The Annals of Mathematical Statistics
1970, Vol. 41, No. 4, 1206–1216

ON A SCHEDULING PROBLEM IN SEQUENTIAL ANALYSIS[1]

BY S. EHRENFELD

New York University

1. Introduction and summary. This paper reconsiders the usual sequential decision problem of choosing between two simple hypotheses H_0 and H_1, in terms of iirv when there is a time delay, assumed to have a known exponential distribution, in obtaining observations.

A basic assumption underlying much of the current analysis, is that results of taking observations are considered as immediately available. In other words, it is assumed that there is no *time delay* between the decision to take an observation and obtaining the result of the observation. This, of course, can be a tremendous limitation to the applicability of the theory. In actuality, such time lags can be substantial and taking an observation often involves experimentation. One important example, where this time delay often considerably inhibits the use of sequential analysis is medical experimentation. Here, a long time may elapse between the application and the result of a treatment.

The theory of sequential analysis is considered, explicitly taking account of time lags. At any point in time the decision maker must decide whether to stop and take action now or to continue and *schedule* more experiments. If he continues he must also decide how many more experiments to schedule. The problem basically then is to find an optimal procedure incorporating a stopping, scheduling and terminal action rule. There is an interesting interplay among these three; and optimal stopping rules, currently used for some problems, may not be optimal when scheduling factors are considered. The usual losses related to decision errors are specified and linear cost assumptions, with regard to amount of experimentation and time until a final decision, are made. Time until a terminal decision is an important variable. If it is decided to continue observation then scheduling many experiments will result in a small expected waiting time until the next result. However, this advantage must be balanced with the cost of scheduling these experiments. Finally, all the previous must be weighed with the loss of taking immediate action with only the currently available information.

Bayes procedures are derived. The information state, at any time, because of the exponential assumption, will be described by (n, π) where π is the current posterior probability of H_0 and n the numbers of results outstanding from tests already scheduled. As indicated in Section 2, when a known exponential time delay distribution is assumed, possible decision changes should be made only when test results are obtained. In Section 3, the usually used Sequential Probability Ratio Test (SPRT) stopping rule is studied. Here there are two values $0 < \pi_1 \leqq \pi_2 < 1$ and SPRT specifies that (n, π) is a continuation state as long as $\pi_1 < \pi < \pi_2$. It is

Received February 26, 1969.

[1] This research was supported by the National Science Foundation under Grant GK-14073.

shown that there is a bounded function $z(\pi)$ such that the optimal scheduling quantity, for a continuation state (n, π), is $y(n, \pi) = \max[0, z(\pi) - n]$. That is, if $n > z(\pi)$ we schedule no experiments. On the other hand, if $n \leqq z(\pi)$ then $z(\pi) - n$ more experiments are scheduled. The functional equation approach of Dynamic Programming is used and provides a computational method for approximating $z(\pi)$.

The general case, where the problem is to find an optimal stopping and scheduling rule, is studied in Section 4. Various results about the optimal stopping region, in the (n, π) plane, are derived and it is shown that the optimal procedure stops with probability one. The optimal stopping region is a kind of generalized SPRT described by functions $0 < \pi_1(n) \leqq \pi_2(n) < 1$ such that (n, π) is a continuation state as long as $\pi_1(n) < \pi < \pi_2(n)$. Also, it is shown that there exist two bounded functions $z_1(\pi) \leqq z_2(\pi) \leqq M < \infty$ such that if (n, π) is a continuation point the optimal scheduling quantity is $y(n, \pi) = z_1(\pi) - n$ if $n \leqq z_1(\pi)$, and $y(n, \pi) = 0$ if $n \geqq z_2(\pi)$.

When a continuous approximation to the problem is used, allowing n to take on a continuous range of values, a stronger result is proven. Here, the two functions $z_1(\pi)$ and $z_2(\pi)$ may be taken as equal. The results for optimal scheduling rules have similarities to some problems studied in Inventory theory, see [5].

2. Notation and background. We consider choosing sequentially between two simple hypotheses H_0 and H_1 on the basis of iirv $X_1, X_2, \cdots$. Let $L(i,j)$ denote the loss of choosing H_j when the true hypothesis is H_i. The usual simple loss structure is assumed where $L(0,0) = L(1,1) = 0$, $L(0,1) = w_0 > 0$ and $L(1,0) = w_1 > 0$.

Let $c > 0$ denote the cost per test and $a > 0$ the cost per unit time until a terminal choice is made. It is also assumed that the time delay in obtaining an observation has an exponential distribution with a known mean value of $\theta > 0$.

