On a Rule-Based Interpretation of Default Conditionals

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Abstract

In nonmonotonic reasoning, a default conditional $\alpha \rightarrow \beta$ has most often been informally interpreted as a defeasible version of a classical conditional, usually the material conditional. There is however an alternative interpretation, in which a default is regarded essentially as a rule, leading from premises to conclusion. In this paper, we present a family of logics, based on this alternative interpretation. A general semantic framework under this rule-based interpretation is developed, and associated proof theories for a family of weak conditional logics is specified. Nonmonotonic inference is easily defined in these logics. Interestingly, the logics presented here are weaker than the commonly-accepted base conditional approach for defeasible reasoning. However, this approach resolves problems that have been associated with previous approaches.

1 Introduction

A major approach in nonmonotonic reasoning is to represent a default as an object that one can reason about, either as a conditional in some object language, or as a nonmonotonic consequence operator. Thus for example "an adult is (typically or normally) employed" might be represented $a \rightarrow e$, where $\rightarrow$ represents a default or normality conditional, distinct from the material conditional $\supset$. In such approaches, one can typically derive other defaults from a given set of defaults. There has been widespread agreement concerning just what principles should constitute a minimal logic. It has been suggested that this minimal logic forms a "conservative core" of defaults that should be present in any approach to nonmonotonic reasoning. However, the default conditional characterized by this logic is quite weak, at least compared with the material conditional, in that it does not (in fact, *should not*) fully support principles such as strengthening of the antecedent, transitivity, and modus ponens.

Since one would want to obtain these latter properties by default, such logics are extended nonmonotonically by a "closure" operation or step. This closure operation has been defined, for example, in terms of a preferred subset of the models of a theory. In the resulting set of models, one obtains strengthening of the antecedent, transitivity, or (effectively) modus ponens, wherever feasible. Thus, there are two components to default reasoning within such a system. First, there is a standard, monotonic logic of conditionals that expresses relations among defaults that are deemed to always hold. Second, there is a nonmonotonic mechanism for obtaining defaults (and default consequences) where justified. In essence, these approaches treat the default conditional like its classical counterpart, the material conditional, where feasible or by default.

While this work captures an important notion of default entailment, it is not without difficulties. As described in the next section, some principles of the suggested core logic are not uncontentious; as well, there are examples of default reasoning in which one obtains undesirable results. Lastly, there are more recent approaches, notably addressing causality, in which one requires a weaker notion of default inference, rejecting, for example, contrapositive default inferences. In response to these points, this paper suggests that there is a second, distinct, interpretation of default conditionals in which a default is regarded more like a rule with properties more in line with a rule of inference than a weakened classical conditional.

This paper describes an approach under this second interpretation. First, an exceptionally weak logic of conditionals is developed; from this basis, a family of conditional logics is defined. Given a default conditional $\alpha \rightarrow \beta$, the underlying intuition is that α supplies a context in which, all other things being equal, β normally holds or, more precisely, in the context of α, the proposition expressed by $\alpha \wedge \beta$ is more "normal" than that expressed by $\alpha \wedge \neg\beta$. This is written[1] as $\|\alpha\|^{\mathcal{M}} \cap \|\neg\beta\|^{\mathcal{M}} < \|\alpha\|^{\mathcal{M}} \cap \|\beta\|^{\mathcal{M}}$ or equivalently $\|\alpha \wedge \neg\beta\|^{\mathcal{M}} < \|\alpha \wedge \beta\|^{\mathcal{M}}$. Notably, all of the logics that are considered are weaker than the aforementioned "conservative core". It proves to be the case however that a nonmonotonic operation is very easily defined; this nonmonotonic step essentially specifies that a property is irrelevant with respect to a default unless it is known to be relevant. Thus, given a default $\alpha \rightarrow \beta$, one would want to also accept the strengthening $(\alpha \wedge \gamma) \rightarrow \beta$ whenever "reasonable". This nonmonotonic step corresponds to formalising the conclusion that if $\|\alpha\|^{\mathcal{M}} \cap \|\neg\beta\|^{\mathcal{M}} < \|\alpha\|^{\mathcal{M}} \cap \|\beta\|^{\mathcal{M}}$ holds, then $\|\gamma\|^{\mathcal{M}} \cap (\|\alpha\|^{\mathcal{M}} \cap \|\neg\beta\|^{\mathcal{M}}) < \|\gamma\|^{\mathcal{M}} \cap (\|\alpha\|^{\mathcal{M}} \cap \|\beta\|^{\mathcal{M}})$ also holds, unless there is a reason to not accept this relation (for example it may be that $\|\gamma\|^{\mathcal{M}} \cap \|\alpha\|^{\mathcal{M}} \cap \|\beta\|^{\mathcal{M}} < \|\gamma\|^{\mathcal{M}} \cap \|\alpha\|^{\mathcal{M}} \cap \|\neg\beta\|^{\mathcal{M}}$ is known to hold). This nonmonotonic step easily admits inferences that have required significant formal machinery in other approaches. As well, it is shown that the aforementioned difficulties that arise in interpreting a default as a weak classical conditional do not occur here. Nonmonotonic inference is then defined in the standard way for such approaches: given a default theory T and a classical (i.e. non-default) formula α, β is a default inference from α with respect to T, just if $\alpha \rightarrow \beta$ is true in each of the nonmonotonic "extensions" of T.

This distinction between treating a default as a conditional or as a rule has been noted previously; see for example [Geffner and Pearl, 1992]. As well, work on inheritance networks [Horty, 1994b] can be viewed as investigating proof theories for the latter interpretation. Work on causality such as [McCain and Turner, 1997] also falls in the rule-based framework. However a logic (that is, with both semantics and proof theory) capturing this interpretation has not been investigated previously, nor has a fully general nonmonotonic closure operator been developed under this interpretation. Last, it is suggested in the conclusion

[1] $\|\alpha\|^{\mathcal{M}}$ is the set of worlds in a model $\mathcal{M}$ in which α is true. Formal details are given in Section 3.

that this alternative interpretation may be widely applicable. In particular, it is suggested that this approach may be an appropriate vehicle for representing counterfactual reasoning [Lewis, 1973], which previously has been treated via the stronger interpretation.

The next section reviews previous work in conditional approaches to nonmonotonic reasoning. Section 3 informally reviews the approach while Section 4 describes a family of weak conditional logics. Section 5 considers the incorporation of a nonmonotonic extension to a conditional knowledge base. We conclude with Section 6. Proofs of theorems are contained in an appendix.

2 Conditional Logics and Nonmonotonic Reasoning

2.1 Conditional Logics for Representing Defaults

In recent years, much attention has been paid to conditional approaches to default reasoning. Such approaches address defeasible conditionals whose meaning is based on notions of preference among worlds or interpretations. Thus, the default that a bird normally flies can be represented propositionally as $b \rightarrow f$.[2] These approaches are typically expressed using a modal logic in which the connective $\rightarrow$ is a binary modal operator. The intended meaning of $\alpha \rightarrow \beta$ is approximately "in the least worlds (or most preferred worlds) in which α is true, β is also true". Possible worlds (or, again, interpretations) are arranged in at least a partial preorder, reflecting a notion of normality or preferredness on the worlds. Given a set of defaults Γ, default entailment with respect to Γ, $\mid\!\sim_{\Gamma}$, can be defined as follows [Delgrande, 1987]:

$$\text{If} \quad \Gamma \vdash \alpha \rightarrow \beta \quad \text{then} \quad \alpha \mid\!\sim_{\Gamma} \beta. \tag{1}$$

There has been a remarkable convergence or agreement on what inferences ought to be common to all such systems, and in the literature a seeming diversity of conditional approaches essentially allows the same inferences. These include approaches based on intuitions from probability theory such as ϵ-entailment [Pearl, 1988] (or 0-entailment or p-entailment [Adams, 1975]), from qualitative possibilistic logic [Dubois *et al.*, 1994], as well as modal-logic based approaches such as preferential entailment [Kraus *et al.*, 1990], *C4* [Lamarre, 1991], *CT4* [Boutilier, 1994a], and $\mathcal{S}$ [Burgess, 1981]. Consequently it has been suggested that the resulting set of inferences may be taken as specifying a *conservative core* [Pearl, 1989] that arguably should be common to all default inference systems. One expression of this logic of conditionals is as follows. Following [Lamarre, 1991] we call this logic *C4*, since it is the conditional logic based on a *S4*-like accessibility relation.[3] The logic includes classical propositional logic (that is, includes modus ponens and substitution of provable

[2] An alternative is to treat the conditional as a nonmonotonic inference operator, $b \mid\!\sim f$. In a certain sense these approaches can be considered equivalent [Boutilier, 1994a]. Here, for simplicity, we remain within the conditional logic framework.

[3] Lamarre's axiomatisation is not exactly as given here. As well, he allows iterated modalities, which we do not consider, as they add little of interest to the logic.

equivalents as well as all truth functional tautologies as axioms) and the following rules and axioms:[4]

RCEA/LLE: From $\vdash \alpha \equiv \beta$ infer $\vdash_{C4} (\alpha \rightarrow \gamma) \equiv (\beta \rightarrow \gamma)$.

RCM/RW: From $\vdash \beta \supset \gamma$ infer $\vdash_{C4} (\alpha \rightarrow \beta) \supset (\alpha \rightarrow \gamma)$

ID/Ref: $\vdash_{C4} \alpha \rightarrow \alpha$

CC/And: $\vdash_{C4} ((\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma)) \supset (\alpha \rightarrow \beta \wedge \gamma)$

RT/Cut: $\vdash_{C4} ((\alpha \rightarrow \beta) \wedge (\alpha \wedge \beta \rightarrow \gamma)) \supset (\alpha \rightarrow \gamma)$

ASC/CM: $\vdash_{C4} ((\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma)) \supset (\alpha \wedge \beta \rightarrow \gamma)$

CA/Or: $\vdash_{C4} ((\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)) \supset (\alpha \vee \beta \rightarrow \gamma)$

Note however, these principles are not uncontentious; for example, [Poole, 1991] can be viewed as arguing against **CC**. Likewise, [Neufeld, 1989] suggests against **CA** in some cases.

The semantics of these approaches is usually phrased in terms of a modal framework, in which possible worlds are ranked by a notion of relative normality or unexceptionalness. The underlying modal logic is generally taken to be *S4* [Hughes and Cresswell, 1996] (also called *KT4* [Chellas, 1980]), in which accessibility between worlds is given by a reflexive, transitive binary relation. A conditional $\alpha \rightarrow \beta$ is true at a world w just when, for every world accessible from w, there is an accessible world in which $\alpha \wedge \beta$ is true and $\alpha \supset \beta$ is true at all worlds that are less or equally exceptional, or if there are no accessible α worlds. Thus, "a bird flies", $b \rightarrow f$, is true if, in the least b-worlds (if such exist), $b \supset f$ is true. Since a penguin is a bird (either $\Box(p \supset b)$ or $p \rightarrow b$) but a penguin doesn't fly ($p \rightarrow \neg f$), this means that the least exceptional penguin-worlds are more exceptional than the least bird-worlds.

The resulting logic is weak. For example, the following relations which hold for the material conditional do not hold for the weak conditional:

Strengthening: From $\alpha \rightarrow \gamma$ infer $\alpha \wedge \beta \rightarrow \gamma$.

Transitivity: From $\alpha \rightarrow \beta$ and $\alpha \rightarrow \gamma$ infer $\alpha \rightarrow \gamma$.

Contraposition: From $\alpha \rightarrow \gamma$ infer $\neg\gamma \rightarrow \neg\alpha$.

Modus ponens: From $\alpha \rightarrow \beta$ and α infer β.

[4]Two systems of nomenclature have arisen, one associated with conditional logic and one with nonmonotonic consequence operators. We list both (when both exist) when first presenting an axiom or rule. For example the conditional logic rule for substitution of logical equivalents in the antecedent is called **RCEA**; its nonmonotonic consequence operator, Left Logical Equivalence is abbreviated **LLE**. Hence we first list the rule as **RCEA/LLE**. After first usage, we use the notation specific to conditional logic, drawing from [Chellas, 1975, Nute, 1980].

Nor would one want these principles to always hold for defaults. Thus, for strengthening, we do not want to conclude, given that a bird flies, that a bird with a broken wing flies. For transitivity, while an adult is normally employed full-time, and a university student is normally an adult, it is not the case that university students are normally employed full-time. For contraposition, just because I normally eat lunch at Joe's place, this doesn't mean that if I don't eat at Joe's then I normally skip lunch [Boutilier, 1994a]. And, last, for modus ponens we want to allow that a normality conditional can be violated.

On the other hand, one would want these properties to hold by default. Thus given that birds normally can fly, and we are presented with a green bird, we would like to conclude that it flies. Similarly, given that a robin is a bird, we would like to be able to conclude that a robin flies. Clearly, this is not something that can be done *within* the logic; that is, given that a bird is asserted to fly by default, one cannot thereby conclude via (1) that a green bird flies by default.[5] That is, simply put, the inference $b \rightarrow f \vdash b \wedge g \rightarrow f$ does not obtain or, alternatively, $\{b \rightarrow f,\ \neg(b \wedge g \rightarrow f)\}$ is satisfiable. The problem is that there is nothing that requires preferred worlds in which a bird flies to include among them green-bird worlds.

2.2 Nonmonotonic Extensions to Conditional Logics

Given the above considerations, various means of strengthening the logic have been proposed in order to incorporate strengthening or transitivity in a principled fashion. We focus in this subsection on two well-known approaches for nonmonotonically extending, or taking the (conditional) closure of, a conditional knowledge base, called *rational closure* and *conditional entailment*.

For rational closure [Lehmann and Magidor, 1992], it can be noted that, as with the base logic *C4* described in the previous subsection, there have been an number of approaches, founded on different intuitions that again converge to essentially the same system. These approaches include rational closure, as well as System **Z** and 1-entailment [Pearl, 1990], approaches based on modal logic [Crocco and Lamarre, 1992, Boutilier, 1994b] and possibilistic entailment [Benferhat *et al.*, 1992], and conditional objects [Dubois and Prade, 1991]. We describe *rational closure* as an exemplar of this set of approaches.

