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On a Semi-Relativistic Treatment of the Gravitational Radiation from a Mass Thrusted into a Black Hole

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A semi-relativistic treatment estimating the gravitational radiation emitted by a particle thrusted into a Schwarzschild black hole with a finite kinetic energy at infinity is presented on the two extreme assumptions: (a) The particle moves along a geodesic in a curved space and (b) the particle radiates as if it were in flat space-time.

The structure of the burst and beaming process of gravitational radiation are studied.

The merit of this approach lies in its simplicity and in providing a direct and complementary understanding of the results obtained by a fully relativistic treatment.

§ 1. Introduction

The recent progress in the development of a new family of gravitational wave antennae[1] and the possibility of achieving the accuracy required to observe predicted levels of gravitational wave signals coming from galactic sources,[2] have made a new analysis of the detailed structure of bursts of gravitational waves necessary.

In this paper we propose a semi-relativistic treatment for the estimation of the gravitational radiation emitted by a particle thrusted into a Schwarzschild black hole with a finite kinetic energy at infinity. Following the approach used in Ref. 3) we have made two extreme assumptions: (a) the particle moves along a geodesic in the Schwarzschild geometry, and (b) the particle radiates as if it were in flat space-time. However, contrary to Ref. 3), in which the stress is mainly on the energy spectrum of the radiation, here we are interested in the temporal structure of the burst. Therefore, we introduce an approximation technique by which the details of the radiating process can be readily studied and easily compared with the results obtained by using a fully relativistic treatment. The fully relativistic treatment of this same process is presented in Ref. 4).

§ 2. Perturbations induced by a particle thrusted into a black hole in the semi-relativistic treatment

Let us consider a particle with initial (asymptotic) velocity $-v_0(v_0>0)$ falling straight into a Schwarzschild black hole along the z-axis. The coordi-

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nates are taken to be the usual Schwarzschild coordinates (t, r, θ, ϕ) and $z=r\cos\theta$. (We shall use gravitational units, where $G=c=1$.) Integrating the geodesic equation we obtain the trajectory

$$-t/2M=\ln\left|\frac{x-\gamma_0^{-1}\sqrt{1+a^2x^2}}{x+\gamma_0^{-1}\sqrt{1+a^2x^2}}\right|$$
$$+[ax\sqrt{1+a^2x^2}+(2a^2-1)\ln|ax+\sqrt{1+a^2x^2}|]/a^2v_0\,, \tag{2·1}$$

where $\gamma_0=(1-v_0^2)^{-1/2}$, $a=\gamma_0 v_0$, $x=(r/2M)^{1/2}$, and M is the mass of the Schwarzschild black hole. Then the velocity $v_z=\dot{r}$ and the acceleration $\dot{v}_z=\ddot{r}$ are given, respectively, by

$$\dot{r}=-(1-1/x^2)\sqrt{1+a^2x^2}/(\gamma_0 x), \tag{2·2}$$

$$\ddot{r}=-(1-1/x^2)[(1-2a^2)-3/x^2]/(4x^2\gamma_0^2 M), \tag{2·3}$$

where the dot denotes time differentiation, and the relativistic factor $\gamma=(1-v_z^2)^{-1/2}$ is given by

$$\gamma=\gamma_0[1-(1-1/x^2)^2/x^2+a^2(2-1/x^2)/x^2]^{-1/2}\,. \tag{2·4}$$

In order to estimate the gravitational wave emission we solve the linearized Einstein equation in flat space-time. In the Lorentz gauge we have

$$\eta^{\alpha\beta}\psi_{\mu\nu,\alpha\beta}=-16\pi T_{\mu\nu}\,, \tag{2·5}$$

where $\psi_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h_\alpha{}^\alpha$ and $h_{\mu\nu}$ is the metric perturbation. This generalizes the quadrupole formula used in Ref. 3) and can in fact be applied to the case of particles moving with relativistic velocities as well. The energy-momentum tensor $T^{\mu\nu}$ of a point particle is given by