The cost, apart from terminal decision losses, can be given as $aT + cN$ where T is the total time until a terminal decision is made and N is the total number of scheduled tests. Letting the a priori probability $\pi = \Pr(H_0)$, the risk for any decision procedure δ can be expressed as,

$$R_\delta(\pi) = \pi[E_0(\text{cost}) + w_0 \alpha] + (1-\pi)[E_1(\text{cost}) + w_1 \beta] \tag{2.1}$$

where α and β are the usual probability of error types. The procedure δ incorporates a stopping rule, terminal action rule and a scheduling rule.

In the case where no observations are made the Bayes risk is denoted by $B(\pi) = \min[\pi w_0, (1-\pi)w_1]$. In general, as observations $x_1, \cdots, x_k$ are obtained the posterior probability of H_0 changes and is denoted by

$$\pi(x_1, \cdots, x_k) = \pi f_0(x_1, \cdots, x_k)/Q_\pi(x_1, \cdots, x_k) \tag{2.2}$$

where $f_i(x_1, \cdots, x_k)$ is the joint probability distribution of $x_1, \cdots, x_k$ under H_i and

$$Q_\pi(x_1, \cdots, x_k) = \pi f_0(x_1, \cdots, x_k) + (1-\pi) f_1(x_1, \cdots, x_k). \tag{2.3}$$

Also, it is known that, see [10]

$$\pi(x_1, \cdots, x_k) = \pi_{x_1, \cdots, x_{k-1}}(x_k) = \pi(x_1, \cdots, x_{k-1}) f_0(x_k)/Q_{\pi(x_1, \cdots, x_{k-1})}(x_k).$$

In other words, the posterior probability of H_0 changes from π to $\pi(x_1)$ then to $\pi_{x_1}(x_2)$ and so forth.

Because the time delay distribution is assumed to be exponential the information state at any time will be described by (n, π) where π is the current a posteriori probability of H_0 and n is the number of results outstanding from tests already scheduled. In this paper, the restricted though natural class of procedures depending on (n, π) will be studied. A non-rigorous discussion, motivating this assumption, is given in what follows. In order to indicate the background let N_t be the number of experiments scheduled until time t, and n_t the number of results outstanding from tests already scheduled at times $t_1, \cdots, t_n$. The number of results already obtained is then $r_t = N_t - n_t$ with observations denoted by $x_1, x_2, \cdots, x_r$ and delay times $\tau_1, \cdots, \tau_r$. The history until t is further specified by the times at which the N_t experiments were scheduled and the r_t results obtained. The history can then be summarized by $(r_t, n_t; x_1, \cdots, x_r; t_1, \cdots, t_n;\ \tau_1, \cdots, \tau_r)$. The optimal procedure, at time t, will depend on the expected future cost, as a function of the past history up to t. Since the time delay distribution was assumed known the values of $\tau_1, \cdots, \tau_r$ are irrelevant to the future. This is usually not the case when the time delay distribution is not known. Using an argument similar to [8] the observations $x_1, \cdots, x_r$ are relevant, for the future, in terms of $\pi(x_1, \cdots, x_r) = \pi_t$ which is the posterior probability of H_0 at time t. Hence, the relevant history is summarized by $(n_t, \pi_t, t_1, \cdots, t_n)$. Finally, since the time delay distribution was assumed to have a known exponential distribution it follows that $t_1, \cdots, t_n$ is irrelevant for assessing when the results of the n_t, previously scheduled, experiments will arrive. Hence, the relevant history up to time t, can be described by (n_t, π_t). Thus, any optimal procedure will depend only on (n_t, π_t). Suppose the optimal procedure has continued until t and we are at state (n_t, π_t) at which no results have arrived. Let t' be the largest $t' < t$ such that experiments were scheduled and/or results from previously scheduled experiments arrived. During the time interval $(t', t]$ the state was (n_t, π_t). Since the optimal procedure only depends on (n_t, π_t) no action except continuation is called for at t, using the optimal procedure, since none was called for during $(t', t]$. Hence, it follows that changes in decision, such as stopping or scheduling should only be made when test results are obtained. In what follows, the current state of information at any time will sometimes be designated by (n, π) with the subscript t deleted.