The essential idea is that, in a semantic sense, a world is assumed to be as unexceptional as consistently possible. Thus, given that a bird flies, all other things being equal a world where that bird flies will be ranked below one where it does not. Consequently, since there is no reason to suppose that greenness has any bearing on flight, one assumes that green-bird-flying worlds are ranked with the least bird-flying worlds if consistently possible. Hence one would expect to find that at the least green-bird worlds that fly is true; similarly, at the least nongreen-bird worlds we would also expect to find that fly is true. Hence a green bird (normally) flies as does a non-green bird. Define[6] $\beta \prec \alpha$ by

$$\Diamond(\alpha \vee \beta)\ \wedge\ ((\alpha \vee \beta) \rightarrow \neg\alpha). \qquad (2)$$

Thus, informally, there are $\alpha \vee \beta$ worlds, and at the least $\alpha \vee \beta$ worlds, $\neg\alpha$ is true; hence β is

[5] It is worth noting however that **ASC** gives a restricted version of strengthening, while **RCM** gives a restricted version of transitivity.

[6] cf. [Lewis, 1973, p. 54]

true at such worlds and any α world is not less than these worlds. From this we can define an ordering on formulas of classical logic. The sign $\vdash_{C4}$ stands for logical derivation in *C4* [Lamarre, 1991], as the representative of the systems discussed in the previous section.[7]

Definition 2.1 ([Lehmann, 1989]) *Given a default theory T, the* degree *of a formula α is defined as follows:*

1. *$degree(\alpha) = 0$ iff for no δ do we have $T \vdash_{C4} \delta \prec \alpha$.*

2. *$degree(\alpha) = i$ iff $degree(\alpha)$ is not less than i and $T \vdash_{C4} \beta \prec \alpha$ only if $degree(\beta) < i$*

3. *$degree(\alpha) = \infty$ iff α is assigned no degree in Parts 1 and 2 above.*

From this the closure operation is defined:

Definition 2.2 ([Lehmann, 1989]) *The* rational consequence *relation, with respect to default theory T, is given by:*

$$\alpha \mid\!\sim_T \beta \quad \textit{iff} \quad degree(\alpha) < degree(\alpha \wedge \neg\beta) \textit{ or } degree(\alpha) = \infty.$$

We obtain that for default theory T that if $T \vdash_{C4} \alpha \to \beta$ then $\alpha \mid\!\sim_T \beta$. (That is, since $T \vdash_{C4} \alpha \to \beta$, one can show that $\alpha \prec (\alpha \wedge \neg\beta)$ using (2), from which $degree(\alpha) < degree(\alpha \wedge \neg\beta)$ follows.)

Consider the following example:

Example 2.1

$$T = \{b \to f,\ b \to w,\ p \to b,\ p \to \neg f\}.$$

Hence, a bird flies and has wings, while a penguins is a bird that does not fly. We obtain inferences such as

$$b \wedge g \mid\!\sim_T f,\ \ b \wedge \neg g \mid\!\sim_T f,\ \ b \mid\!\sim_T \neg p,\ \ \text{and}\ \ p \wedge g \mid\!\sim_T \neg f.$$

Let us consider the first inference, working backwards to see how it is obtained: $b \wedge g \mid\!\sim_T f$ holds just if $degree(b \wedge g) < degree(b \wedge g \wedge \neg f)$. In fact here we have $degree(b \wedge g) = 0$ and $degree(b \wedge g \wedge \neg f) = 1$. We have $degree(b \wedge g \wedge \neg f) = 1$ in turn because $T \vdash_{C4} b \prec (b \wedge g \wedge \neg f)$, which is to say (via (2)) that $T \vdash_{C4} ((b \wedge g \wedge \neg f) \vee b) \to \neg(b \wedge g \wedge \neg f)$. For this last relation, the antecedent of the conditional is equivalent to b and the consequent to $\neg b \vee \neg g \vee f$; hence this is equivalent to $T \vdash_{C4} b \to (\neg b \vee \neg g \vee f)$ which in turn is derived from $T \vdash_{C4} b \to f$ by **RCM** and using $\vdash f \supset (\neg b \vee \neg g \vee f)$.

On the other hand, one does not obtain the result $p \mid\!\sim w$ even though this inference would appear to be sanctioned by the defaults $p \to b$ and $b \to w$. The reason for this is that $degree(p) = degree(p \wedge \neg w) = 1$, as can be verified from the definitions. Thus, in the rational closure, one does not obtain inheritance of properties (in this case w) across exceptional subclasses (in this case p). The failure to allow full inheritance of properties has

[7] We use $\vdash_{C4}$ in place of *preferential entailment* by appeal to [Boutilier, 1992, Thm 4.18], which provides the correspondence result.

been addressed, for example in [Benferhat *et al.*, 1993, Lehmann, 1995] via the *lexicographic closure* of a set of defaults; as well, see [Benferhat *et al.*, 1995]. However these extensions are syntax-dependent, and come at the expense of higher complexity than the original formulation.

A second, well-known approach is *conditional entailment* [Geffner and Pearl, 1992]. Conditional entailment was formulated in part to reconcile approaches exemplified by conditional logics on the one hand, and earlier approaches such as circumscription [McCarthy, 1980] on the other. In conditional entailment, defaults are arranged in a partial order, determined in part by the specificity of a rule's antecedent. This priority order over the set of defaults $\Delta_{\mathcal{L}}$ is defined such that every set Δ of defaults in conflict with a default r contains a default r' that is less than that default in the ordering. Given this ordering on rules, an ordering on worlds can then be defined: If $\Delta(w)$ and $\Delta(w')$ are the defaults falsified by worlds w and w' respectively, then w is preferred to w' iff $\Delta(w) \neq \Delta(w')$, and for every rule in $\Delta(w) \setminus \Delta(w')$ there is a rule in $\Delta(w') \setminus \Delta(w)$ which has higher priority. As usual, β is a default consequence of α just if β is true in the most preferred α worlds. We obtain the same consequences given for Example 2.1 as for the rational closure; moreover we obtain that $p \mid\sim w$. However full inheritance of properties is not supported. Consider the following example [Geffner and Pearl, 1992]:

Example 2.2

$$T = \{b \rightarrow f,\ p \rightarrow s,\ s \rightarrow b,\ p \rightarrow \neg f\}.$$

(Thus, a bird flies; a penguin is a shorebird; a shorebird is a bird; but a penguin doesn't fly.) The inference $p \mid\sim_T b$, which would be expected to hold from the defaults $p \rightarrow s$ and $s \rightarrow b$, does not obtain. The difficulty here is that based on the default $b \rightarrow f$, one can draw the contrapositive inference that a non-flyer is a non bird; this, along with the default $p \rightarrow \neg f$ allows the possibility of a penguin being a non-bird, thereby blocking the desired inference of bird from penguin.

Both rational closure and conditional entailment formalise important and interesting phenomena in nonmonotonic reasoning, and have found widespread application in the literature. However there are problems with both approaches when considered as a general approach to formalising reasoning with defaults or normality conditionals. Inheritance reasoning has already been mentioned; other examples are given below. Nonetheless, it may be argued that these problems don't represent a limitation of the approaches per se, but rather indicate that these approaches are inapplicable to certain interpretations of defaults. Rational closure, for example, employs a very strong minimization criterion that is not always appropriate. Consider the following elaboration of an example given in [Horty, 1994a]:

Example 2.3

$$\top \rightarrow \neg f,\ a \rightarrow f,\ \top \rightarrow n,\ s \rightarrow \neg n.$$

(Normally one does not eat with the fingers (f), but one does when eating asparagus at a meal (a); normally one uses a napkin (n), but not when one is eating while standing (s).)

The rational closure of these conditionals gives that, if one is not standing ($\neg s$), one does not eat asparagus ($\neg a$). Clearly this interaction between unrelated defaults is undesirable.

In addition, consider the following example [Geffner and Pearl, 1992]:

Example 2.4

$$a \rightarrow e,\ u \rightarrow a,\ u \rightarrow \neg e,\ f \rightarrow a.$$

(That is, an adult is normally employed, a university student is normally an adult but is not employed, and a fan of Frank Sinatra is normally an adult.) In both conditional entailment and rational closure we obtain the default inference that a fan of Frank Sinatra is not a university student, $f \mid\!\sim \neg u$. This is too strong to be a plausible inference, since there is nothing in the example that would relate Frank Sinatra fans to university students. As well, if the conditional $u \rightarrow \neg e$ is dropped from the theory, one now loses the default inference that a Frank Sinatra fan is not a university student. In this instance, it seems very strange that a nonmonotonic inference between Frank Sinatra fans and university students should be mediated by a person's being employed or not.

2.3 Reconsidering Default Conditionals

As suggested, at least some of the preceding examples do not necessarily reflect a problem with the approaches per se. Rather, our thesis is that there are (at least) two distinct interpretations that can be given to a default. First, there is the intuition that a default is essentially a weak version of the material conditional, and should behave as such a conditional, except that it is defeasible. This intuition is seen most clearly in the expression of defaults by circumscriptive abnormality theories [McCarthy, 1986]. In this case a default $\alpha \rightarrow \beta$ is represented as the formula $\alpha \wedge \neg Ab_i \supset \beta$. The circumscription of Ab_i asserts (very roughly) that Ab_i is false if consistently possible. Obviously, if Ab_i is asserted to be false, we obtain the material conditional.

Conditional entailment explicitly adopts the intuition that a default is a weak version of the material conditional. That is, the default $\alpha \rightarrow \beta$ is basically the same as $\top \rightarrow (\alpha \supset \beta)$ (i.e. the material counterpart normally holds) together with specificity information implicit in α [Geffner and Pearl, 1992, p. 232]. There are certainly instances (for example in diagnosing abnormalities in a circuit [Reiter, 1987]) where one wants, all other things being equal, a default to behave as a material conditional.

However, there are also situations where one does not want this behaviour. For example, consider the theory that asserts of a person that if they were to get a good review at work, they would be happy. On the other hand, if they were to break their leg, they would not be happy:

Example 2.5

$$T = \{r \rightarrow h,\ bl \rightarrow \neg h\}.$$

In rational closure and conditional entailment, as well as in the corresponding circumscriptive abnormality theory, one obtains the inference $r \mid\!\sim \neg bl$: if someone gets a good review then they won't break their leg. This is clearly undesirable. Moreover, it is not clear how such a theory could be repaired to avoid this conclusion; breaking the conflict by, for example, adding $r \wedge bl \rightarrow \neg h$ doesn't solve the problem.

Thus, in these approaches, given defaults $\alpha \rightarrow \gamma$, $\beta \rightarrow \neg\gamma$, one nonmonotonically infers the default $\alpha \rightarrow \neg\beta$. As the above example indicates, this is not always desirable. Other such patterns are readily identifiable; consider an example (of unknown source) with defaults $\alpha \rightarrow \gamma$ and $\gamma \rightarrow \beta$. In the above cited approaches (rational closure, conditional entailment, and via a circumscriptive abnormality theory), one obtains as a nonmonotonic inference that $\alpha \wedge \neg\gamma \rightarrow \beta$. Thus, for a specific example, given the defaults that a university student is normally an adult ($u \rightarrow a$) and an adult is normally employed ($a \rightarrow e$), the inferred default $u \wedge \neg a \rightarrow e$ would allow one to conclude that a university student that was not an adult is employed. Informally the reason for this is as follows: From the defaults $u \rightarrow a$ and $a \rightarrow e$, one nonmonotonically obtains the "transitive" result $u \rightarrow a$. But given the default $u \rightarrow a$, one can similarly nonmonotonically strengthen the antecedent, yielding $u \wedge \neg a \rightarrow e$. Clearly this is incorrect, as the inferred default that a university student is employed is mediated by their being an adult.

These considerations indicate that conditional closures, as represented by rational closure and conditional entailment (and by implication applying also to extensions of these works and related work) at times produce undesired conclusions. However, the monotonic consequences of the "conservative core" can also lead to unintuitive conclusions; consider the following example (also of unknown source):

> A crime has been committed, of which the two suspects are John and Mary. In deciding who to arrest, the detective decides that if the murder weapon is found in John's room, then John will be arrested; if found in Mary's room then Mary will be arrested. If the weapon if found with John's fingerprints, then John will be arrested, and if Mary's then Mary.

We can symbolize this by:

Example 2.6

$$rJ \rightarrow J, \;\; rM \rightarrow M, \;\; fJ \rightarrow J, \;\; fM \rightarrow M, \;\; \Box\neg(J \wedge M).$$

What if the gun is found in John's room but with Mary's fingerprints (or vice versa)? Assume that to settle this conflict, it is decided that fingerprints decide the culprit. So we add

Example 2.6 continued

$$rJ \wedge fM \rightarrow M \quad \text{and} \quad rM \wedge fJ \rightarrow J.$$

With these defaults we can derive $rJ \rightarrow \neg fM$, that is if the murder weapon is found in John's room then it does not have Mary's fingerprints on it. This clearly is contrary to one's intuitions.

Note that $rJ \rightarrow \neg fM$ is a monotonic consequence of the logic, in that it is derivable from the original theory in the logic *C4*. So here, we have an unintuitive result in the logic itself – which is to say, to the extent that this result is counter to one's expectations, the blame cannot be allocated to any nonmonotonic closure step, since there is no nonmonotonicity in this example. (As well, we have already noted [Poole, 1991] and [Neufeld, 1989] as having identified implausible results in the underlying logic.)

Further, consider the preceding examples in the light of this last example. Assume that we have some generic approach where the underlying intuition is to treat a default as the material conditional wherever possible. In Example 2.5, there is no reason to not treat the two defaults as their classical counterparts, say $r \supset h$ and $bl \supset \neg h$. (This in essence is what the aforecited closure operations, and approaches extending them, do.) Given the premiss r, it is perfectly sensible to conclude h; using the contrapositive of the second formula, $h \supset \neg bl$ one then obtains $\neg bl$. The problem arises, informally, in treating $bl \rightarrow \neg h$ as a classical material conditional, from which one obtains the contrapositive.

This also informally explains example (2.4). In (2.4) one can derive (monotonically) the default $a \rightarrow \neg u$. Since one already has the default $f \rightarrow a$, there is nothing to prevent the "transitive" nonmonotonic inference yielding $f \mid\!\sim \neg u$. If one accepts that a default transitivity should hold unless explicitly blocked, then the difficulty with this example can again be attributed to the underlying logic.