$$T^{\mu\nu}=m\int_{-\infty}^{+\infty}\frac{dz^\mu}{ds}\frac{dz^\nu}{ds}\delta^4(x-z(s))ds\,, \tag{2·6}$$

where m and s are the mass and the proper time of the particle, respectively, and $\delta^4(x-z(s))$ is the Dirac delta function. From Eqs. (2·5) and (2·6) we obtain

$$\psi^{\mu\nu}=-4m[u^\mu u^\nu/Rl_\alpha u^\alpha]_{\rm ret}\,, \tag{2·7}$$

where $u^\mu=dz^\mu/ds$ is the 4-velocity, $R=|\boldsymbol{x}-\boldsymbol{z}(s)|$ is the spatial distance between the field point and the position of the particle $z^\mu(s)$ at its retarded time $z^0(s)=x^0-R$, and $l^\alpha=(1,\boldsymbol{n})$ is the null vector joining these two points, $\boldsymbol{n}$ being given by $\boldsymbol{n}=(\boldsymbol{x}-\boldsymbol{z}(s))/R$. $\psi_{\mu\nu}$ is just the gravitational Lienard-Wiechert potential. Several features and implications of Eq. (2·7) are discussed in the Appendix.

Now, for motion along the z-axis we have

$$u^\mu=\gamma(1,0,0,v), \tag{2·8}$$

and the only non-vanishing components of $\psi_{\mu\nu}$ are

$$\psi_{00}=4m[\gamma/R(1-v\cos\theta)]_{\mathrm{ret}}\,,$$

$$\psi_{0z}=-4m[\gamma v/R(1-v\cos\theta)]_{\mathrm{ret}}\,,$$

$$\psi_{zz}=4m[\gamma v^2/R(1-v\cos\theta)]_{\mathrm{ret}}\,, \tag{2·9}$$

where $\cos\theta$ is the z-component of the unit vector $\boldsymbol{n}$. In contrast with the usual assumption that the Lorentz condition applied to Eq. (2·6) implies the motion along a flat space-time geodesic, in the following we assume that the velocity in Eq. (2·9) is given by Eq. (2·2). This assumption is valid in the limit of weak gravitational fields. For a field point which is sufficiently far away from the Schwarzschild black hole we have $r=|x|\gg|z(s)|$. Assuming that $|\gamma^2\dot{v}z/v|\ll 1$, R and $z^0(s)=x^0-R$ can be replaced by r and x^0-r respectively. Inspection of Eqs. (2·2), (2·3) and (2·4) shows that this assumption holds over the major part of the trajectory (i.e., $x\gtrsim 1.5$), if a ($=\gamma_0 v_0$) is less than the order of unity. Our treatment will apply to astrophysical systems fulfilling this condition. (In fact it is fulfilled in most of the relevant astrophysical situations.) The relevant information about gravitational waves is contained in the transverse traceless (TT) components of $\psi_{\mu\nu}$ (see the discussion given in § 4). The general form of $\psi_{\mu\nu}^{TT}$ is given in the Appendix, Eqs. (A·2) and (A·4). From these equations we obtain

$$\psi^{TT}_{\hat\theta\hat\phi}=0\,,$$

$$\psi^{TT}_{\hat\theta\hat\theta}=-\psi^{TT}_{\hat\phi\hat\phi}=\frac{2m}{r}\left[\frac{\gamma v^2\sin^2\theta}{1-v\cos\theta}\right]_{\mathrm{ret}}, \tag{2·10}$$

where $\psi^{TT}_{\hat a\hat b}=(a,\,b=\theta,\,\phi)$ are the components of the orthonormal diad basis ($e^{\hat\theta}$, $e^{\hat\phi}$). The effective energy-momentum tensor of the gravitational wave is obtained from the first derivative of $\psi^{TT}_{\mu\nu}$. From Eq. (A·3) we have

$$\psi^{TT}_{\hat\theta\hat\theta,\alpha}=-\psi^{TT}_{\hat\phi\hat\phi,\alpha}=-\frac{2m}{r}\left[\frac{\sin^2\theta\gamma v\dot{v}}{(1-v\cos\theta)^3}(1+\gamma^2(1-v\cos\theta))\,l_\alpha\right]_{\mathrm{ret}}. \tag{2·11}$$