3. Optimal scheduling with an SPRT stopping rule. An SPRT stopping rule specifies two values $0 < \pi_1 \leqq \pi_2 < 1$. We will assume that $0 < \pi_1 \leqq \pi_0 \leqq \pi_2 < 1$ with $\pi_0 = w_1/(w_0 + w_1)$. The information state (n, π) calls for continuation so long as $\pi_1 < \pi < \pi_2$. Let $R_\delta(n, \pi)$ be the risk, using procedure δ starting from state (n, π). Consider truncated procedures T_k where we stop after k observations if we have not stopped before with the SPRT stopping rule. Let $\rho_k(n, \pi)$ equal the minimum risk which can be achieved when $\delta \in T_k$ and we start at (n, π). That is,

$$\rho_k(n, \pi) = \inf_{\delta \in T_k} R_\delta(n, \pi). \tag{3.1}$$

The best terminal action rule is to choose H_1 when the stopped value of $\pi \leqq \pi_1$ and choose H_0 when $\pi \geqq \pi_2$. Also, $\rho_k(n, \pi)$ is decreasing in n and $\rho_k(n, \pi) - \rho_k(n+y, \pi) \leqq cy$ since we can always go from (n, π) to $(n+y, \pi)$ at a cost cy, by scheduling y tests.

The decision process can be described by changes of information state as results of previously scheduled experiments arrive and the successive intervals between such changes. At a continuation state (n, π) we schedule y experiments and start the next time interval with $(n+y, \pi)$. Then, from the exponential assumption the waiting time until the next observation has an exponential distribution with mean $\theta/(n+y)$. The next state reached is $(n+y-1, \pi(X))$, where X has the unconditional distribution $Q_\pi(x)$.

The expected loss between two such time points is $cy + d/(n+y)$, with a $\theta = d$, where the first term is the cost of scheduling y tests and the second is the added expected loss due to the waiting time until a decision is reached. Finally, if we stop at (n, π) we have a loss $B(\pi)$ due to errors in the terminal action rule. Using a standard procedure one obtains a functional equation relating $\rho_{k+1}(n, \pi)$ and $\rho_k(n, \pi)$. We have,

$$(3.2)\quad \begin{aligned} \rho_{k+1}(n, \pi) &= B(\pi) && \text{if } \pi \notin (\pi_1, \pi_2), \\ &= \min_{y \geqq 0}\left[cy + \frac{d}{n+y} + E_\pi(\rho_k(n+y-1, \pi(X)))\right] && \text{otherwise;} \end{aligned}$$

where E_π refers to an unconditional expectation and the minimum relates to the fact that the optimal amount y must be scheduled. Letting $n+y = z$ and

$$(3.3)\quad H_k(z, \pi) = cz + d/z + E_\pi(\rho_k(z-1, \pi(X)))$$

one obtains,

$$(3.4)\quad \begin{aligned} \rho_{k+1}(n, \pi) &= B(\pi) && \text{for } \pi \notin (\pi_1, \pi_2); \\ &= \min_{z \geqq n} H_k(z, \pi) - cn && \text{otherwise.} \end{aligned}$$

As can be seen the essential facts depend on the behavior of the functions $H_k(z, \pi)$. Under typical conditions in sequential analysis, $\rho_k(n, \pi)$ approaches a function $\rho(n, \pi)$ as k approaches infinity.

Let $Y = \ln[f_1(x)/f_0(x)]$ where $f_i(x)$ is the probability density of X under H_i, and W equal the number of state changes until $\pi \notin (\pi_1, \pi_2)$.

LEMMA 1. *If $P_\pi(Y = 0) < 1$ then $\rho_k(n, \pi)$ approaches a unique bounded function $\rho(n, \pi)$ satisfying functional equation* (3.7).

PROOF. The value of $\rho_{k+1}(n, \pi)$ can be bounded from above by finding the expected loss using the optimum k truncated procedure and then scheduling one experiment for the $k+1$th stage if required. Then, comparing with $\rho_k(n, \pi)$ we can obtain a constant $C_1 > 0$ such that

$$(3.5)\quad \rho_{k+1}(n, \pi) - \rho_k(n, \pi) \leqq C_1 P_\pi(W \geqq k).$$

Now, using the condition of the Lemma and Stein's theorem, see [10], one obtains constants $b > 0$ and $0 < r < 1$ with $P_\pi(W \geqq k) \leqq br^k$. A similar procedure can be used to find a lower bound. Combining them we have $C > 0$ such that,

$$|\rho_{k+1}(n,\pi) - \rho_k(n,\pi)| \leqq Cbr^k \tag{3.6}$$

which is sufficient to prove the result. A direct proof, using (3.4) is also possible. Hence, we finally have the functional equation,