Consequently, there are situations in which nonmonotonic operations based on the default "core" logic lead to unintuitive results. Moreover, these unintuitive results arguably arise from properties of the underlying logic. This is illustrated in (2.4) and (2.5) where, in one fashion or another, one does not want to apply the contrapositive of a default; rather a default is to be applied in a "forward" fashion only. Under this second interpretation a default is regarded more as an (object-level) *rule*, whose properties would be closer to those of a rule of inference. Hence, given a conditional $\alpha \rightarrow \beta$, if the antecedent α happens to be true, we conclude β by default. Given $\neg\beta$ we specifically do not want to conclude $\neg\alpha$.

Work along these lines is not new. This distinction between regarding a default as a weak variant of a material conditional and as a rule has been noted in [Geffner and Pearl, 1992]. Inheritance networks [Horty, 1994b] can be viewed as proof theoretic accounts of this interpretation, as can subsequent work in argumentation systems [Simari and Loui, 1992, Dung, 1995]. In fact, [Dung and Son, 2001] combines argumentation with a specificity relation on rules, to obtain an approach with properties (in the stratified case) among those of the "conservative core" logic, here called *C4*. A further motivation for exploring the rule-based interpretation of conditionals is that there has been recent interest in conditional accounts of causality (for example [McCain and Turner, 1997]), in which reasoning via a default contrapositive is explicitly rejected. Thus, from "closing the switch causes the light to go on" we do not want to conclude that the light being off causes the switch to open. Thus, from "a causes b" we don't necessarily want to conclude "$\neg b$ causes $\neg a$." Hence a basic approach to rule-based conditional reasoning may shed light on the foundations of such causality-based approaches.

3 Defaults as Rules

The general approach developed here is the same as those described in the previous section: we begin by specifying a conditional logic of defaults and subsequently provide a principled, nonmonotonic means to extend the logic to account for irrelevant properties. Our point of departure is that we informally treat defaults as having properties more like rules of inference, in that defaults are intended to be applied in a "forward" direction only.

Our interpretation, roughly, is that the antecedent of a default establishes a context in which the consequent (normally) holds, or holds all other things being equal. Consider the default that normally asparagus is eaten with the fingers, $a \to f$. Our interpretation is that, of the set of worlds in which a is true, the subset of worlds with f also true is more normal (usual, preferable) than the subset with $\neg f$. This suggests a notation along the lines of $a : (\neg f < f)$. But to say of the set of worlds in which a is true, that the subset with f true is more normal than the subset with $\neg f$ true, is no different than saying that the set of $a \wedge f$ worlds is more normal than the set of $a \wedge \neg f$ worlds. This suggests a notation along the lines of $a \wedge \neg f < a \wedge f$.

Hence one can consider that, for default $\alpha \to \beta$, the formula α establishes a context, and in this context it is the case that β is more normal (typical, etc.) than $\neg\beta$. We express this semantically by introducing a binary relation of relative normality $<_w$ between propositions, where a proposition corresponding to a formula is the set of those possible worlds in which the formula is true. Formal details are give in the next section, but basically in a model $\mathcal{M}$ and for a world w, the proposition expressed by a formula α is denoted by $\|\alpha\|_w^{\mathcal{M}}$, and consists of those worlds considered possible at w in which α is true. Given with this notation, we can express that formula $\alpha \to \beta$ is true in a model $\mathcal{M}$ at world w by

$$\|\alpha\|_w^{\mathcal{M}} \cap \|\neg\beta\|_w^{\mathcal{M}} <_w \|\alpha\|_w^{\mathcal{M}} \cap \|\beta\|_w^{\mathcal{M}}, \tag{3}$$

that is, the proposition $\|\alpha\|_w^{\mathcal{M}} \cap \|\beta\|_w^{\mathcal{M}}$ is more normal (typical, etc.) than $\|\alpha\|_w^{\mathcal{M}} \cap \|\neg\beta\|_w^{\mathcal{M}}$ at world w. It seems reasonable that this binary relation of relative normality $<_w$ be asymmetric and transitive, and so we generally assume that these conditions hold.

We note that the form of (3) has appeared regularly in the literature, going back at least to [Lewis, 1973].[8] The difference is that usually the interpretation of (3) is along the lines of "the *least* worlds where $\alpha \wedge \neg\beta$ is true are less normal than the least $\alpha \wedge \beta$ worlds". Thus for example in [Lewis, 1973, pp. 54-56] $P \preceq_i Q$ is used to express that "the proposition P is at least as possible, at the world i, as the proposition Q." However, as the exposition makes clear, what this notation really means is that the *least* P-worlds are at least as possible, at the world i, as the *least* Q-worlds. Hence more perspicuous notation would express this relation as something like $min(P) \preceq_i min(Q)$. In any case, our notation will refer to the *proposition* expressed by these formulas. Consequently, our relation $<_w$ cannot be regarded as an accessibility relation in the normal sense, since it is a relation on *sets* of worlds.

Filling in the (formal) details yields a weak logic of conditionals **C**, significantly weaker than the so-called "conservative core", in which weak conditionals of the form $\alpha \to \beta$ can be interpreted. The operator $\to$ is a binary modal operator defined not in terms of accessibility among possible worlds, but rather directly in terms of pairs of propositions. For the sake of increased expressibility, it is convenient to also introduce a notion of necessity, expressed by $\Box\alpha$ for "α is necessarily true" or semantically "α is true at all worlds considered possible". Our notion of necessity is given a physical interpretation (as opposed to a more usual epistemic interpretation). Hence, this notion of necessity may be thought of as expressing a

[8] We use $<_w$ in the opposite sense of Lewis.

domain-specific integrity constraint. Thus we might use $\Box(s \supset c)$ to express propositionally that a spoon s is necessarily a piece of cutlery c.

Since the base logic is very weak, we also consider various strengthenings of the logic for expressing statements of normality. However, all of these strengthenings are still weaker than the so-called "core" set of defaults represented here by *C4*. Nonetheless, we show that these logics have desirable properties; as well, the undesirable inference illustrated in Example 2.6 is not obtained.

Moreover, it proves to be the case that nonmonotonic reasoning is definable in a very simple and straightforward manner. Consider again our example that one normally eats asparagus with the fingers, $a \rightarrow f$. One would also want to be able to incorporate irrelevant properties, when reasonable. Thus it would seem that barring information to the contrary, one should (nonmonotonically) accept that normally white asparagus is eaten with the fingers, $a \wedge w \rightarrow f$.[9] Semantically this would mean that, given $\|a \wedge \neg f\|_w^{\mathcal{M}} <_w \|a \wedge f\|_w^{\mathcal{M}}$, one would like to extend a model to have $\|a \wedge \neg f\|_w^{\mathcal{M}} \cap \|w\|_w^{\mathcal{M}} <_w \|a \wedge f\|_w^{\mathcal{M}} \cap \|w\|_w^{\mathcal{M}}$ and so $\|a \wedge w \wedge \neg f\|_w^{\mathcal{M}} <_w \|a \wedge w \wedge f\|_w^{\mathcal{M}}$. How to do this, at least in broad outline, is straightforward: Basically, the (semantic) relation $X <_w Y$ asserts that in the "context" (set of possible worlds) $X \cup Y$, partitioned by X,Y, we have that Y is more normal than X. Our nonmonotonic assumption is that this obtains in all "feasible" subcontexts (where, of course "feasible" needs to be defined). That is, for proposition Z, unless there is reason to conclude otherwise, we assert that $Z \cap X <_w Z \cap Y$. The next section develops the formal details of the family of weak conditional logics for defaults, while the following section addresses nonmonotonic reasoning in this family of logics.

4 The Approach

4.1 Formal Preliminaries

We assume some familiarity with modal logics. $\mathcal{L}_{PC}$ is the language of classical propositional logic defined, for simplicity, over a finite alphabet $\mathbf{P} = \{a, b, c, \ldots\}$ of *atomic sentences*. Formulas of $\mathcal{L}_{PC}$ are constructed from the logical symbols $\neg$, $\vee$, $\wedge$, $\supset$, and $\equiv$ in the standard manner. The symbol $\top$ is taken to be some propositional tautology, and $\perp$ is defined as $\neg\top$. Our language for expressing weak conditionals, $\mathcal{L}$, is $\mathcal{L}_{PC}$ extended with the binary operator $\rightarrow$ and the unary operator $\Box$. The operator $\rightarrow$ is the *weak conditional*, in contrast to the material conditional $\supset$; the operator $\Box$ expresses necessity. For convenience, arguments of both $\rightarrow$ and $\Box$ are members of $\mathcal{L}_{PC}$; that is, we do not allow nested occurrences of $\rightarrow$ nor $\Box$.[10] As is usual, we use $\Diamond$ to abbreviate $\neg\Box\neg$.

Formulas are denoted by the Greek letters α, β, $\alpha_1, \ldots$ and sets of formulas by upper case

[9] In Germany, where white asparagus is the norm, the example should be changed to green asparagus.

[10] Nested occurrences of $\Box$ are disallowed for two reasons. First, the operator $\Box$ is not required, and the development of $\rightarrow$ can be carried out without it; rather it is included for increased expressibility of the language so, for example, we can say not just that penguins are normally birds, but rather that penguins are necessarily birds. Second, our interest lies in the default operator $\rightarrow$, and a full consideration of iterated $\Box$ modalities would add nothing to this endeavour while requiring a reasonable amount of off-topic formal development.

Greek letters Γ, Δ, Γ_1, $\ldots$. To simplify notation, unary connectives will bind most tightly, and the weak conditional $\rightarrow$ will bind most weakly. Thus for example $\neg\alpha \wedge \beta \rightarrow \beta \vee \gamma$ is interpreted as $((\neg\alpha) \wedge \beta) \rightarrow (\beta \vee \gamma)$. Parentheses of course will be used to override operator precedence and for clarity. The symbol $\vdash$, possibly subscripted with the name of a system, is used to indicate derivation of a formula from a set of formulas. A subscript indicates derivation with respect to a particular system; hence $\Gamma \vdash_{\mathbf{C}} \alpha$ indicates that α is deducible from Γ in the logic **C**. The unsubscripted symbol $\vdash$ denotes inference in classical propositional logic.

The semantics, developed in the next section, is based on the notion of a *possible world*, where a possible world can be thought of a complete, consistent description of how the world could conceivably be. Every formula then will be either true or false at a world w in a model $\mathcal{M}$. *Propositions* correspond to sets of possible worlds; the proposition expressed by a sentence α will consist of the set of possible worlds in which α is true. Propositions are denoted by the upper case letters X, Y, $\ldots$. Since we assume a finite alphabet $\mathbf{P}$, we are guaranteed a 1-1 correspondence between propositions expressible by formulas and sets of possible worlds.[11] That is, we equate the proposition expressed by a sentence with a set of possible worlds; conversely, given a proposition (set of worlds) X, one can easily determine a representative formula, for example by representing a world by a conjunction of those literals true at the world and then taking the disjunction over the worlds in the set X.

4.2 The Base Logic

Unlike the approaches described in Section 2, we do not employ a Kripke structure on possible worlds for the interpretation of the conditional $\rightarrow$. Rather, each world in a model is associated with a binary notion of *relative normality*, denoted $<$, between sets of possible worlds, or propositions. Sentences are interpreted with respect to a model, as follows.

Definition 4.1 *A* comparative conditional model *is a tuple* $\mathcal{M} = \langle W, N, <, P \rangle$ *where:*

1. *W is a set (of states or possible worlds);*
2. $N : W \mapsto 2^W \setminus \emptyset$.
3. $< \subseteq W \times 2^W \times 2^W$ *with properties described below;*
4. $P : \mathbf{P} \mapsto 2^W$.

P maps atomic sentences onto sets of worlds, being those worlds at which the sentence is true. For $w \in W$, $N(w)$ gives the set of those worlds considered possible at w. We require that

$$w \in N(w) \quad \text{for every} \quad w \in W.$$

The relation $<$ associates with each world $w \in W$ a binary notion of *relative normality* between propositions; we write $X <_w Y$ to assert that, according to world w, proposition Y

[11] See [Lakemeyer and Levesque, 2000] for a discussion and analysis of problems that arise in the infinite case.

is more normal than X. Given our intended usage of $<_w$ we require that arguments to $<_w$ be disjoint:

If $X <_w Y$ then $X \cap Y = \emptyset$.

As well, since $<_w$ is a relation of relative normality between propositions, the propositions in question must be possible at a world. That is, we require that if $X <_w Y$ then $X, Y \subseteq N(w)$. Since $<_w$ is to be interpreted as capturing a notion of normality between propositions, the following four properties are necessary, in that if any were omitted, then $<_w$ would be arguably too weak to capture this notion of normality. These properties are as follows:

1. Min $\emptyset_1$: $\emptyset <_w X$ for every $\emptyset \neq X \subseteq N(w)$.
2. Min $\emptyset_2$: $X \not<_w \emptyset$ for every $X \subseteq N(w)$.
3. Asymmetry: If $X <_w Y$ then $Y \not<_w X$.
4. Transitivity: If $X <_w Y$, $Y <_w Z$ then $X <_w Z$ provided X, Y, Z are pairwise disjoint.

The condition min $\emptyset_1$ asserts that for any proposition X, where $X \neq \emptyset$, we have that $\emptyset <_w X$, that is, that $\emptyset$ is the minimum of $<$. The second condition, min $\emptyset_2$, states that no proposition is $<_w$ the incoherent proposition. Clearly if $<_w$ is to reflect a notion of relative normality, it should be asymmetric. Thus, if $X <_w Y$ holds then $Y <_w X$ should not hold. Finally, we would want that $<_w$ be transitive. The proviso that X, Y, and Z be pairwise disjoint is added to the condition since $<_w$ is defined only for disjoint sets.