Then

$$(T_{\mu\nu})_{GW}=\frac{1}{32\pi}\langle\overset{TT}{\psi}_{\alpha\beta,\mu}\,,\ \overset{TT}{\psi}{}^{\alpha\beta}{}_{,\nu}\rangle=\frac{m^2}{4\pi r^2}\langle A^2\,l_\mu l_\nu\rangle\,, \tag{2·12}$$

where

$$A=\sin^2\theta\gamma v\dot{v}[1+\gamma^2(1-v\cos\theta)]/(1-v\cos\theta)^3\,.$$

Since the fully relativistic calculation is done using the Regge-Wheeler-Zerilli formulation of the perturbation of the Schwarzschild geometry,[4)~7)] it is convenient to expand our results in terms of tensor spherical harmonics. From Eq.

(2·10) $\overset{TT}{\psi}_{ij}$ can be written as

$$\overset{TT}{\psi}_{ij} = \frac{2m}{r} \psi \varepsilon_{ij} , \tag{2·13}$$

where the polarization tensor ε_{ij} is defined by

$$\varepsilon_{ij} dx^i \otimes dx^j = e^{\hat{\theta}} \otimes e^{\hat{\theta}} - e^{\hat{\phi}} \otimes e^{\hat{\phi}} . \tag{2·14}$$

Then $\overset{TT}{\psi}_{ij}$ is expanded as[7)]

$$\psi^{TT}{}_{ab} = r^2 \sum_{l=2}^{\infty} f_l(r, t) N_l \left[Y_{lm|a|b} + \frac{l(l+1)}{2} Y_{lm} \gamma_{ab} \right], \tag{2·15}$$

where γ_{ab} $(a, b = \theta, \phi)$ is the metric on the unit sphere, the vertical bar (|) denotes covariant differentiation with respect to γ_{ab} and the normalization constant N_l and the expansion coefficient $f_l(r, t)$ are given by

$$N_l = [2(l-2)!/(l+2)!]^{1/2} \qquad \text{for } l \geq 2 , \tag{2·16}$$

$$\begin{aligned} f_l(r, t) &= \frac{2m}{r} N_l \int (\psi \varepsilon^{ab} Y^*_{lm|a|b}) d\Omega \\ &= \frac{2m}{r} N_l \int (\psi \varepsilon^{ab})_{|a|b} Y^*_{lm} d\Omega . \end{aligned} \tag{2·17}$$

Since ψ is independent of the azimuthal angle ϕ, Eq. (2·17) reduces to

$$f_l(r, t) = \sqrt{(2l+1)\pi}\, N_l \frac{2m}{r} \int_0^{\pi} (\psi \varepsilon^{ab})_{|a|b} P_l(\cos\theta) \sin\theta d\theta . \tag{2·18}$$

Now, since $|\gamma^2(\dot{v}z/v)| \ll 1$, this equation is further simplified to

$$f_l(r, t) = [(2l+1)]^{1/2} N_l \frac{2m\gamma v^2}{r} h_l(v), \tag{2·19}$$

where $\gamma = \gamma(t-r)$, $v = v(t-r)$ and $h_l(v)$ is

$$\begin{aligned} h_l(v) &= \int_{-1}^{1} P_l(x) \frac{d^2}{dx^2} [(1-x^2)^2/(1-vx)] dx \\ &= \left[\frac{l(l-1)}{v^2} - l(l+1) \right] \int_{-1}^{1} \frac{P_l(x)}{(1-vx)} dx + \frac{2l}{v} \int_{-1}^{1} \frac{P_{l-1}(x)}{1-vx} dx . \end{aligned} \tag{2·20}$$