$$\begin{aligned} \rho(n,\pi) &= B(\pi) && \text{for} \quad \pi \notin (\pi_1, \pi_2), \\ &= \min\nolimits_{z \geqq n} H(z,\pi) - cn && \text{otherwise}; \end{aligned} \tag{3.7}$$

where,

$$H(z,\pi) = cz + d/z + E_\pi(\rho(z-1, \pi(X))). \tag{3.8}$$

LEMMA 2. *$\rho(n,\pi)$ is a convex function in n.*

PROOF. We use induction on k with functions $\rho_k(n,\pi)$. It can be observed that $\rho_0(n,\pi) = B(\pi)$. Then,

$$\begin{aligned} \rho_1(n,\pi) &= B(\pi) && \text{if} \quad \pi \notin (\pi_1, \pi_2); \\ &= \min\nolimits_{z \geqq n}(cz + d/z) - cn + E_\pi(B(\pi(X))) && \text{otherwise}. \end{aligned} \tag{3.9}$$

It is easily calculated that $\rho_1(n,\pi)$ is convex in n. Now, we use induction on k. From the convexity of $cz + d/z$ and $E_\pi(\rho_k(z-1, \pi(X)))$ one obtains the convexity of $H_k(z,\pi)$ and $\min_{z \geqq n} H_k(z,\pi) - cn$, hence of $\rho_{k+1}(n,\pi)$.

The optimal scheduling rule can now be obtained. Let $z(\pi)$ be the integer which minimizes $H(z,\pi)$ over the range $z \geqq 0$. That is,

$$H(z(\pi),\pi) = \min\nolimits_{z \geqq 0} H(z,\pi) = K(\pi). \tag{3.10}$$

Then, if $n \leqq z(\pi)$ the minimum of $H(z,\pi)$ over the range $z \geqq n$ is taken on at $z(\pi)$. On the other hand, because of the convexity of $H(z,\pi)$, when $n > z(\pi)$ the minimum of $H(z,\pi)$ is taken on at $z = n$. Hence, since $z = n + y$ the optimal scheduling amount is

$$y(n,\pi) = \max[0, z(\pi) - n]. \tag{3.11}$$

That is, when $n \leqq z(\pi)$ schedule $z(\pi) - n$ to bring the number of active experiments up to $z(\pi)$. When $n > z(\pi)$ do not schedule any more experiments. The values of $cz + d/z$ are decreasing in the range of $z \leqq (d/c)^{\frac{1}{2}}$. Hence, $H(z,\pi)$ is decreasing for this range of z and so $z(\pi) \geqq (d/c)^{\frac{1}{2}}$. Also, it can be noted that when $\pi_1 < \pi < \pi_2$ and $n \leqq z(\pi)$ we have $\rho(n,\pi) = K(\pi) - cn$ a linear function of n. Finally, it is observed that, by dividing both sides of the functional equation (3.7) by c, the results only depend on the constants w_0, w_1, c and d via w_0/c, w_1/c and d/c. Also, equation (3.4) provides a computational means for successively approximating $z(\pi)$.

4. The general case. In this case one must find an optimal scheduling and stopping rule. As in Section 3 procedures truncated at k are considered. Let $g_k(n,\pi)$ be the minimum risk achievable, starting at (n,π), using any procedure truncated at k. Also, let $G_k(z,\pi) = cz + d/z + E_\pi(g_k(z-1, \pi(X)))$. From general optimal stopping theory, a recursion relation between $g_{k+1}(n,\pi)$ and $g_k(n,\pi)$ is

$$g_{k+1}(n,\pi) = \min\left[B(\pi), \min_{z \geqq n} G_k(z,\pi) - cn\right]. \tag{4.1}$$

The first minimum relates to the stopping rule, while the second concerns scheduling. It may be observed that $g_k(n,\pi)$ is decreasing with k. That is, $g_{k+1}(n,\pi) \leqq g_k(n,\pi)$ since anything achievable with a k truncated plan can certainly be done with a $k+1$ truncated procedure. This was not necessarily the case when a specified SPRT stopping rule was used as in Section 3. Because of this monotonicity $g_k(n,\pi)$ approaches a unique bounded function $g(n,\pi)$ satisfying

$$g(n,\pi) = \min\left[B(\pi), \min_{z \geqq n} G(z,\pi) - cn\right] \tag{4.2}$$

where

$$G(z,\pi) = cz + d/z + E_\pi(g(z-1, \pi(X))). \tag{4.3}$$

The information states (n,π) with $0 \leqq \pi \leqq 1$ and $n = 0, 1, 2, \cdots$ can be divided into three disjoint sets B_0, B_1 and C where C are the continuation states and B_i the stopping sets where H_i $(i = 0, 1)$ is chosen. As in stopping rule theory these sets can be characterized in terms of the functional equation (4.2).