That α is true at world w in model $\mathcal{M}$ is written $\mathcal{M}, w \models \alpha$. We identify the proposition expressed by a sentence α with the set of possible worlds in which α is true, denoted $\|\alpha\|^{\mathcal{M}}$. That is,

$$\text{For} \quad \alpha \in \mathcal{L}_{PC}, \quad \|\alpha\|^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \models \alpha\}.$$

We extend this notion to that of a proposition expressed by α according to world w by

$$\|\alpha\|^{\mathcal{M}}_w = \{w \in N(w) \mid \mathcal{M}, w \models \alpha\} = \|\alpha\|^{\mathcal{M}} \cap N(w).$$

Truth of a formula at a world in a model is as for propositional logic, with additions for $\Box$ and $\rightarrow$:

Definition 4.2

1. *$\mathcal{M}, w \models p$ for $p \in \mathbf{P}$ iff $w \in P(p)$.*
2. *$\mathcal{M}, w \models \alpha \wedge \beta$ iff $\mathcal{M}, w \models \alpha$ and $\mathcal{M}, w \models \beta$.*
3. *$\mathcal{M}, w \models \neg\alpha$ iff $\mathcal{M}, w \not\models \alpha$.*
4. *$\mathcal{M}, w \models \Box\alpha$ iff $\|\alpha\|^{\mathcal{M}}_w = N(w)$.*
5. *$\mathcal{M}, w \models \alpha \rightarrow \beta$ iff $\|\alpha\|^{\mathcal{M}}_w \cap \|\neg\beta\|^{\mathcal{M}}_w <_w \|\alpha\|^{\mathcal{M}}_w \cap \|\beta\|^{\mathcal{M}}_w$.*

Thus $\alpha \to \beta$ is true just if the proposition expressed by $\alpha \wedge \beta$ is more normal than that expressed by $\alpha \wedge \neg\beta$. If α is true at every world in a model, then α is *valid* in that model. If α is true at every world in a class of models, then α is said to be valid in that class of models. If α is true at every world in every model, then α is *valid*, written $\models \alpha$.

For a corresponding proof theory, consider the deductive system closed under classical propositional logic (that is, including modus ponens and substitution of provable equivalents as well as all truth functional tautologies as axioms) along with the following rules of inference and axioms:

Nec: From $\vdash \alpha$ infer $\vdash_{\mathbf{C}} \Box\alpha$.

K: $\vdash_{\mathbf{C}} \Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$.

T: $\vdash_{\mathbf{C}} \Box\alpha \supset \alpha$.

CEA: $\vdash_{\mathbf{C}} \Box(\alpha \equiv \alpha') \supset ((\alpha \to \beta) \equiv (\alpha' \to \beta))$.

CECA: $\vdash_{\mathbf{C}} \Box(\alpha \supset (\beta \equiv \beta')) \supset ((\alpha \to \beta) \equiv (\alpha \to \beta'))$.

RR: $\vdash_{\mathbf{C}} \Diamond\alpha \supset (\alpha \to \alpha)$.

NA: $\vdash_{\mathbf{C}} \neg(\bot \to \alpha)$

CEM: $\vdash_{\mathbf{C}} (\alpha \to \beta) \supset \neg(\alpha \to \neg\beta)$

Trans: $\vdash_{\mathbf{C}} \Box\left(\bigwedge_{1\leq i<j\leq 3} \neg(\alpha_i \wedge \alpha_j)\right) \supset ([(\alpha_1 \vee \alpha_2 \to \alpha_2) \wedge (\alpha_2 \vee \alpha_3 \to \alpha_3)] \supset (\alpha_1 \vee \alpha_3 \to \alpha_3))$

We call the smallest logic based on the above axiomatisation **C**. **Nec**, **K**, and **T** characterise $\Box$. **CEA** (Conditional Equivalent Antecedents) gives substitution of necessary equivalents in the antecedent of a conditional. **CECA** (Conditional Equivalent Consequents, given Antecedents) asserts the same thing with respect to consequents, but is somewhat more general, in that the consequents need to be equivalent just in the "context" given by the antecedent. **RR** is restricted reflexivity; here as with other conditionals we disallow the incoherent proposition $\bot$ to be the antecedent of a true conditional. As we will see in the next section, **RR** corresponds to the semantic condition Min $\emptyset_1$. **NA** (Nihil ex Absurdo) asserts that the incoherent proposition never normally implies anything. [Benferhat *et al.*, 1992] expresses this axiom nicely, that "while $\bot$ (classically) entails anything, it should preferentially entail nothing". This axioms corresponds to the semantic condition Min $\emptyset_2$. **CEM** is the principle of the excluded middle for a weak conditional; in the semantics this is reflected by asymmetry of $<$. Similarly **Trans** reflects transitivity of $<$ in the semantics.

We obtain the following basic results:

Theorem 4.1

1. *From* $\vdash \alpha \equiv \alpha'$ *infer* $\vdash_{\mathbf{C}} (\alpha \to \beta) \equiv (\alpha' \to \beta)$
2. *From* $\vdash \beta \equiv \beta'$ *infer* $\vdash_{\mathbf{C}} (\alpha \to \beta) \equiv (\alpha \to \beta')$

3. $\vdash_{\mathbf{C}} (\alpha \rightarrow \beta) \equiv (\alpha \rightarrow (\alpha \wedge \beta))$

4. $\vdash_{\mathbf{C}} (\alpha \rightarrow \beta) \equiv (\alpha \rightarrow (\alpha \supset \beta))$

5. $\vdash_{\mathbf{C}} \Diamond\alpha \supset (\Box(\alpha \supset \beta) \supset (\alpha \rightarrow \beta))$

6. $\vdash_{\mathbf{C}} (\Diamond\alpha \wedge \Box\beta) \supset (\alpha \rightarrow \beta)$

7. $\vdash_{\mathbf{C}} \neg(\alpha \rightarrow \bot)$

8. $\vdash_{\mathbf{C}} \Box\alpha \equiv \neg(\neg\alpha \rightarrow \neg\alpha)$

The first two results express substitution of logical equivalents in the antecedent and consequent of a conditional. These rules have been called **RCEA** and **RCEC** in the conditional logic literature. The next two results effectively express the range of equivalent forms a conditional may take on with respect to the consequent. The following two results, Parts 5 and 6, connect the modalities $\Box$ and $\rightarrow$. The former is analogous to a weakening of the principle **RCE** in the conditional logic literature, while the latter is slightly weaker than **MOD**. The penultimate result asserts that no proposition normally implies the incoherent proposition. The final result shows that we could in fact define necessity in terms of the weak conditional. However we retain $\Box$ as a primitive modality because we also look at a weaker logic $\mathbf{C}^-$ in which this equivalence does not hold.

Of those axioms in the so-called "core" logic (Section 2), **RCM**, **RT**, **ASC**, **CC**, and **CA** (and their nonmonotonic consequence operator counterparts: Right Weakening, Cut, Cautious Monotonity, And, and Or) are not valid in **C**. Nonetheless, despite its (monotonic) inferential weakness, the logic **C** allows a rich set of nonmonotonic inferences, as covered in the next section. However, first we explore properties of the (monotonic) logic.

4.3 Properties of the Logic

Soundness of the logic is shown by a straightforward inductive argument. For the completeness proof, it is of interest to first consider the weakest logic compatible with the semantic framework given in Definition 4.1.[12] Completeness is given with respect to the weakest realistic semantic framework; the corresponding weakened axiomatic system is called $\mathbf{C}^-$. Then the roles of nontriviality, asymmetry and transitivity in the semantics are clearly reflected in the axioms that are added to $\mathbf{C}^-$, yielding the system **C**.

Definition 4.3 *A* weak comparative conditional model *is a comparative conditional model (Definition 4.1),* $\mathcal{M} = \langle W, N, <, P \rangle$ *except that we drop the conditions of nontriviality (Min* $\emptyset_1$*, Min* $\emptyset_2$*), asymmetry, and transitivity of* $<$*.*

(Since $<_w$ is assumed to be neither asymmetric nor transitive at this point, the notation $<_w$ is somewhat misleading, since $<_w$ is not any type of order.) Consider the logic over $\mathcal{L}$ closed under classical propositional logic and **Nec**, **K**, **T**, along with the following axioms:

[12] This is of interest with respect to the semantics of families of conditional logics, and not necessarily with respect to default reasoning. The reader not interested in semantics of conditional logics can safely skip this section, taking it from Corollary 4.1.

CEA: $\vdash_{\mathbf{C}} \Box(\alpha \equiv \alpha') \supset ((\alpha \to \beta) \equiv (\alpha' \to \beta))$.

CECA: $\vdash_{\mathbf{C}} \Box(\alpha \supset (\beta \equiv \beta')) \supset ((\alpha \to \beta) \equiv (\alpha \to \beta'))$.

The smallest logic based on the above axiomatisation is called $\mathbf{C}^-$. It is easily shown that $\mathbf{C}^-$ is sound with respect to weak comparative conditional models.

Completeness is demonstrated by constructing a *canonical model* [Chellas, 1980, Hughes and Cresswell, for the logic $\mathbf{C}^-$, that is, a model such that every non-theorem of $\mathbf{C}^-$ is false at some world in the model. While the overall approach is as given in the aforecited references, the particular construction for this class of logics is novel. A canonical model will be a structure $\mathcal{M} = \langle W, N, <, P \rangle$ where W is the set of maximal consistent sets of sentences. That is, for $w \in W$ in $\mathcal{M}$, we have that $w \subset \mathcal{L}$ and for every $\alpha \in \mathcal{L}$ we have $\alpha \in w$ iff $\neg\alpha \notin w$; as well if $\alpha \in w$ and $\alpha \vdash_{\mathbf{C}^-} \beta$ then $\beta \in w$. See the aforecited references for details on how these maximal consistent sets of sentences can be constructed.[13] In a canonical model, and analogously to $\| \cdot \|^{\mathcal{M}}$, $\| \cdot \|^{\mathcal{M}}_w$, define

$$|\alpha| = \{w \in W \mid \alpha \in w\}.$$

Similarly, define

$$|\alpha|_w = \{w \in W \mid \alpha \in w\} \cap N(w).$$

That is, $|\alpha|$ is the set of maximum consistent sets of sentences containing α, and $|\alpha|_w$ is the set of maximum consistent sets of sentences containing α according to world w.

Definition 4.4 *Define the* canonical model *for* $\mathbf{C}^-$ *by:*
$\mathcal{M} = \langle W, N, <, P \rangle$ *where:*

1. *W is the set of maximal $\mathbf{C}^-$-consistent sets of sentences of $\mathcal{L}$.*

2. $w' \in N(w)$ *iff* $\{\alpha \mid \Box\alpha \in w\} \subseteq w'$.

3. *For every $w \in W$ and $U, V \subseteq N(w)$,*
   $U <_w V$ *iff:*
   $U \cap V = \emptyset$ *and*
   for every $\alpha, \beta \in \mathcal{L}$, if $|\alpha \wedge \neg\beta|_w = U$ and $|\alpha \wedge \beta|_w = V$ then $\alpha \to \beta \in w$.

4. $P(p_i) = \{w \mid p_i \in w\} = |p_i|$.

The canonical model is clearly a weak comparative conditional model. Moreover, in the canonical model, the sentences contained in a world are precisely those that are true at a world. That is, we have the following result:

[13] Essentially, for some enumeration of all sentences in the language, a sentence is iteratively added to the set of formulas if and only if it is consistent with this set.

Theorem 4.2 *Let $\mathcal{M}$ be the canonical model for* $\mathbf{C}^-$. *Then for every $\alpha \in \mathcal{L}$ and $w \in W$ we have:*

$$\mathcal{M}, w \models \alpha \quad \textit{iff} \quad \alpha \in w.$$

That is, for every $\alpha \in \mathcal{L}$ we have that $\|\alpha\|^{\mathcal{M}} = |\alpha|$.

We obtain the completeness result:

Theorem 4.3 *If α is valid in the class of weak comparative conditional models then $\vdash_{\mathbf{C}^-} \alpha$.*

Given this result, we can next consider the addition of properties to the logic that will strengthen $\mathbf{C}^-$ to our "official" base logic $\mathbf{C}$. We obtain the following correspondence between these semantic conditions and their corresponding axioms:

Theorem 4.4
$\mathbf{C}^- + \mathbf{RR}$ *(***NA***,* **CEM***,* **Trans***) is complete with respect to the class of weak comparative conditional models in which $<_w$ satisfies min $\emptyset_1$ (min $\emptyset_2$, asymmetry, transitivity).*

Corollary 4.1 $\mathbf{C}$ *is complete with respect to the class of comparative conditional models.*

4.4 Extensions to the Logic

In the logic $\mathbf{C}$, most properties of the relation $<_w$ stem from its being a strict partial order (viz. asymmetric and transitive), along with the fact that $\emptyset$ is the minimum in $<_w$. In this subsection we look at strengthening $<_w$ by considering properties that seem reasonable for a notion of normality. Consider the following:

1. Continues Down:

   If $X <_w Y$ then $X \setminus Z <_w Y$.

2. Continues Up:

   If $X <_w Y$ then $X <_w Y \cup Z$ provided $X \cap Z = \emptyset$.

3. Restricted Continues Down/Up:

   If $X <_w Y$ then $X \setminus Z <_w Y \cup (X \cap Z)$.

4. Continues Down/Up:

   If $X <_w Y$ then $X \setminus Z <_w Y \cup Z$.

5. Weak Disjoint Union:

   If $X_1 <_w Y_1$, $X_2 <_w Y_2$, $Y_1 \cap Y_2 = \emptyset$ and $(X_1 \cup X_2) \cap (Y_1 \cup Y_2) = \emptyset$ then $X_1 \cup X_2 <_w Y_1 \cup Y_2$.

For Continues Down, if X is less normal than Y, then a stronger proposition than X (viz. $X \setminus Z$) is also less normal than Y. That is, removing worlds from a proposition serves to strengthen it, thereby making it "less normal". Continues Up is a dual: if X is less normal than Y, then a weaker proposition than Y (viz. $Y \cup Z$) is also more normal than X. That is, adding worlds to a proposition serves to make it "more normal". For Restricted Continues Down/Up, if $X < Y$, then "part" of X can be shifted to Y; hence X is strengthened and Y weakened by the same set of worlds. Continues Down/Up combines Continues Down and Continues Up, as well as generalises Restricted Continues Down/Up. Weak Disjoint Union allows for the combination of two independent instances of $<_w$, provided that the result yields a relation in which the arguments are disjoint.