It is also convenient to have the expression for $h_l(v)$ expanded in terms of v,

$$h_l(v) = 2^{l+1} \frac{(l+2)!}{(l-2)!} v^{l-2} \sum_{n=0}^{\infty} \frac{(n+l)!(2n+l-2)!}{n!(2n+2l+1)!} v^{2n} . \tag{2·21}$$

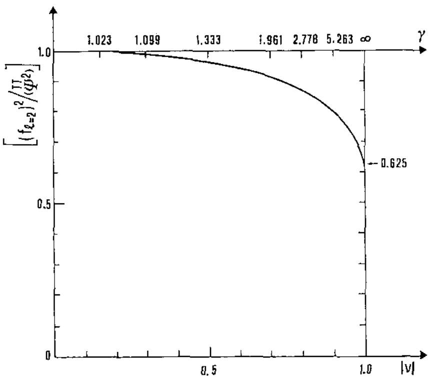

Fig. 1. The square of the ratio of the $l=2$ potential to the total potential is shown as a function of the velocity. Since we assumed $|\gamma^2 \dot{v} z/v| \ll 1$, the high velocity part of the graph is not necessarily correct. The high ratio of the $l=2$ part for $|v| \lesssim 0.5$ suggests that the quadrupole contribution is predominant in most cases.

From this formula for $h_l(v)$, it is easy to see that when $|v| \ll 1$ the only surviving term is the one for $l=2$, which is essentially equivalent to the quadrupole approximation. In any case we expect the $l=2$ term to give an important contribution to the total potential $\overset{TT}{\psi}_{ij}$ (see Fig. 1 for verification of this conjecture). Let us, therefore, concentrate our attention on this term; $h_2(v)$ is given explicitly as

$$\begin{aligned} h_2(v) &= 6\left[\left(-v+\frac{5}{3}v^3\right)+\frac{1}{2\gamma^4}\ln\left|\frac{1+v}{1-v}\right|\right]/v^5 \\ &= 48\sum_{n=0}^{\infty} v^{2n}/[(2n+1)(2n+3)(2n+5)]. \end{aligned} \tag{2·22}$$

Now we define the function R_l:

$$\begin{aligned} R_l(t-r) &= N_l f_l r \\ &= (4\pi)^{1/2}(2l+1)^{1/2}(N_l)^2 m\gamma v^2 h_l(v). \end{aligned} \tag{2·23}$$

This corresponds to the Fourier transformed radial function satisfying the Zerilli equation and is to be compared with the result of the fully relativistic treatment.[4),6)]

§ 3. Results of numerical computations

In Fig. 2, the $l=2$ component of the perturbation is given for selected values of γ_0 as a function of the retarded time $(t-r)/2M$. One of the most striking features is the strong dependence of the value of R_2 on the kinetic energy of the particle at infinity.

Since the value of the potential R_l depends directly on the value of the apparent velocity as given in Eq. (2·23), we have plotted the apparent velocity of the projected particle given by Eq. (2·2) in Fig. 3. It is shown that for γ_0