$$(n,\pi) \in C \quad \text{when} \quad \min_{z \geqq n} G(z,\pi) - cn < B(\pi). \tag{4.4}$$

For continuation states,

$$g(n,\pi) = \min_{z \geqq n} G(z,\pi) - cn. \tag{4.5}$$

On the other hand,

$$(n,\pi) \in B_0 \quad \text{when} \quad \min_{z \geqq n} G(z,\pi) - cn \geqq B(\pi) \quad \text{and} \quad \pi > \pi_0 \tag{4.6}$$

where $\pi_0 = w_1/(w_0 + w_1)$ the solution of $w_0 \pi = (1-\pi) w_1$. Also,

$$(n,\pi) \in B_1 \quad \text{when} \quad \min_{z \geqq n} G(z,\pi) - cn \geqq B(\pi) \quad \text{and} \quad \pi \leqq \pi_0. \tag{4.7}$$

For stopping points, $g(n,\pi) = B(\pi)$. More specifically, when $(n,\pi) \in B_1$ we have $g(n,\pi) = w_0 \pi$ and for $(n,\pi) \in B_0$ it equals $(1-\pi) w_1$.

The functions $g_k(n,\pi)$ are decreasing in n since the decision maker starting at $(n+1,\pi)$ can do as well as one starting at (n,π), in terms of risk, by simply ignoring one of his scheduled tests. Also, using induction it may be shown that $g_k(n,\pi)$ and $g(n,\pi)$ are concave functions of π. From (4.2) it is also generally true that $g(n,\pi) \leqq B(\pi)$.

Some properties of the sets B_0, B_1 and C are:

(i) $(n,0) \in B_1$ and $(n,1) \in B_0$ and $g(n,0) = g(n,1) = 0$. This is so since the loss for stopping when at $(n,0)$ is equal to $B(0) = 0$ and for $(n,1)$ it is equal to $B(1) = 0$.

(ii) If $(n,\pi) \in C$ then $(n+1,\pi) \in C$. Since $(n,\pi) \in C$ we have $g(n,\pi) < B(\pi)$. Then, since $g(n+1,\pi) \leqq g(n,\pi)$ we obtain $g(n+1,\pi) \leqq g(n,\pi) < B(\pi)$ and hence $(n+1,\pi) \in C$.

(iii) If $(n,\pi) \in B_i$ then $(n-1,\pi) \in B_i$ for $i = 0, 1$. Since $(n,\pi) \in B_i$ we have $g(n,\pi) = B(\pi)$. Then,

$$B(\pi) = g(n,\pi) \leqq g(n-1,\pi) \leqq B(\pi). \tag{4.8}$$

Hence, $g(n-1,\pi) = B(\pi)$ which proves the result.

(iv) If $\pi' \leqq \pi \leqq \pi_0$ and $(n, \pi) \in B_1$ then $(n, \pi') \in B_1$. Similarly when $\pi_0 \geqq \pi \geqq \pi'$ and $(n, \pi) \in B_0$ then $(n, \pi') \in B_0$. To prove the first relation let $\pi' = \lambda\pi$ where $0 \leqq \lambda \leqq 1$. Using the concavity of $g(n, \pi)$ and $g(n, \pi) = w_0 \pi$, it follows that,

$$B(\pi') = w_0 \pi' \geqq g(n, \pi') \geqq \lambda g(n, \pi) = w_0 \pi' = B(\pi'). \tag{4.9}$$

Hence, $g(n, \pi') = w_0 \pi'$ so that $(n, \pi') \in B_1$.

From the previous properties it follows that the sets can be characterized by functions $0 \leqq \pi_1(n) \leqq \pi_2(n) \leqq 1$ such that:

$$\begin{aligned} C &= ((n, \pi): \pi_1(n) < \pi < \pi_2(n)) \\ B_1 &= ((n, \pi): \pi \leqq \pi_1(n)) \\ B_0 &= ((n, \pi): \pi \geqq \pi_2(n)). \end{aligned} \tag{4.10}$$

It can be observed that $\pi_1(n)$ decreases with n and $\pi_2(n)$ increases with n. This may be thought of as a generalized SPRT procedure. Further properties of $\pi_1(n)$, $\pi_2(n)$ and $g(n, \pi)$ are incorporated in the following lemmas.