Interestingly, Continues Down, Continues Up, and their combination have appeared in the belief revision literature. Our relation $<$ is what [Alchourrón and Makinson, 1985] call a *(transitive) hierarchy*;[14] while $<$ with Continues Down/Up is a *regular* hierarchy. Their interpretation of $<$ echoes ours for $X < Y$, that "X is less secure or reliable or plausible ... than Y" [Gärdenfors and Rott, 1995, p. 75].

Consider next the following formulas:

WSA: $\Box(\beta \supset \gamma) \supset ((\alpha \to \beta) \supset (\alpha \wedge \gamma \to \beta))$

CW: $(\alpha \wedge \beta \to \gamma) \supset (\alpha \to (\beta \supset \gamma))$

CM: $\Box(\beta \supset \gamma) \supset ((\alpha \to \beta) \supset (\alpha \to \gamma))$

WCA: $(\alpha \to \beta) \supset ((\alpha \vee \gamma) \to (\beta \vee \gamma))$.

D: $((\alpha \wedge \beta \to \gamma) \wedge (\alpha \wedge \neg\beta \to \gamma)) \supset (\alpha \to \gamma)$.

WSA (Weak Strengthening of the Antecedent) is a stronger version of the easy result in **C**: $\Box\gamma \supset ((\alpha \to \beta) \supset (\alpha \wedge \gamma \to \beta))$. It appears to have not been discussed previously in the literature. **CW** (Conditional Weakening, or "Conditionalisation" in nonmonotonic consequence relations) gives a conditional version of one half of the deduction theorem. The formula **CM** allows weakening of the consequent of a conditional. In conditional approaches it more often appears in a weaker form, as a rule of inference, where it is called **RCM** or (as a nonmonotonic inference relation) Right Weakening. As well, this formula can be seen as allowing a form of modus ponens in the consequent of a conditional. Combining **WSA** and **CW** yields the formula **WCA** (Weak **CA**), which allows incorporation of information uniformly in the antecedent and consequent of a conditional. **D** supplies a certain "reasoning by cases" for the conditional.

We obtain the following correspondence between semantic conditions and the axiomatisation:

Theorem 4.5
C + **WSA** *(***CW***,* **CM***,* **WCA***,* **D***) is complete with respect to the class of comparative conditional models in which $<_w$ satisfies Continues Down (Continues Up, Restricted Continues Down/Up, Continues Down/Up, Weak Disjoint Union).*

[14] In [Alchourrón and Makinson, 1985], $<$ is a binary relation on deductively-closed sets of sentences which, in the finite case, serves as well as sets of worlds for expressing propositions.

We obtain the following relations:

Theorem 4.6

1. **CW** *is a theorem of* **C** + **WCA**.
2. **CM** *is a theorem of* **C** + **WCA**.
3. **WCA** *is a theorem of* **C** + **WSA** + **CW**.

For ease of discussion, let $\mathbf{C}^+$ be the logic:

$$\mathbf{C} + \mathbf{WSA} + \mathbf{WCA} + \mathbf{D} \quad = \quad \mathbf{C} + \mathbf{WSA} + \mathbf{CW} + \mathbf{D}.$$

$\mathbf{C}^+$ might seem plausible as an "official" logic for a rule-based interpretation of defaults and statements of normality. However, as we discuss in the next section, even this logic has questionable interactions with the most straightforward nonmonotonic extension to the logic.

We obtain the following easy results:

Theorem 4.7

1. $\vdash_{\mathbf{C}^+} (\alpha \rightarrow (\beta \wedge \gamma)) \supset (\alpha \rightarrow \beta)$
2. *From* $\vdash \beta \supset \gamma$ *infer* $\vdash_{\mathbf{C}^+} (\alpha \rightarrow \beta) \supset (\alpha \rightarrow \gamma)$

The first (called **CM**) is a converse to **CC**/And. The second, (**RCM** or Right Weakening) is a common condition for conditional approaches; it is a weaker version of our **CM**.

The following formulas are not derivable in $\mathbf{C}^+$:

1. $(\alpha \rightarrow (\beta \supset \gamma)) \supset ((\alpha \rightarrow \beta) \supset (\alpha \rightarrow \gamma))$
2. $((\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)) \supset (\alpha \vee \beta \rightarrow \gamma)$

The first of these is called **CK** in the conditional logic literature, **MPC** in that of nonmonotonic inference relations. It is the natural strengthening of **CM** to modus ponens in the consequence of a conditional. The second is called **CA** or **OR**. It is a natural strengthening of **D**. As mentioned, there are authors who see this formula as problematic. As well, various other characterizing formulas in preferential systems are not theorems here, specifically **CC**/And, **RT**/Cut, and **ASC**/Cautious Monotony. The lack of **CC**/And means that we do not have what [Gärdenfors and Makinson, 1994] call an *inference system*.

We next very briefly consider further extensions to the logic.

4.5 Further Extensions to the Logic

We could go on and add other conditions in the semantics. Space considerations dictate against a lengthy discussion, but two conditions are worth noting here:

Disjoint Union: If $(X \cup Y) \cap Z = \emptyset$ then:

$$X <_w Y \quad \text{iff} \quad X \cup Z <_w Y \cup Z.$$

Connectivity: If $X \neq Y$ then either $Y <_w X$ or $X <_w Y$.

Disjoint union has appeared frequently in the literature, for example [Savage, 1972, Fine, 1973, Dubois *et al.*, 1994]. The addition of disjoint union requires that the notion of a model be altered slightly (from a relation $<$ to $\leq$); the resultant semantic framework appears to correspond to the basic definition of a *plausibility structure* [Friedman and Halpern, 2001]. We note however that the addition of disjoint union does not alter the set of valid sentences obtained with **UD** and **D**. The addition of connectivity would make $<_w$ a *qualitative probability* in the terminology of [Savage, 1972].

5 Considerations on Nonmonotonic Reasoning

We claimed at the outset that the logic **C** and its strengthenings would allow a simple approach to nonmonotonic inference. In this section, we give a definition for nonmonotonic inference and discuss specific examples of its application. Since the primary goal of the paper is to explore (monotonic) conditional logics for weak conditionals, we defer a full exploration of nonmonotonic inference to future research.

For nonmonotonic inference, the central idea is that, given a partition $\{X, Y\}$ of a context $X \cup Y \subseteq N(w)$, the relation $X <_w Y$ asserts that Y is more normal (unexceptional, etc.) than X. To obtain nonmonotonic inference, we simply assume that this relation holds in any subcontext, that is $X \cap Z <_w Y \cap Z$, wherever "reasonable".

Informally this notion of "reasonable" is straightforward to specify:

> If we have $X <_w Y$ then assert $X \cap Z <_w Y \cap Z$ just when, for every X', Y' where
>
> $$X \cap Z \subseteq X' \subseteq X \quad \text{and} \quad Y \cap Z \subseteq Y' \subseteq Y$$
>
> we don't have $Y' <_w X'$.

That is, from $X <_w Y$ we assert $X \cap Z <_w Y \cap Z$, unless there is a reason (given by $Y' <_w X'$ above) not to do so. As well we have the constraint that $Y \cap Z \neq \emptyset$. More formally, we have the following:

Definition 5.1 *Let $\mathcal{M} = \langle W, N, <, P\rangle$ be a comparative conditional model in* **C**. *$\mathcal{M}^* = \langle W, N, <^*, P\rangle$, is an* augmented comparative conditional model *of $\mathcal{M}$ if:*

1. *$\mathcal{M}^*$ is a comparative conditional model, and*
2. *$X <^*_w Y$ satisfies the condition:*

   $X <^*_w Y$ *if: $Y \neq \emptyset$ and there are $X' \supseteq X$, $Y' \supseteq Y$ such that*

   (a) $X' <_w Y'$ *and*

   (b) for every X'', Y'' where
   $X \subseteq X''$, $Y \subseteq Y''$ *and* $Y'' <^*_w X''$
   we have:
   $X' \subseteq X''$, $Y' \subseteq Y''$.

$X \not<^*_w Y$ *otherwise.*

It can be noted that in Definition 5.1, if we have Continues Down, then we don't need to bother with X' and X'', since monotonically we get that if $X' < Y'$ then $X < Y'$ for any $X \subseteq X'$. As well, Definition 5.1 in combination with Continues Up may perhaps give too many relations: From $X < Y$ we have $X < Y \cup Z$ by Continues Up and then, all other things being equal, we nonmonotonically conclude $X < (Y \cup Z) \cap Z$ or $X < Z$. Thus from $X < Y$ and arbitrary (but in accordance with the conditions of the definition) Z, we obtain that $X < Z$. Assuming that this is a problem, there are two ways in which this difficulty can be resolved. First, one can decide that Continues Up (and so **CW**) is too strong for our logic of conditionals. Or, second, the Definition 5.1 can be restricted to apply to certain "minimal" sets of worlds.

It seems that for "reasonable" theories, there is just a single augmented comparative conditional model. However, it is possible to construct theories in which this is not the case; consider for example the theory in which we have

$$X \subseteq X', \quad Y \subseteq Y', \quad Z \subseteq Z',$$

and

$$X < Y', \quad Y < Z', \quad Z < X'.$$

There are three augmented comparative conditional models, in which $X < Y$ and $Y < Z$ holds, or $Y < Z$ and $Z < X$ holds, or $Z < X$ and $X < Y$ holds, respectively.

We define $\models^*$ as validity in the class of augmented comparative conditional models; that is $\models^* \alpha$ iff α is true at every world in every augmented comparative conditional model. Nonmonotonic inference is defined as follows:

Definition 5.2 *Let* $\Gamma \subseteq \{\alpha \rightarrow \beta \mid \alpha, \beta \in \mathcal{L}_{PC}\} \cup \{\Box\alpha \mid \alpha \in \mathcal{L}_{PC}\}$.
Define: $\alpha \mathrel{|\!\sim}_{\Gamma} \beta$ *iff* $\models^* \Gamma \supset (\alpha \rightarrow \beta)$.

We say that β is a nonmonotonic inference from α with respect to Γ, or just β is a nonmonotonic inference from α (written $\alpha \mathrel{|\!\sim} \beta$) if the set Γ is clear from the context of discussion.

We illustrate nonmonotonic inference first by a familiar example:

$$b \rightarrow f, \tag{4}$$
$$b \rightarrow w, \tag{5}$$
$$\Box(p \supset b), \tag{6}$$
$$p \rightarrow \neg f. \tag{7}$$

Thus a bird flies and has wings, and a penguin is (necessarily) a bird that does not fly. We obtain the following:

$$\begin{array}{ll} b \wedge w \mathrel{|\!\sim} f, & p \wedge w \mathrel{|\!\sim} \neg f, \\ b \wedge \neg w \mathrel{|\!\sim} f, & p \wedge b \mathrel{|\!\sim} \neg f, \\ b \wedge \neg p \mathrel{|\!\sim} f, & p \wedge b \wedge w \mathrel{|\!\sim} \neg f. \end{array}$$

To see how this type of inference obtains, consider the first of these default inferences, $b \wedge w \mathrel{|\!\sim} f$. In every model of the set of sentences (4)-(7) we have $\|b \wedge \neg f\|_w^{\mathcal{M}} <_w \|b \wedge f\|_w^{\mathcal{M}}$. Clearly $\|b \wedge \neg f \wedge w\|_w^{\mathcal{M}} \subseteq \|b \wedge \neg f\|_w^{\mathcal{M}}$ and $\|b \wedge f \wedge w\|_w^{\mathcal{M}} \subseteq \|b \wedge f\|_w^{\mathcal{M}}$. In every augmented model we consequently obtain $\|b \wedge \neg f \wedge w\|_w^{\mathcal{M}} <_w \|b \wedge f \wedge w\|_w^{\mathcal{M}}$, or rearranging terms $\|(b \wedge w) \wedge \neg f\|_w^{\mathcal{M}} <_w \|(b \wedge w) \wedge f\|_w^{\mathcal{M}}$, whence $b \wedge w \mathrel{|\!\sim} f$.

We also obtain $b \wedge x \wedge y \wedge z \mathrel{|\!\sim} w$ for $x \in \{\top, g, \neg g\}, y \in \{\top, p, \neg p\}, z \in \{\top, f, \neg f\}$. Thus a green ($g$) bird has wings, as does a non-green flying penguin. As well we obtain $p \mathrel{|\!\sim} w$, and so a penguin inherits the property of having wings by virtue of necessarily being a bird. This can be seen as follows: since $b \rightarrow w$ we have that $\|b \wedge \neg w\|_w^{\mathcal{M}} <_w \|b \wedge w\|_w^{\mathcal{M}}$. Since we also have $\Box(p \supset b)$, this means that $\|p \wedge \neg w\|_w^{\mathcal{M}} \subseteq \|b \wedge \neg w\|_w^{\mathcal{M}}$, and $\|p \wedge w\|_w^{\mathcal{M}} \subseteq \|b \wedge w\|_w^{\mathcal{M}}$. It follows that $\|p \wedge \neg w\|_w^{\mathcal{M}} <_w \|p \wedge w\|_w^{\mathcal{M}}$ in every augmented model. Note that if we replaced (6) by $p \rightarrow b$, we would no longer obtain $p \mathrel{|\!\sim} w$; this is because the above line of argument would fail since we would not have $\|p\|_w^{\mathcal{M}} \subseteq \|b\|_w^{\mathcal{M}}$ and so could not infer $\|p \wedge \neg w\|_w^{\mathcal{M}} \subseteq \|b \wedge \neg w\|_w^{\mathcal{M}}$ nor $\|p \wedge w\|_w^{\mathcal{M}} \subseteq \|b \wedge w\|_w^{\mathcal{M}}$. However, if we were to replace (6) by $p \rightarrow b$, we would still obtain the weaker $b \wedge p \mathrel{|\!\sim} w$. This appears to be reasonable, given that a normality conditional $\alpha \rightarrow \beta$ does not imply a strict specificity relation between α and β whereas $\Box(\alpha \supset \beta)$ does.

The next example further illustrates reasoning in the presence of exceptions.

$$q \rightarrow p, \quad r \rightarrow \neg p, \quad q \rightarrow g \tag{8}$$

So a Quaker is a pacifist while a Republican is not, and a Quaker is generous. We obtain $q \wedge \neg r \mathrel{|\!\sim} p$ and $q \wedge r \mathrel{|\!\sim} g$. Thus in the last case, while a Quaker that is a Republican is, informally, an exceptional Quaker, it is nonetheless still generous by default.