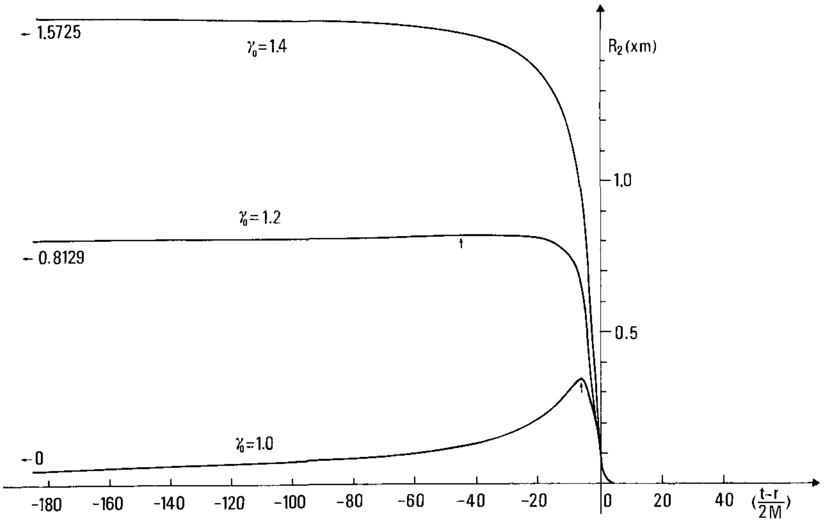

Fig. 2. The $l=2$ potentials for $\gamma_0=1.0$, 1.2 and 1.4 are shown as functions of retarded time. The arrows are the moments when the potentials become maximum.

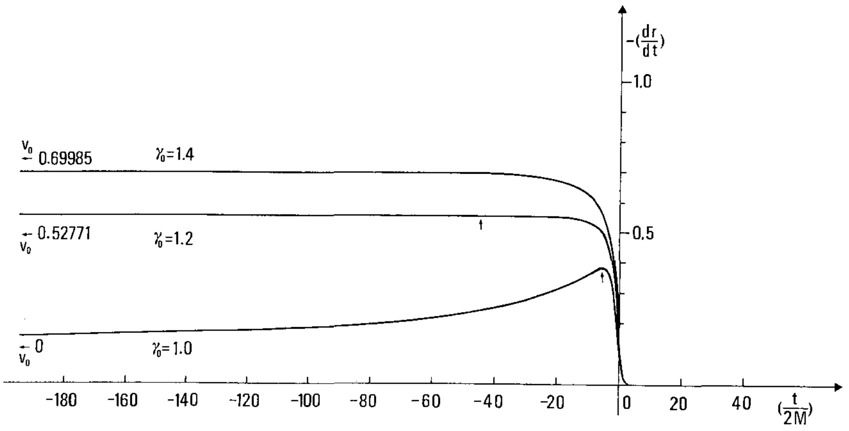

Fig. 3. The velocity is shown as a function of the Schwarzschild time t. The arrow denotes a maximum velocity for each γ_0. When $\gamma_0>(3/2)^{1/2}$, this maximum disappears as explained in the Appendix.

$>(3/2)^{1/2}$ the apparent velocity decreases monotonically with time as a consequence of relativistic time dilation (see the Appendix). For $\gamma_0<(3/2)^{1/2}$ the velocity of the particle reaches a maximum and then asymptotically approaches zero for $t\to+\infty$. Similar behaviour is found for the potential R_2.

In Fig. 4, we have plotted the relevant components of the Riemann tensor for

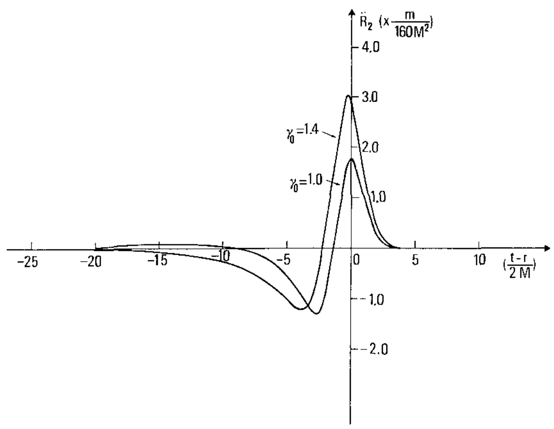

Fig. 4. $(d^2/dt^2)R_2(t-r)$ is shown for $\gamma_0=1.0$ and 1.4 as a function of the retarded time. The corresponding Riemann tensor is

$$(R^{\alpha}_{0\beta 0})_{l=2}=-\frac{1}{2r}\frac{d^2}{dt^2}R_2(t-r)$$
$$\times(Y_{20}{}^{|\alpha}{}_{|\beta}+3\delta_{\beta}{}^{\alpha}Y_{20});\quad(\alpha,\beta=\theta,\phi).$$

$\gamma_0=1.0$ and $\gamma_0=1.4$.