LEMMA 3. *$\pi_1(n) > 0$ and $\pi_2(n) < 1$.*

PROOF. We show $\pi_1(n) > 0$. First, if

$$\min_{z \geqq n}(cz + d/z) - cn \geqq w_0 \pi \tag{4.11}$$

then

$$\min_{z \geqq n} G(z, n) - cn = \min_{z \geqq n}(cz + d/z + E_\pi(g(l-1, \pi(x)))) - cn \geqq w_0 \pi. \tag{4.12}$$

Hence, $(n, \pi) \in B_1$. Thus, equation (4.11) forms a bound, away from zero, for $\pi_1(n)$. The relation $\pi_2(n) < 1$ is proven in similar fashion.

LEMMA 4. *$g(n, \pi) \to 0$ as $n \to \infty$.*

PROOF. Pick any k and let $n > k$ and consider the procedure which does not schedule and waits for the kth result and then stops. Comparing the risk for this procedure with $g(n, \pi)$ gives,

$$0 \leqq g(n, \pi) \leqq \frac{d}{n} + \frac{d}{n-1} + \cdots + \frac{d}{n-k+1} + E[B(\pi(X_1, X_2, \cdots, X_k))] \tag{4.13}$$

Hence,

$$0 \leqq \lim_{n \to \infty} g(n, \pi) \leqq E_\pi[B(\pi(X_1, \cdots, X_k))] \qquad \text{for all } k. \tag{4.14}$$

The result follows by letting $k \to \infty$ because when $X_1, X_2, \cdots$ are iirv the standard sequential analysis theory shows that the upper bound in (4.14) goes to zero. This can be seen by considering $\pi_k = \pi(X_1, \cdots, X_k)$ as $k \to \infty$. If H_0 is true then if $0 < \pi < 1$ $\pi_k \to 1$ as $k \to \infty$ with probability 1, see [9]. Similarly, when H_1 is true then $\pi_k \to 0$ with probability 1. Hence, $B(\pi_k) \to 0$, with probability 1 when H_0 is true with probability π and H_1 with probability $1 - \pi$.

In the case with the SPRT stopping rule, discussed in Section 3, a similar result does not hold. That is, $\rho(n, \pi)$ does not in general approach zero as $n \to \infty$. In this case, it may be shown that $\lim_{n\to\infty} \rho(n, \pi) = E_\pi[B(\pi(X_1, \cdots, X_k)]$ where k is the number of observations when stopping occurs. Consider any case where the stopping boundaries $0 < \pi_1 \leqq \pi_2 < 1$ are chosen so that stopping always occurs exactly at either π_1 or π_2, see [10]. In such circumstances, $B(\pi(X_1, \cdots, X_k)) \geqq \min[B(\pi_1), B(\pi_2)] > 0$ and so $\lim_{n\to\infty} \rho(n, \pi) \geqq \min[B(\pi_1), B(\pi_2)] > 0$.

We next show the basic result that no tests should be scheduled beyond a certain value of n.

THEOREM 1. *There exists $N < \infty$ such that if $(n, \pi) \in C$ then $y(n, \pi) = 0$ for $n \geqq N$.*

PROOF. Since $g(n, \pi)$ is concave in π and bounded it is continuous. Further, $g(n, \pi)$ decreases monotonically to zero as $n \to \infty$. Hence, by Dini's theorem for any $\delta > 0$ there is an n_0 such that

$$|g(n, s) - g(n+1, s)| \leqq \delta \quad \text{for all} \quad n \geqq n_0 \quad \text{and all} \quad 0 \leqq s \leqq 1. \tag{4.15}$$

Choosing $0 < \delta < c$ and using the forward difference, Δ, operator, $\Delta[h(x)] = h(x+1) - h(x)$, with respect to the variable z,

$$\Delta[G(z, \pi)] = \Delta\left[cz + \frac{d}{z}\right] + \Delta[E_\pi(g(z-1, \pi(X))] \geqq c - \frac{d}{z(z+1)} - \delta > 0 \tag{4.16}$$

when $z \geqq \max(N_0, n_0) = N$ where N_0 is chosen as the smallest integer such that $N_0(N_0+1) > d/(c-\delta) > 0$.

LEMMA 5. *The optimal procedure stops with probability one.*

This result is obvious since $g(n, \pi) \leqq B(\pi)$, and if there were a positive probability of an infinite number of observations the expected testing cost would be infinite.

Another view of this lemma is to consider the value of N in Theorem 1. The optimal procedure certainly stops when $\pi = \pi(X_1, \cdots, X_k)$ moves out of the interval $(\pi_1(N), \pi_2(N))$. Since $\pi_1(N) > 0$ and $\pi_2(N) < 1$ from Lemma 3, the usual theory of sequential analysis gives the result.