The next set of examples may not add many new examples of nonmonotonic inferences per se; however it does illustrate the application of nonmonotonic principles involving normality defaults where there is no suggestion of a probability-based interpretation of the default. Consider again Example 2.3:

$$\begin{array}{ccc} \top & \rightarrow & \neg f \\ a & \rightarrow & f \\ \top & \rightarrow & n \\ s & \rightarrow & \neg n \end{array}$$

We get:

$$\begin{array}{lll} n \mathrel{|\!\sim} \neg f, & \neg n \mathrel{|\!\sim} \neg f & \\ n \wedge a \mathrel{|\!\sim} f, & \neg n \wedge a \mathrel{|\!\sim} \neg f & \\ \neg f \mathrel{|\!\sim} n, & f \mathrel{|\!\sim} n & \\ a \mathrel{|\!\sim} n, & f \wedge a \mathrel{|\!\sim} n, & \neg f \wedge a \mathrel{|\!\sim} n \end{array}$$

We do not obtain the undesirable inference $\neg s \mathrel{|\!\sim} \neg a$ found in Example 2.3. Further, for Example 2.4 we do not obtain $f \mathrel{|\!\sim} \neg u$. Last, we note that while we obtain full incorporation of irrelevant properties, we do not obtain full default transitivity. Thus

$$a \rightarrow b, \quad b \rightarrow c \tag{9}$$

does not yield $a \mathrel{|\!\sim} c$ (nor, incidentally, do we obtain $\neg b \mathrel{|\!\sim} \neg a$). However we do get $a \wedge b \mathrel{|\!\sim} c$. If we replaced $a \rightarrow b$ with $\Box(a \supset b)$ we would get $a \mathrel{|\!\sim} c$. If we replaced $b \rightarrow c$ with $\Box(b \supset c)$ we would again get $a \mathrel{|\!\sim} c$, in **C** (in fact we could derive $a \rightarrow c$ in those logics containing **CM**, as given in Theorem 4.4).

6 Conclusion

We have argued that there are two interpretations of a default conditional: first as a weak (typically material) implication, and second as something akin to a rule of inference. The former interpretation is explicit in, for example, circumscriptive abnormality theories, and implicit in approaches such as conditional entailment and rational closure. It is clear that there are many, and varied, applications in which the first interpretation is appropriate. However we have also noted that there are various reasons to suppose that this is not the only such interpretation: First, other authors have argued against principles of the "core" logic underlying this first interpretation (specifically, against **CC**/And and **CA**/Or). Second, there are examples (such as Example 2.6) in the underlying (monotonic) logic that give results that are too strong. Third, there are various examples of inferences in approaches such as rational closure or in conditional entailment that are either too weak or too strong. Last, there are emerging areas (such as causal reasoning) in which a "weak material implication" interpretation is not appropriate. While this distinction has been recognized previously, what is new here is the development of a family of logics, with a novel semantic theory and proof theory along with a specification of nonmonotonic inference, for the "rule-based" interpretation.

All of the logics presented here are quite weak, at least compared to the so-called "conservative core" or, equivalently, the system **P** of [Kraus *et al.*, 1990]. We argue however that such lack of inferential capability is characteristic of a "rule-based" interpretation of a conditional. Moreover it proves to be the case that nonmonotonic reasoning is defined very easily in these logics, and allows a rich set of inferences concerning the incorporation of irrelevant properties and of property inheritance.

An open question concerns how informal, commonsense defaults should be classified – whether as a defeasible classical (material) conditional or as a rule. As discussed, the great majority of past work has favoured the "defeasible classical conditional" interpretation. Such previous work includes non-default conditionals, such as Lewis' system-of-spheres semantics for counterfactual assertions [Lewis, 1973]. However, a case can be made that such non-default examples, formerly interpreted as belonging to the first category, may be better interpreted as belonging to the "rule" category. Consider the following example of counterfactual assertions in which the following statements, concerning a past party, are given [Lewis, 1973]:

> "If John had gone it would have been a good party" and
>
> "If John and Mary had gone it would have not been a good party".

From these assertions, we deduce in Lewis' "official" counterfactual logic *VC* that "if John had gone, Mary would not have gone".[15] This, to most readers, is a strange result: John's going and Mary's going are (presumably) independent events. Arguably this result ought not to obtain, and so perhaps counterfactuals, as previously modelled by Lewis' sphere semantics, may also be better interpreted via the "rule" interpretation.

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A Proofs

Proof 4.1

1. This is a consequence of **Nec** and **CEA**: If $\vdash \alpha \equiv \alpha'$ then $\vdash_{\mathbf{C}} \Box(\alpha \equiv \alpha')$ by **Nec**, from which we obtain via **CEA** and modus ponens.

2. This is a consequence of **Nec** and **CECA**: If $\vdash \beta \equiv \beta'$ then $\vdash \alpha \supset (\beta \equiv \beta')$ from which we get $\vdash_{\mathbf{C}} \Box(\alpha \supset (\beta \equiv \beta'))$ by **Nec**, from which we obtain via **CECA** and modus ponens.

3. We have $\vdash \alpha \supset (\beta \equiv (\alpha \wedge \beta))$ in classical propositional logic. **Nec** and **CECA** yields $\vdash_{\mathbf{C}} (\alpha \to \beta) \equiv (\alpha \to (\alpha \wedge \beta))$.

4. We have $\vdash \alpha \supset (\beta \equiv (\alpha \supset \beta))$ in classical propositional logic. **Nec** and **CECA** yields $\vdash_{\mathbf{C}} (\alpha \to \beta) \equiv (\alpha \to (\alpha \supset \beta))$.

5. An instance of **CECA** is $\vdash_{\mathbf{C}} \Box(\alpha \supset (\alpha \equiv \beta)) \supset ((\alpha \to \alpha) \equiv (\alpha \to \beta))$. From **Nec** and **K** we have $\vdash_{\mathbf{C}} \Box(\alpha \supset (\alpha \equiv \beta)) \equiv \Box(\alpha \supset \beta)$. Via substitution of equivalents, from the initial formula we get $\vdash_{\mathbf{C}} \Box(\alpha \supset \beta) \supset ((\alpha \to \alpha) \equiv (\alpha \to \beta))$. A consequence of this last formula, using propositional logic, is $\vdash_{\mathbf{C}} (\alpha \to \alpha) \supset (\Box(\alpha \supset \beta) \supset (\alpha \to \beta))$. **RR** is $\vdash_{\mathbf{C}} \Diamond\alpha \supset \alpha \to \alpha$. Transitivity of $\supset$ applied to the preceding two formulas yields our result.

6. In the underlying modal logic for $\Box$ we have $\vdash_{\mathbf{C}} \Box(\beta \supset (\alpha \supset \beta))$, whence $\vdash_{\mathbf{C}} \Box\beta \supset \Box(\alpha \supset \beta)$. Part 5 of this theorem is $\vdash_{\mathbf{C}} \Diamond\alpha \supset (\Box(\alpha \supset \beta) \supset (\alpha \to \beta))$; from this and $\vdash_{\mathbf{C}} \Box\beta \supset \Box(\alpha \supset \beta)$, using propositional logic we obtain $\vdash_{\mathbf{C}} (\Diamond\alpha \wedge \Box\beta) \supset (\alpha \to \beta)$.

7. Consider first where we have $\diamond\alpha$. An instance of Theorem 4.1, Part 6 is $\vdash_{\mathbf{C}} \diamond\alpha \supset (\alpha \to \top)$ and an instance of **CEM** is $\vdash_{\mathbf{C}} (\alpha \to \top) \supset \neg(\alpha \to \bot)$. From propositional logic we conclude $\vdash_{\mathbf{C}} \diamond\alpha \supset \neg(\alpha \to \bot)$.

   Second assume that $\neg\diamond\alpha$, or $\Box\neg\alpha$. We get $\vdash_{\mathbf{C}} \Box(\alpha \equiv \bot)$ in the underlying modal logic for $\Box$. An instance of **CEA** is $\vdash_{\mathbf{C}} \Box(\alpha \equiv \bot) \supset ((\alpha \to \bot) \equiv (\bot \to \bot))$. An instance

[15] That is, we have $\{J \to G,\ J \wedge M \to \neg G\} \vdash_{VC} J \to \neg M$.

of **NA** is $\vdash_{\mathbf{C}} \neg(\bot \to \bot)$. From these sentences we derive $\vdash_{\mathbf{C}} \neg \diamond \alpha \supset \neg(\alpha \to \bot)$ using propositional logic.

Combining the above two parts we obtain that $\vdash_{\mathbf{C}} \neg(\alpha \to \bot)$.

8. To show $\vdash_{\mathbf{C}} \Box\alpha \supset \neg(\neg\alpha \to \neg\alpha)$, begin with the instance of **CEA**: $\vdash_{\mathbf{C}} \Box(\neg\alpha \equiv \bot) \supset (\neg\alpha \to \neg\alpha \equiv \bot \to \neg\alpha)$, whence $\vdash_{\mathbf{C}} \Box(\neg\alpha \equiv \bot) \supset (\neg(\bot \to \neg\alpha) \supset \neg(\neg\alpha \to \neg\alpha))$. From the equivalence $\vdash_{\mathbf{C}} \Box\alpha \equiv \Box(\neg\alpha \equiv \bot)$ we obtain $\vdash_{\mathbf{C}} \Box\alpha \supset (\neg(\bot \to \neg\alpha) \supset \neg(\neg\alpha \to \neg\alpha))$. Given axiom **NA**, this is equivalent to $\vdash_{\mathbf{C}} \Box\alpha \supset \neg(\neg\alpha \to \neg\alpha)$.

   The converse, $\vdash_{\mathbf{C}} \neg(\neg\alpha \to \neg\alpha) \supset \Box\alpha$ is an instance of the contrapositive of **RR**.

■

In the following proofs, we make extensive use of the fact that in the canonical model, for a set of worlds X, there is a formula γ such that $|\gamma| = X$. This is a simple consequence of the fact that the set of atomic sentences $\mathbf{P}$ is finite, and so W also is finite. For example, for $X \subseteq W$ where, $X \neq \emptyset$, one could choose $\gamma = \bigvee_{i=1}^{|X|} \gamma_i$ where, for $w_i \in X$, γ_i is a conjunction of the literals true in w_i. For $X = \emptyset$ one could choose $\bot$ or the negation of any propositional tautology. In particular, this means that for $w \in W$ in the canonical model, there is a formula ψ such that $|\psi|_w = N(w)$.

Proof 4.2 The proof is by induction on the construction of a formula of $\mathcal{L}$.

1. For the base case, consider a formula $p_i \in P$. We have:

   $\mathcal{M}, w \models p_i$ iff $w \in P(p_i)$ iff $p_i \in w$.

2. For a formula of the form $\alpha \wedge \beta$:

   $\mathcal{M}, w \models \alpha \wedge \beta$ iff $\mathcal{M}, w \models \alpha$ and $\mathcal{M}, w \models \beta$ iff $\alpha \in w$ and $\beta \in w$ (by the induction hypothesis) iff $\alpha \wedge \beta \in w$ (by the definition of a maximal consistent sets of sentences).

3. For a formula of the form $\neg\alpha$:

   $\mathcal{M}, w \models \neg\alpha$ iff $\mathcal{M}, w \not\models \alpha$ iff $\alpha \notin w$ (by the induction hypothesis) iff $\neg\alpha \in w$ (by the definition of a maximal consistent sets of sentences).

4. For a formula of the form $\Box\alpha$:

   (a) Suppose that $\Box\alpha \in w$.
   Then for every $w' \in N(w)$ we have that $\alpha \in w'$ by the definition of $N(w)$.
   By the induction hypothesis we obtain that $\mathcal{M}, w' \models \alpha$.
   This means that for every $w' \in N(w)$ we have that $\mathcal{M}, w' \models \alpha$, whence $\mathcal{M}, w \models \Box\alpha$.

   (b) Suppose that $\Box\alpha \notin w$.
   Hence, since w is a maximal consistent set, we have that $\neg\Box\alpha \in w$.
   By Lemma 2.3 of [Hughes and Cresswell, 1984] we have that $\{\beta \mid \Box\beta \in w\} \cup \{\neg\alpha\}$ is consistent; hence there is $w' \in W$ such that $\{\beta \mid \Box\beta \in w\} \cup \{\neg\alpha\} \subseteq w'$.

Hence $w' \in N(w)$ by the definition of $N(w)$ in the canonical model. However $\alpha \notin w'$.

Therefore $\mathcal{M}, w' \not\models \alpha$ by the induction hypothesis, and so $\mathcal{M}, w \not\models \Box\alpha$ by the truth conditions of $\Box$.

At this point we interpolate a lemma, whose proof follows that of the preceding step, and that will be used for the next part of the theorem.

Lemma A.1 *Let $\mathcal{M}$ be the canonical model and for $w \in W$ let ψ be a formula such that $|\psi| = N(w)$. If $\vdash \psi \supset \delta$ then $\Box\delta \in w$.*

Proof A.1 To begin we show that if $|\psi| = N(w)$ then $\Box\psi \in w$, by proving the contrapositive. To this end, assume that $\Box\psi \notin w$. Since w is a maximal consistent set, we have that $\neg\Box\psi \in w$. By Lemma 2.3 of [Hughes and Cresswell, 1984] we have that $\{\beta \mid \Box\beta \in w\} \cup \{\neg\psi\}$ is consistent. Thus there is $w' \in W$ such that $\{\beta \mid \Box\beta \in w\} \cup \{\neg\psi\} \subseteq w'$. Hence $w' \in N(w)$ by the definition of $N(w)$ in the canonical model. However $\psi \notin w'$, and therefore $w' \notin |\psi|$. Consequently $|\psi| \neq N(w)$.

Assume that $|\psi| = N(w)$. From the preceding we have that $\Box\psi \in w$.

Since $\vdash \psi \supset \delta$, by **Nec** we have that $\vdash_{\mathbf{C}^-} \Box(\psi \supset \delta)$ and so $\Box(\psi \supset \delta) \in w$. Since w is closed under deductive consequence, we have that $\Box\delta \in w$. ■

5. Last, consider $\alpha \to \beta$:

   (a) Suppose that $\alpha \to \beta \in w$. We claim that $|\alpha \wedge \neg\beta|_w <_w |\alpha \wedge \beta|_w$.

