*

(4) *The class of right Armendarizlike rings is not closed under subrings.*

(5) *The class of right Armendarizlike rings is not closed under homomorphic images.*

Proof. (1) Let $D = \{R_i, \alpha_{ij}\}$ be a direct system of right Armendarizlike rings R_i for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \to R_j$ for each $i \leq j$ satisfying

$\alpha_{ij}(1) = 1$, where I is a directed partially ordered set. Set $R = \varinjlim R_i$ be the direct limit of D with $\iota_i : R_i \to R$ and $\iota_j \alpha_{ij} = \iota_i$. Let $a, b \in R$. Then $a = \iota_i(a_i)$, $b = \iota_j(b_j)$ for some $i, j \in I$ and there exists $k \in I$ such that $i \leq k$, $j \leq k$. Define $a + b = \iota_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j))$ and $ab = \iota_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j))$, where $\alpha_{ik}(a_i)$ and $\alpha_{jk}(b_j)$ are in R_k. Then R forms a ring with $0 = \iota_i(0)$ and $1 = \iota_i(1)$.

Next let $f(x) = \sum_{i=0}^{m} a_i x^i, 0 \neq g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ be polynomials such that $f(x)g(x) = 0$. Then there exists $k \in I$ such that $f(x), g(x) \in R_k[x]$ via ι_i's and α_{ij}'s; hence we get $f(x)g(x) = 0$ in $R_k[x]$. Since R_k is right Armendarizlike, there exists $c_k \in R_k$ such that $g(x)c_k \neq 0$ and $a_i b_j c_k = 0$ for all i, j. Put $c = \iota_k(c_k)$. Then $g(x)c \neq 0$ and $a_i b_j c = 0$ for all i, j, concluding R being right Armendarizlike. The proof for left Armendarizlike rings is similar.

(2) Let $f(x)g(x) = 0$ for $f(x) = \sum_{j=0}^{m} (a(j)_i)x^j, 0 \neq g(x) = \sum_{k=0}^{n} (b(k)_i)x^k \in R[x]$. Letting $f_i(x) = \sum_{j=0}^{m} a(j)_i x^j$ and $g_i(x) = \sum_{k=0}^{n} b(k)_i x^k$ we can write $f(x) = (f_i(x))$ and $g(x) = (g_i(x))$. Suppose that each ring R_i is right Armendarizlike. Since $g(x) \neq 0$ there exists an index $s \in I$ such that $g_s(x) \neq 0$. Then since R_s is right Armendarizlike, there exists $r_s \in R_s$ such that $g_s(x)r_s \neq 0$ and $a(i)_s b(j)_s r_s = 0$ for all i, j. Let $r = (r_i) \in R$ be the sequence with r_s in the s-th coordinate and zero elsewhere. Then $g(x)r \neq 0$ and $a(i)b(j)r = 0$ for all i, j and so R is right Armendarizlike.

Conversely, suppose that R is right Armendarizlike. If R_{i_0} is not right Armendarizlike for some $i_0 \in I$, then for all $r_{i_0} \in R_{i_0}$, there exist $f_{i_0}(x), g_{i_0}(x)$ in $R_{i_0}[x]$ with $g_{i_0}(x) \neq 0$ such that $f_{i_0}(x)g_{i_0}(x) = 0$ but $g_{i_0}(x)r_{i_0} = 0$ or $a(i_0)_i b(i_0)_j r_{i_0} \neq 0$ for some i, j. Taking $f(x) = (f_i(x)), g(x) = (g_i(x))$ such that $f(x)$ and $g(x)$ are the sequences with $f_{i_0}(x)$ in the i_0-th coordinate and 1 elsewhere and $g_{i_0}(x)$ in the i_0-th coordinate and zero elsewhere, respectively. Then since $f(x)g(x) = 0$ and R is right Armendarizlike, there exists $r \in R$ such that r is the sequence with nonzero r_{i_0} in the i_0-th coordinate and zero elsewhere and $g(x)r \neq 0$ and $a(i)_k b(j)_k r = 0$ for all i, j, k, a contradiction. The proof for left Armendarizlike rings is similar.

(3) The proof is almost similar to one of (2).

(4) By help of Example 2.2(1) and [14, Example 1.12(1)], there exists a right Armendarizlike ring whose subring need not be right Armendarizlike.

(5) Let R be the ring of quaternions with integer coefficients. Then R is a domain, so right Armendarizlike. However for any odd prime integer q, the ring R/qR is isomorphic to $\mathrm{Mat}_2(\mathbb{Z}_q)$ by the argument in [11, Exercise 2A]. Thus R/qR is not right Armendarizlike by Example 2.4. $\square$

By the above Proposition 3.3(4), the class of right Armendarizlike rings is not closed under subrings. But we find a kind of subring that preserves the right Armendarizlike property.

Proposition 3.4. *Let e be a central idempotent of R. Then the following statements are equivalent*:

(1) *R is right Armendarizlike*;

(2) *Both eR and $(1-e)R$ are right Armendarizlike.*

Proof. (1)⇒(2) is obvious since e is central.

(2)⇒(1): It follows from Proposition 3.3(3), since $R = eR \oplus (1-e)R$. □

Next consider a natural conjecture: If R/I and I are both right Armendarizlike for any ideal I of a ring R, then R is right Armendarizlike, where I is considered as a ring without identity. However there exist counterexamples as follows.

Example 3.5. (1) Let R be the 2 by 2 upper triangular matrix ring over a field F and $I = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$. Then I and $R/I \cong F \oplus F$ are both Armendarizlike, but R is not right Armendarizlike by Example 2.4.