5. A continuous approximation. In the previous sections n was assumed to take on only integer values. In the continuous version we let n be any non-negative real number. The optimal return functions for truncated procedures is denoted by $v_k(n, \pi)$, where $v_0(n, \pi) = B(\pi)$ and,

$$v_{k+1}(n, \pi) = \min[B(\pi), \min_{z\geqq n} V_k(z, \pi) - cn] \tag{5.1}$$

where

$$V_k(z, \pi) = cz + d/z + E_\pi(v_k(z-1, \pi(X))). \tag{5.2}$$

The functions $v_k(n, \pi)$ are monotonically decreasing with k and thus approach uniformly $v(n, \pi)$ the minimum risk attainable by any procedure, starting at (n, π), and

$$v(n, \pi) = \min[B(\pi), \min_{z\geqq n} V(z, \pi) - cn] \quad \text{with} \tag{5.3}$$

$$V(z, \pi) = cz + d/z + E_\pi(v(z-1, \pi(X))). \tag{5.4}$$

Stopping sets and continuation set can be defined similar to that discussed in Section 4. Again, $v(n,\pi)$ is a concave function in π. Also, $v(n,\pi) \leqq g(n,\pi)$. The stopping sets and continuation set are denoted by B_1', B_1' and C'. Let $\pi_0'(n)$ and $\pi_2'(n)$ denote the functions characterizing the optimal stopping rule. Unless $d=0$, the sets B_0', B_1' are not convex. This follows from the fact that $(0, \pi_1'(0)) \in B_1$ and $(n, 0) \in B_1$ for all n. If B_1 were convex, all convex combinations would also be in B_1 which by facts (ii) and (iii), Section 4 can only be if $\pi_1'(n)$ is constant for all n. A similar argument holds for B_0. This would then give the SPRT stopping rule which is optimum when $d=0$. However, for small values of n the sets have a kind of convexity property.

LEMMA 6. *The sets* $R_i = B_i \cap [n \leqq (d/c)^{\frac{1}{2}}]$ $(i=0,1)$ *are convex.*

PROOF. We show the result for R_1. Let (n_1, π_1) and (n_2, π_2) be in R_1 and denote $(n,\pi) = \lambda(n_1,\pi_1) + (1-\lambda)\,(n_2,\pi_2)$ for any $0 \leqq \lambda \leqq 1$. Since, $(n_i, \pi_i) \in B_1$ we have,

$$v(n_1,\pi_1) = w_0\pi_1 \quad \text{and} \quad v(n_2,\pi_2) = w_0\pi_2. \tag{5.5}$$

Let $K'(\pi) = \min_{z \geqq 0} v(z,\pi)$ and, $z'(\pi)$ the value of z where the minimum occurs. Since $V(z,\pi)$ is decreasing in the range $0 \leqq z \leqq (d/c)^{\frac{1}{2}}$, the value of $z'(\pi) \geqq (d/c)^{\frac{1}{2}}$. Hence, for $n \leqq (d/c)^{\frac{1}{2}}$,

$$\min_{z \geqq n} V(z,\pi) = K'(\pi). \tag{5.6}$$

Thus,

$$\begin{aligned} v(n_1,\pi_1) &= w_0\pi_1 \leqq K'(\pi_1) - cn_1 \\ v(n_2,\pi_2) &= w_0\pi_2 \leqq K'(\pi_2) - cn_2. \end{aligned} \tag{5.7}$$

We note that $K'(\pi)$ is concave in π.

Now, taking convex combination $(\lambda, 1-\lambda)$ on both sides of (5.7) we have,

$$w_0\pi \leqq \lambda K'(\pi_1) + (1-\lambda)K'(\pi_2) - cn \leqq K'(\pi) - cn. \tag{5.8}$$

Hence,

$$w_0\pi \leqq \min_{z \geqq 0} V(z,\pi) - cn \leqq \min_{z \geqq n} V(z,\pi) - cn. \tag{5.9}$$

Thus, $w_0\pi = v(n,\pi)$ and $(n,\pi) \in B_1$. Since $n_i \leqq (d/c)^{\frac{1}{2}}$ we also have $n \leqq (d/c)^{\frac{1}{2}}$ and so, $(n,\pi) \in R_1$.