   From Definition 4.4 we will have $|\alpha \wedge \neg\beta|_w <_w |\alpha \wedge \beta|_w$ just if we can show, for every $\gamma, \delta \in \mathcal{L}_{PC}$ that if $|\gamma \wedge \neg\delta|_w = |\alpha \wedge \neg\beta|_w$ and $|\gamma \wedge \delta|_w = |\alpha \wedge \beta|_w$ then $\gamma \to \delta \in w$.

   Consequently assume that for $\gamma, \delta \in \mathcal{L}_{PC}$ we have

$$|\gamma \wedge \neg\delta|_w = |\alpha \wedge \neg\beta|_w, \qquad |\gamma \wedge \delta|_w = |\alpha \wedge \beta|_w. \tag{10}$$

   Let ψ be a formula such that $|\psi| = N(w)$.

   From (10) we have that $\vdash \psi \supset ((\alpha \wedge \beta) \supset (\gamma \wedge \delta))$ and so $\vdash \psi \supset ((\alpha \wedge \beta) \supset \delta)$.[16]

   Analogously, we have that $\vdash \psi \supset ((\alpha \wedge \neg\beta) \supset (\gamma \wedge \neg\delta))$ and so $\vdash \psi \supset ((\alpha \wedge \neg\beta) \supset \neg\delta)$.

   From the preceding we obtain that $\vdash \psi \supset (\alpha \supset (\beta \equiv \delta))$.

   Lemma A.1 then implies that $\Box(\alpha \supset (\beta \equiv \delta)) \in w$.

   By assumption we have $\alpha \to \beta \in w$. Since also $\Box(\alpha \supset (\beta \equiv \delta)) \in w$, and since w is closed under logical consequence in $\mathbf{C}^-$, an application of **CECA** gives $\alpha \to \delta \in w$.

[16] For proofs of results involving maximum consistent sets of formulas, see for example [Chellas, 1980] or [Hughes and Cresswell, 1984].

Next, we have that $|\gamma|_w = |\gamma \wedge \neg\delta|_w \cup |\gamma \wedge \delta|_w = |\alpha \wedge \neg\beta|_w \cup |\alpha \wedge \beta|_w = |\alpha|_w$.

Consequently, since $|\gamma|_w = |\alpha|_w$ and ψ is a formula such that $|\psi| = N(w)$, we have $\vdash \psi \supset (\alpha \equiv \gamma)$.

Applying Lemma A.1, this implies that $\Box(\alpha \equiv \gamma) \in w$. Since we have also shown that $\alpha \to \delta \in w$, an application of **CEA** yields $\gamma \to \delta \in w$.

This then is sufficient to prove our initial claim, that $|\alpha \wedge \neg\beta|_w <_w |\alpha \wedge \beta|_w$.

By the induction hypothesis we have that $|\alpha|_w = \|\alpha\|^{\mathcal{M}}_w$, $|\neg\alpha|_w = \|\neg\alpha\|^{\mathcal{M}}_w$, $|\beta|_w = \|\beta\|^{\mathcal{M}}_w$, and so it follows that $|\alpha \wedge \beta|_w = \|\alpha \wedge \beta\|^{\mathcal{M}}_w$, and $|\alpha \wedge \neg\beta|_w = \|\alpha \wedge \neg\beta\|^{\mathcal{M}}_w$. Consequently $\|\alpha \wedge \neg\beta\|^{\mathcal{M}}_w <_w \|\alpha \wedge \beta\|^{\mathcal{M}}_w$, and thus $\mathcal{M}, w \models \alpha \to \beta$ by the truth condition for $\to$.

(b) Conversely, suppose that $\alpha \to \beta \notin w$.

If follows from Definition 4.4 that $|\alpha \wedge \neg\beta|_w \not<_w |\alpha \wedge \beta|_w$. By the induction hypothesis, in a manner analogous to the preceding, $\|\alpha \wedge \neg\beta\|^{\mathcal{M}}_w \not<_w \|\alpha \wedge \beta\|^{\mathcal{M}}_w$, and thus $\mathcal{M}, w \not\models \alpha \to \beta$.

■

Since it will be useful, we extract the following result from the preceding proof:

Corollary A.1 *Let $\mathcal{M}$ be the canonical model for $\mathbf{C}^-$. For every $w \in W$ we have*

$$\alpha \to \beta \in w \quad \textit{iff} \quad |\alpha \wedge \neg\beta|_w <_w |\alpha \wedge \beta|_w.$$

Proof A.1 The proof of Theorem 4.2, Part 5(a) begins with the assumption that $\alpha \to \beta \in w$ and the claim that $|\alpha \wedge \neg\beta|_w <_w |\alpha \wedge \beta|_w$. This claim is proven toward the end of this step in the proof. ■

Proof 4.3 We show the contrapositive, that if a formula $\alpha \in \mathcal{L}$ is consistent, then α is satisfiable; that is there exists a world w in some model $\mathcal{M}$ such that $\mathcal{M}, w \models \alpha$.

For consistent formula α, consider the canonical model $\mathcal{M}$. Let w be a maximal consistent set of formulas such that $w \in |\alpha|$. Then we have from Theorem 4.2 that $\mathcal{M}, w \models \alpha$, which was to be shown.

■

Lemma A.2 *Let $\mathcal{M}$ be the canonical model and let $w \in W$. Let $\emptyset \neq X \subseteq N(w)$ and let α be a formula such that $|\alpha|_w = X$. Then $\diamond\alpha \in w$.*

Proof A.2 Let $w' \in X$. Thus $\alpha \in w'$, and $w' \in N(w)$.

It follows that $\{\beta \mid \Box\beta \in w\} \cup \{\alpha\}$ is consistent. (Otherwise we would have $\{\beta \mid \Box\beta \in w\} \vdash \neg\alpha$. From compactness we would have that for some finite set $\{\beta_1, \ldots, \beta_n\} \subseteq \{\beta \mid \Box\beta \in w\}$ that $\beta_1 \wedge \cdots \wedge \beta_n \supset \neg\alpha$, whence $\Box\beta_1 \wedge \cdots \wedge \Box\beta_n \supset \Box\neg\alpha$, whence $\Box\neg\alpha \in w$, from which $|\neg\alpha| = N(w)$, contradicting $\emptyset \neq |\alpha| = X \subseteq N(w)$.)

Since $\{\beta \mid \Box\beta \in w\} \cup \{\alpha\}$ is consistent, this means that $\Box\neg\alpha \notin w$; we then obtain that $\neg\Box\neg\alpha \in w$ by the construction of w, which is the same as $\diamond\alpha \in w$. ■

Proof 4.4 We have to prove that in the canonical model for logics closed under a given axiom or rule that the corresponding condition holds.[17]

1. We show that in the canonical model for $\mathbf{C}^- + \mathbf{RR}$ that min $\emptyset_1$ holds

   Let $w \in W$ and consider some $X \neq \emptyset$, $X \subseteq N(w)$, and select some α such that $|\alpha|_w = X$. (Recall that this last part is justified by the fact that we deal with a finite language.) Since $X \neq \emptyset$, we have that α is consistent, and so via Lemma A.2 we have $\diamond\alpha \in w$. Hence, since the canonical model for $\mathbf{C}^- + \mathbf{RR}$ is closed under $\mathbf{RR}$, we have that $\alpha \to \alpha \in w$. But this means (Corollary A.1) that $|\alpha \wedge \neg\alpha|_w <_w |\alpha \wedge \alpha|_w$, or $\emptyset <_w |\alpha|_w$, as required.

2. We show that in the canonical model for $\mathbf{C}^- + \mathbf{NA}$ that min $\emptyset_2$ holds.

   Assume that $X <_w \emptyset$, for some $X \subseteq N(w)$ and $w \in W$.

   Consequently from Definition 4.4 we have that for every α, β where $X = |\alpha \wedge \neg\beta|_w$ and $\emptyset = |\alpha \wedge \beta|_w$, that $\alpha \to \beta \in w$.

   Specifically, select a formula α such that $X = |\alpha|_w$ and take $\beta = \bot$; we have $X = |\alpha \wedge \top|_w = |\alpha|_w$ and $\emptyset = |\alpha \wedge \bot|_w = |\bot|_w$; that is, we have $\alpha \to \bot \in w$.

   But from $\mathbf{NA}$ we have that $\neg(\alpha \to \bot) \in w$, contradicting the fact that w is a consistent set of formulas. Hence $X \not<_w \emptyset$.

3. We show that in the canonical model for $\mathbf{C}^- + \mathbf{CEM}$ that asymmetry holds.

   Assume that $X <_w Y$ for some $w \in W$. Then, for every α, β such that $X = |\alpha \wedge \neg\beta|_w$ and $Y = |\alpha \wedge \beta|_w$ we have that $\alpha \to \beta \in w$ by Corollary A.1. Since $\mathbf{CEM}$ holds at every $w \in W$, we get $\neg(\alpha \to \neg\beta) \in w$.

   Let α, β be specific formulas according to the preceding paragraph, and assume, contrary to what is to be shown, that we also have $Y <_w X$.

   Then for every γ, δ such that $Y = |\gamma \wedge \neg\delta|_w$ and $X = |\gamma \wedge \delta|_w$ we have that $\gamma \to \delta \in w$.

   Specifically, choosing $\gamma = \alpha$ and $\delta = \neg\beta$, we obtain that $\alpha \to \neg\beta \in w$. Since we also have that $\neg(\alpha \to \neg\beta) \in w$, this contradicts the fact that w is consistent. Hence we cannot have $Y <_w X$.

4. We show that in the canonical model for $\mathbf{C}^- + \mathbf{Trans}$ that transitivity holds.

   Assume that $X_1 <_w X_2$, $X_2 <_w X_3$ and that $X_1 \cap X_3 = \emptyset$ for some $w \in W$.

   Since $X_1 <_w X_2$, we have that for every α, β such that $X_1 = |\alpha \wedge \neg\beta|_w$ and $X_2 = |\alpha \wedge \beta|_w$, that $\alpha \to \beta \in w$.

   By Theorem 4.1, Part 3 we get that $\alpha \to (\alpha \wedge \beta) \in w$. In propositional logic we have $\alpha \equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta))$. Applying $\mathbf{CEA}$ to this formula and the preceding formula yields $((\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta)) \to (\alpha \wedge \beta) \in w$.

[17] See [Hughes and Cresswell, 1984] for a good discussion on this point.

Similarly, since $X_2 <_w X_3$, we have that for every γ, δ such that $X_2 = |\gamma \wedge \neg\delta|_w$ and $X_3 = |\gamma \wedge \delta|_w$, that $\gamma \rightarrow \delta \in w$. By the same argument as above, we have that $((\gamma \wedge \delta) \vee (\gamma \wedge \neg\delta)) \rightarrow (\gamma \wedge \delta) \in w$.

Now, since w is closed under **Trans** and $|\alpha \wedge \beta|_w = |\gamma \wedge \neg\delta|_w$, we have that $((\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta)) \rightarrow (\alpha \wedge \beta) \in w$ and $((\gamma \wedge \delta) \vee (\gamma \wedge \neg\delta)) \rightarrow (\gamma \wedge \delta) \in w$ imply that $((\alpha \wedge \neg\beta) \vee (\gamma \wedge \delta)) \rightarrow (\gamma \wedge \delta) \in w$.

By Corollary A.1 this means that $|((\alpha \wedge \neg\beta) \vee (\gamma \wedge \delta)) \wedge \neg(\gamma \wedge \delta)|_w <_w |((\alpha \wedge \neg\beta) \vee (\gamma \wedge \delta)) \wedge (\gamma \wedge \delta)|_w$. Hence $|(\alpha \wedge \neg\beta) \wedge \neg(\gamma \wedge \delta)|_w <_w |\gamma \wedge \delta|_w$.

Since $|\alpha \wedge \neg\beta|_w \cap |\gamma \wedge \delta|_w = \emptyset$ by assumption (viz. $X_1 \cap X_3 = \emptyset$), we get $|\alpha \wedge \neg\beta|_w \subseteq |\neg(\gamma \wedge \delta)|_w$ and so $|(\alpha \wedge \neg\beta) \wedge \neg(\gamma \wedge \delta)|_w <_w |\gamma \wedge \delta|_w$ is equivalent to $|\alpha \wedge \neg\beta|_w <_w |\gamma \wedge \delta|_w$, or $X_1 <_w X_3$.

■

Lemma A.3 *In the canonical model, for $w \in W$, if $|\beta|_w \subseteq |\gamma|_w$ then $\Box(\beta \supset \gamma) \in w$.*

Proof A.3 If $|\beta|_w \subseteq |\gamma|_w$ then $|\beta \supset \gamma|_w = N(w)$. Since $\vdash (\beta \supset \gamma) \supset (\beta \supset \gamma)$ an application of Lemma A.1 implies that $\Box(\beta \supset \gamma) \in w$. ■

Proof 4.5

1. We show that in the canonical model for **C** + **WSA** that Continues Down holds.

   Assume that for $w \in W$ and $X, Y \subseteq N(w)$, we have $X <_w Y$ in the canonical model for **C** + **WSA**, and consider some $Z \subseteq N(w)$.

   We have from Definition 4.4 that: for every α, β where $X = |\alpha \wedge \neg\beta|_w$ and $Y = |\alpha \wedge \beta|_w$, $\alpha \rightarrow \beta \in w$.

   Let $\bar{Z} = (N(w) \setminus Z) \cup |\beta|_w$ and let γ be a formula such that $|\gamma|_w = \bar{Z}$.

   Hence $|\beta|_w \subseteq |\gamma|_w$; consequently from Lemma A.3 we obtain that $\Box(\beta \supset \gamma) \in w$.

   Since as well we have $\alpha \rightarrow \beta \in w$ and w is closed under **WSA** and modus ponens, we obtain that $(\alpha \wedge \gamma) \rightarrow \beta \in w$.

   As a result, we get that $|\alpha \wedge \gamma \wedge \neg\beta|_w <_w |\alpha \wedge \gamma \wedge \beta|_w$ from Corollary A.1.