(2) Let $T = \{a, b\}$ be the semigroup with multiplication $a^2 = ab = a, b^2 = ba = b$. Put $S = \mathbb{Z}_2 T$ be the four-element semigroup ring without identity. Then we claim that S is Armendariz. Suppose that $f(x)g(x) = 0$ with $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in S[x]$. Note that $a + b \in r_S(f(x))$. If every coefficient of $g(x)$ is $a + b$, then $a_i(a + b) = 0$ ($a_i(a + b)a = 0$ and $g(x)a = g(x) \neq 0$) for all i. If there exists b_j such that $b_j \in \{a, b\}$, then a quick calculation yields $f(x) = 0$ because $\ell_S(a) = 0 = \ell_S(b)$. By these two cases, $a_i b_j = 0$ for all i, j. Thus S is Armendariz (right Armendarizlike) as a ring without identity.

Next we attach an identity to S, obtaining the ring $R = S \times \mathbb{Z}$. Consider the polynomials $s(x) = (a, 1) + (b, 1)x, t(x) = (a, 0) + (b, 0)x \in R[x]$. Then $s(x)t(x) = 0$, but $s(x)c = 0$ implies $c = 0$ and $dt(x) = 0$ implies $d = 0$, concluding that R is neither left nor right Armendaizlike. But letting $I = S$, $R/I \cong \mathbb{Z}$ and I are both right Armendarizlike.

In the preceding examples, I is non-reduced ($I^2 = 0$ in (1), and $(a + b)^2 = 0$ in (2)). So in the following we use the condition "reduced" for I.

Theorem 3.6. *For a ring R suppose that R/I is a right Armendarizlike ring for some proper ideal I of R. If I is reduced (as a ring without identity), then R is right Armendarizlike.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i, 0 \neq g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ with $f(x)g(x) = 0$. We first assume $g(x) \notin I[x]$. In this case we apply the proof of [16, Theorem 11]. Since R/I is right Armendarizlike, there exists $r \in R$ such that $g(x)r \notin I[x]$ and $a_i b_j r \in I$ for all i, j. Then clearly $g(x)r \neq 0$. We proceed by induction on m. If $m = 0$, then we are done and so suppose $m \geq 1$. We claim that $a_0 b_j r = 0$ for all $j \in \{0, 1, \ldots, n\}$, based on $a_0 b_0 = 0$. Assume on the contrary that $a_0 b_j r \neq 0$ for some j. Then we can take ℓ in $\{1, 2, \ldots, n\}$ such that ℓ is the smallest positive integer such that $a_0 b_\ell r \neq 0$. So for $j \in \{0, \ldots, \ell - 1\}$, $a_0 b_j r = 0$ and it follows that $b_j r I a_0 = 0$ since $b_j r I a_0 \subseteq I$, $(b_j r I a_0)^2 = 0$ and I is reduced. So $(a_{\ell-j} b_j r)(a_0 b_\ell r)^2 = a_{\ell-j} b_j r (a_0 b_\ell r) a_0 b_\ell r \in a_{\ell-j} b_j r I a_0 b_\ell r = a_{\ell-j}(b_j r I a_0) b_\ell r = 0$ implies $(a_{\ell-j} b_j r)(a_0 b_\ell r)^2 = 0$. The coefficient of the term x^ℓ in $f(x)g(x) = 0$ is $0 = a_0 b_\ell + a_1 b_{\ell-1} + \cdots + a_\ell b_0 = a_0 b_\ell + \sum_{j=0}^{\ell-1} a_{\ell-j} b_j$.

Multiplying $r(a_0b_\ell r)^2$ to the preceding equation on the right side, we obtain $0=(a_0b_\ell+\sum_{j=0}^{\ell-1}a_{\ell-j}b_j)r(a_0b_\ell r)^2=(a_0b_\ell r)^3$. Since $a_0b_\ell r\in I$ and I is reduced, $a_0b_\ell r=0$, a contradiction. Thus $a_0b_jr=0$ for all $j\in\{0,1,\ldots,n\}$ and so we have that $f_1(x)g(x)r=0$ with $f_1(x)=a_1+a_2x+\cdots+a_mx^{m-1}$. But the degree of $f_1(x)$ is less than m and so, by the induction hypothesis, there exists $s\in R$ such that $g(x)rs\neq 0$ and $a_ib_jrs=0$ for all i,j with $1\leq i\leq m$ and $0\leq j\leq n$. Therefore $g(x)rs\neq 0$ and $a_ib_jrs=0$ for all i,j with $0\leq i\leq m$ and $0\leq j\leq n$.

Next assume $g(x)\in I[x]$. Then a_ib_j and b_ja_i are both contained in I for all i,j. So by the proof of [4, Lemma 1], we obtain that $a_ib_j=0$ for all i,j. Therefore R is right Armendarizlike. □

We apply Theorem 3.6 to the following situation.

Example 3.7. Consider the ideal $J=\{0,2,4\}$ of $\mathbb{Z}_6$ and the ideal $S=J[\{t_i\mid i\in\mathbb{Z}\}]$ of the polynomial ring $T=\mathbb{Z}_6[\{t_i\mid i\in\mathbb{Z}\}]$ over $\mathbb{Z}_6$. Next consider the automorphism σ of T defined by sending each t_i to t_{i+1}, and the ideal $S[x;\sigma]$ of the skew polynomial ring $T[x;\sigma]$.

Set R be the ring $S[x;\sigma]\times\mathbb{Z}_6$ obtained by attaching an identity to $S[x;\sigma]$. Letting $I=S[x;\sigma]\times 0$, I is an ideal of R that is reduced as a ring without identity. Since $R/I\cong\mathbb{Z}_6$, R is Armendarizlike by Theorem 3.6.

Analyzing the computation of Example 3.7, we can extend the ideal J in $\mathbb{Z}_6$ to reduced ideals in $\mathbb{Z}_n$.

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