In order to characterize the optimal scheduling rule we focus on the functions $V_k(z,\pi)$ and $v_k(n,\pi)$. Also, we will assume that $\pi(X)$ has an absolutely continuous distribution over $0 \leqq \pi(X) \leqq 1$ for $0 < \pi < 1$. Let,

$$R_k(n,\pi) = \left(\frac{\partial^2 v_k(n,\pi)}{\partial^2 n}\right)^+ = \lim_{h \to 0^+} \frac{v_k(n+2h) - 2v_k(n+h) + v_k(n)}{h^2}$$

$$D_k(z,\pi) = \frac{\partial V_k(z,\pi)}{\partial z} \tag{5.10}$$

$$d_k(n,\pi) = \frac{\partial v_k(n,\pi)}{\partial n}.$$

Some properties of these functions are described in the next theorem.

THEOREM 2. *The functions* $R_k(n,\pi)$ *are bounded and satisfy* $R_k(n,\pi) \geqq 0$. *Also,* $D_k(n,\pi)$ *exists for all* (n,π) *and* $d_k(n,\pi)$, *for any specified* π, *exists for all but a finite number of* n.

PROOF. First, if (n,π) is a stopping point and an interior point of $B_1 \cup B_2$ then $v_k(n,\pi) = w_0\pi$ or $w_1(1-\pi)$. In either case, $R_k(n,\pi) = 0$. Suppose (n',π) is a continuation point or a boundary point then (n,π) is for all $n > n'$ a continuation point and

(5.11) $$v_k(n,\pi) = \min_{z \geqq n} V_k(z,\pi) - cn \quad \text{for} \quad n > n'.$$

We will prove the theorem by induction. We have,

(5.12) $$V_k(z,\pi) = cz + d/z + E_\pi(v_{k-1}(z-1,\pi(X))).$$

Suppose the theorem is true for $k-1$; then essentially from the Lebesgue dominated convergence theorem,

(5.13) $$\frac{\partial^2 V_k(z,\pi)}{\partial z^2} = \frac{2d}{z^3} + E_\pi\left[\left(\frac{\partial^2 v_{k-1}(z-1,\pi(x))}{\partial z^2}\right)^+\right] \geqq 0.$$

Also, we have assumed that $d_{k-1}(z-1,\pi(x))$, for any specified $\pi(x)$, exists for all but a finite number of z. Hence, again basically resulting from the Lebesgue theorem to differentiation under the integral sign, with respect to function with a parameter, see [11] we have that $D_k(z,\pi)$ exists for all z. Hence, combining all these observations, the optimal truncated scheduling plan is characterized by function $z_k'(\pi) \geqq (d/c)^{\frac{1}{2}}$ such that,

(5.14) $$y_k(n,\pi) = \max[0, z_k'(\pi) - n].$$

Hence, if $n < z_k'(\pi)$ then

(5.15) $$v_k(n,\pi) = V_k(z_k'(\pi),\pi) - cn \quad \text{and,}$$

(5.16) $$\left(\frac{\partial^2 v_k(n,\pi)}{\partial n^2}\right)^+ = 0.$$

If, $n \geqq z_k'(\pi)$ then

(5.17) $$v_k(n,\pi) = d/n + E_\pi(v_{k-1}(n-1,\pi(X))) \quad \text{and,}$$

(5.18) $$\left(\frac{\partial^2 v_k(n,\pi)}{\partial n^2}\right)^+ = \frac{2d}{n^3} + E_\pi\left[\left(\frac{\partial^2 v_{k-1}(n-1,\pi(X))}{\partial n^2}\right)^+\right] \geqq 0.$$

Boundedness follows from the induction assumption and $n \geqq (d/c)^{\frac{1}{2}}$. The induction process is completed by inspecting the behavior of $v_0(n,\pi)$ and $v_1(n,\pi)$.

The above property and the uniform approximation of $v(n,\pi)$ and $V(z,\pi)$ by $v_k(n,\pi)$ is sufficient to characterize the optimal scheduling rule as of the form $y(n,\pi) = \max[0, z'(\pi)-n]$. The value of $z'(\pi)$ is the value of z where $V(z,\pi)$ is minimized and clearly if $n < z'(\pi)$ then $z'(\pi)-n$ tests should be scheduled. It remains only to show that $V(z,\pi)$ is non-decreasing for $z > z'(\pi)$. Suppose, there were values $z_2 > z_1 > z'(\pi)$ such that $V(z'(\pi), \pi) \leqq V(z_2, \pi) < V(z_1, \pi)$. This is impossible since $V(z, \pi)$ can be uniformly approximated, as closely as desired, by $V_k(z, \pi)$ with some suitable k and the inequalities cannot occur for $V_k(z, \pi)$ by Theorem 2.

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