   Now, $|\alpha \wedge \gamma \wedge \beta|_w = |\alpha \wedge \beta|_w = Y$, where the first equality follows since $|\beta|_w \subseteq |\gamma|_w$. As well, $|\alpha \wedge \gamma \wedge \neg\beta|_w = |\alpha \wedge \neg\beta|_w \cap |\gamma|_w = X \cap (N(w) \setminus Z) = X \setminus Z$. But this means that $X \setminus Z <_w Y$.

   Consequently, for arbitrary X, Y, Z, we get that if $X <_w Y$ then $X \setminus Z <_w Y$.

2. We show that in the canonical model for **C** + **CW** that Continues Up holds.

   Assume that we have $X <_w Y$ for some $w \in W$ in the canonical model for **C** + **CW**, and consider some $Z \subseteq N(w)$ such that $X \cap Z = \emptyset$.

   We have from Definition 4.4 that: for every α, β where $X = |\alpha \wedge \neg\beta|_w$ and $Y = |\alpha \wedge \beta|_w$, $\alpha \rightarrow \beta \in w$.

Let ψ, γ be formulas such that $|\psi|_w = Z \cup |\alpha|_w$ and $|\gamma|_w = (N(w) \setminus Z) \cup |\alpha|_w$. From set theory we get $|\alpha|_w = |\psi|_w \cap |\gamma|_w$ and so $|\alpha|_w = |\psi \wedge \gamma|_w$. Lemma A.3 then yields $\Box(\alpha \equiv (\psi \wedge \gamma)) \in w$.

We have $\psi \wedge \gamma \to \beta \in w$ since $\Box(\alpha \equiv (\psi \wedge \gamma)) \in w$ and $\alpha \to \beta \in w$, and using **CEA**.

Since w is closed under **CW** and modus ponens, we obtain that $\psi \to (\gamma \supset \beta) \in w$.

From Corollary A.1 we get that $|\psi \wedge \gamma \wedge \neg\beta|_w <_w |\psi \wedge (\neg\gamma \vee \beta)|_w$.

Now,

$$|\psi \wedge \gamma \wedge \neg\beta|_w = |\alpha \wedge \neg\beta|_w = X, \qquad \text{and}$$

$$|\psi \wedge (\neg\gamma \vee \beta)|_w = |(\psi \wedge \gamma \wedge \beta) \vee (\psi \wedge \neg\gamma)|_w = |\alpha \wedge \beta|_w \cup |\psi \wedge \neg\gamma|_w.$$

Since

$$|\neg\gamma|_w = N(w) \setminus ((N(w) \setminus Z) \cup |\alpha|_w) = Z \cap |\neg\alpha|_w.$$

we therefore have that

$$|\psi \wedge \neg\gamma|_w = (Z \cup |\alpha|_w) \cap (Z \cap |\neg\alpha|_w) = Z \cap |\neg\alpha|_w$$

Consequently we have the following, where each step follows from set theory:

$$\begin{aligned}
|\alpha \wedge \beta|_w \cup |\psi \wedge \neg\gamma|_w &= Y \cup (Z \cap |\neg\alpha|_w) \\
&= (Y \cup Z) \cap (Y \cup |\neg\alpha|_w) \\
&= (Y \cup Z) \cap |\neg\alpha \vee \beta|_w \\
&= (Y \cup Z) \cap (N(w) \setminus |\alpha \wedge \neg\beta|_w) \\
&= (Y \cup Z) \cap (N(w) \setminus X) \\
&= Y \cup Z
\end{aligned}$$

The final step above is justified by the fact that $X \cap Z = \emptyset$.

Hence, for arbitrary $X, Y, Z \subseteq N(w)$, with $X \cap Z = \emptyset$, we get that if $X <_w Y$ then $X <_w Y \cup Z$.

3. We show that in the canonical model for **C** + **CM** that Restricted Continues Down/Up holds.

   Assume that we have $X <_w Y$ in the canonical model for **C** + **CM**, and consider some $Z \subseteq N(w)$.

   We have from Definition 4.4 that: for every α, β where $X = |\alpha \wedge \neg\beta|_w$ and $Y = |\alpha \wedge \beta|_w$, that $\alpha \to \beta \in w$.

   Let γ be a formula such that $|\gamma|_w = |\beta|_w \cup Z$.

   We have that $|\beta|_w \subseteq |\gamma|_w$; consequently via Lemma A.3 we obtain that $\Box(\beta \supset \gamma) \in w$.

Since we have $\alpha \to \beta \in w$ and w is closed under **CM** and modus ponens, we obtain that $\alpha \to \gamma \in w$.

As a result, we get that $|\alpha \wedge \neg\gamma|_w <_w |\alpha \wedge \gamma|_w$ from Corollary A.1.

We have the following, where each step follows from set theory:

$$\begin{aligned}
|\alpha \wedge \neg\gamma|_w &= |\alpha|_w \cap (N(w) \setminus (|\beta|_w \cup Z)) \\
&= |\alpha|_w \cap ((N(w) \setminus |\beta|_w) \cap (N(w) \setminus Z)) \\
&= (|\alpha|_w \cap (N(w) \setminus |\beta|_w)) \cap (N(w) \setminus Z) \\
&= |\alpha \wedge \neg\beta|_w \setminus Z \\
&= X \setminus Z.
\end{aligned}$$

As well, $|\alpha \wedge \gamma|_w = |\alpha|_w \cap (|\beta|_w \cup Z) = |\alpha \wedge \beta|_w \cup (|\alpha|_w \cap Z) = Y \cup ((X \cup Y) \cap Z) = Y \cup (X \cap Z)$.

Consequently, for arbitrary X, Y, Z, we get that if $X <_w Y$ then $X \setminus Z <_w Y \cup (X \cap Z)$.

4. We show that in the canonical model for **C** + **WCA** that Continues Down/Up holds.

   Assume that we have $X <_w Y$ in the canonical model for **C** + **WCA**, and consider some $Z \subseteq N(w)$.

   We have from Definition 4.4 that: for every α, β where $X = |\alpha \wedge \neg\beta|_w$ and $Y = |\alpha \wedge \beta|_w$, that $\alpha \to \beta \in w$.

   Let γ be a formula such that $|\gamma|_w = |\beta|_w \cup Z$.

   Since we have $\alpha \to \beta \in w$ and w is closed under **WCA** and modus ponens, we obtain that $(\alpha \vee \gamma) \to (\beta \vee \gamma) \in w$.

   As a result, we get that $|(\alpha \vee \gamma) \wedge \neg\beta \wedge \neg\gamma|_w <_w |(\alpha \vee \gamma) \wedge (\beta \vee \gamma)|_w$ from Corollary A.1, or $|\alpha \wedge \neg\beta \wedge \neg\gamma|_w <_w |(\alpha \wedge \beta) \vee \gamma|_w$ by set theory.

   But this is just $X \setminus Z <_w Y \cup Z$. Consequently, for arbitrary $X, Y, Z \in N(w)$, we get that if $X <_w Y$ then $X \setminus Z <_w Y \cup Z$.

5. We show that in the canonical model for **C** + **D** that Weak Disjoint Union holds.

   Assume that we have $X_1 <_w Y_1$ and $X_2 <_w Y_2$ in the canonical model for **C** + **D**, where X_1, Y_1, X_2, Y_2 are pairwise disjoint.

   Let α, β, γ be formulas such that

   $$\begin{aligned}
   &|\alpha|_w = X_1 \cup Y_1 \cup X_2 \cup Y_2; \\
   &|\beta|_w = X_1 \cup Y_1; \\
   &|\gamma|_w = Y_1 \cup Y_2.
   \end{aligned}$$

   Since by assumption $X_1 <_w Y_1$ we have that $|\alpha \wedge \beta \wedge \neg\gamma|_w <_w |\alpha \wedge \beta \wedge \gamma|_w$. We obtain via Corollary A.1 that $(\alpha \wedge \beta) \to \gamma \in w$.

   A similar argument for X_2 and Y_2 gives that $(\alpha \wedge \neg\beta) \to \gamma \in w$.

Since w is closed under **D** and modus ponens, we obtain that $\alpha \to \gamma \in w$.

Thus from Corollary A.1 we obtain that $|\alpha \wedge \neg\gamma|_w <_w |\alpha \wedge \gamma|_w$ or $X_1 \cup X_2 <_w Y_1 \cup Y_2$.

Since this holds for all choices of α, β, γ, our result obtains.

■

Proof 4.6

1. An instance of **WCA** is

$$(\alpha \wedge \beta) \to \gamma \;\supset\; ((\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta)) \to (\gamma \vee (\alpha \wedge \neg\beta))$$

   which is $(\alpha \wedge \beta) \to \gamma \;\supset\; \alpha \to (\gamma \vee (\alpha \wedge \neg\beta))$.

   Since we have $\vdash \alpha \supset [(\gamma \vee (\alpha \wedge \neg\beta)) \equiv (\gamma \vee \neg\beta)]$ from propositional logic, **Nec** yields $\vdash_{\mathbf{C}\ +\mathbf{WCA}} \Box(\alpha \supset [(\gamma \vee (\alpha \wedge \neg\beta)) \equiv (\gamma \vee \neg\beta)])$.

   An instance of **CECA** is

$$\vdash_{\mathbf{C}\ +\mathbf{WCA}} \Box(\alpha \supset [(\gamma \vee (\alpha \wedge \neg\beta)) \equiv (\gamma \vee \neg\beta)]) \supset (\alpha \to (\gamma \vee (\alpha \wedge \neg\beta)) \;\equiv\; \alpha \to (\gamma \vee \neg\beta)).$$

   Hence we get $\vdash_{\mathbf{C}\ +\mathbf{WCA}} \alpha \to (\gamma \vee (\alpha \wedge \neg\beta)) \;\equiv\; \alpha \to (\gamma \vee \neg\beta)$.

   Given that we initially established that $\vdash_{\mathbf{C}\ +\mathbf{WCA}} (\alpha \wedge \beta) \to \gamma \;\supset\; \alpha \to (\gamma \vee (\alpha \wedge \neg\beta))$, substitution of logical equivalents yields $\vdash_{\mathbf{C}\ +\mathbf{WCA}} (\alpha \wedge \beta) \to \gamma \;\supset\; \alpha \to (\gamma \vee \neg\beta)$, which is the same as $\vdash_{\mathbf{C}\ +\mathbf{WCA}} (\alpha \wedge \beta) \to \gamma \;\supset\; \alpha \to (\beta \supset \gamma)$.

2. We show first that $\vdash_{\mathbf{C}\ +\mathbf{WCA}} (\alpha \to \beta) \supset (\alpha \to (\beta \vee \gamma))$.

   An instance of **WCA** is $\vdash_{\mathbf{C}\ +\mathbf{WCA}} (\alpha \to \beta) \supset ((\alpha \vee (\alpha \wedge \neg\beta \wedge \gamma)) \to (\beta \vee (\alpha \wedge \neg\beta \wedge \gamma)))$. Simplifying (via **CEA**) yields $\vdash_{\mathbf{C}\ +\mathbf{WCA}} (\alpha \to \beta) \supset (\alpha \to (\beta \vee (\alpha \wedge \gamma)))$.

   An instance of **CECA** is

$$\vdash_{\mathbf{C}\ +\mathbf{WCA}} \Box[\alpha \supset (\beta \vee (\alpha \wedge \gamma) \equiv (\beta \vee \gamma))] \;\supset\; [(\alpha \to (\beta \vee (\alpha \wedge \gamma))) \equiv (\alpha \to (\beta \vee \gamma))].$$

   We have that $\vdash_{\mathbf{C}\ +\mathbf{WCA}} \Box(\alpha \supset (\beta \vee (\alpha \wedge \gamma) \equiv (\beta \vee \gamma)))$ (which results from **Nec** applied to a propositional logic tautology).

   Applying this to the preceding formula yields via modus ponens

$$\vdash_{\mathbf{C}\ +\mathbf{WCA}} (\alpha \to (\beta \vee (\alpha \wedge \gamma))) \equiv (\alpha \to (\beta \vee \gamma)).$$

   Since we have already shown that $\vdash_{\mathbf{C}\ +\mathbf{WCA}} (\alpha \to \beta) \supset (\alpha \to (\beta \vee (\alpha \wedge \gamma)))$, we obtain $\vdash_{\mathbf{C}\ +\mathbf{WCA}} (\alpha \to \beta) \supset (\alpha \to (\beta \vee \gamma))$.

   To obtain the correct syntactic form for **CM**, if we have premiss $\Box(\beta \supset \delta)$, substitute $\neg\beta \wedge \delta$ for γ in the above formula, yielding $\vdash_{\mathbf{C}\ +\mathbf{WCA}} \Box(\beta \supset \delta) \supset ((\alpha \to \beta) \supset (\alpha \to \delta))$.

3. Since $\vdash (\beta \supset (\beta \vee \neg\gamma))$ in propositional logic, **Nec** yields $\vdash_{\mathbf{C\ +WSA+CW}} \Box(\beta \supset (\beta \vee \neg\gamma))$.

   From this we obtain, using **WSA**, that $\vdash_{\mathbf{C\ +WSA+CW}} (\alpha \to \beta) \supset (\alpha \wedge (\beta \vee \neg\gamma) \to \beta)$.

   Given the equivalence in propositional logic, $\vdash (\alpha \wedge (\beta \vee \neg\gamma)) \equiv (\alpha \vee \gamma) \wedge ((\alpha \wedge \beta) \vee \neg\gamma)$, we obtain $\vdash_{\mathbf{C\ +WSA+CW}} (\alpha \to \beta) \supset [(\alpha \vee \gamma) \wedge ((\alpha \wedge \beta) \vee \neg\gamma) \to \beta]$ via **CECA**.

   An application of **CW** to the consequent of the preceding formula yields: $\vdash_{\mathbf{C\ +WSA+CW}} (\alpha \to \beta) \supset ((\alpha \vee \gamma) \to ((\alpha \wedge \beta) \vee \neg\gamma) \supset \beta)$, or simplifying, $\vdash_{\mathbf{C\ +WSA+CW}} (\alpha \to \beta) \supset ((\alpha \vee \gamma) \to (\beta \vee \gamma))$. ■

■

Proof 4.7 Both results follow straightforwardly from **Nec** and **CM**. ■

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