In the burst structure shown in these figures, we can distinguish two different parts:

(a) One part of the burst is emitted when the particle is far away from the black hole (the "precursor" of Ref. 7)) with values of $(t-r)/2M\lesssim-2.5$.

(b) The other part is emitted when the particle approaches the black hole, with values of $-2.5\lesssim(t-r)/2M\lesssim5$ (the "main burst" of Ref. 7)), in which the largest values of the Riemann tensor are obtained.

These results are compared and contrasted with the ones obtained by the fully relativistic treatment in Ref. 4). While this semi-relativistic treatment gives an excellent approximation, both of the precursor and of the main burst, it is clearly inadequate for the description of the third component of the burst (the "ringing

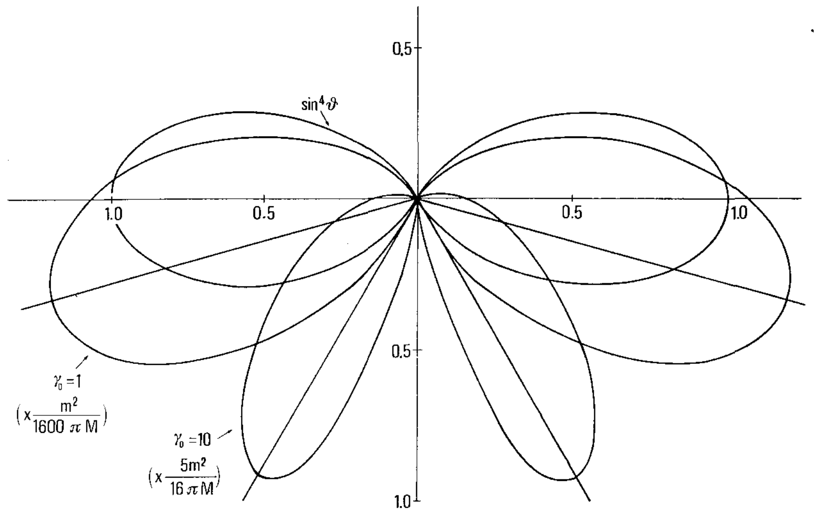

Fig. 5. The angular distribution of the total radiated energy is shown. One sees the characteristic forward beaming of the radiation. However the beaming does not actually become so sharp, because the radiation comes mainly from the "deceleration" process where the velocity eventually becomes small.

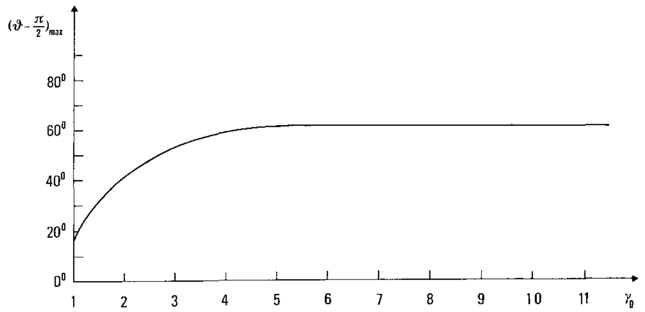

Fig. 6. The angle for which the radiation is a maximum is shown. One clearly sees the slow down of the peak angle, which converges to a finite angle even if γ_0 becomes very large.

tail" of Ref. 7)), since the oscillations of the black hole induced by the infalling particle are clearly missing in our approximation scheme.

It is also of interest to use this semi-relativistic approach in order to estimate the relative contribution of the higher multipoles as a function of γ_0 and to estimate the beaming of the radiation. The angular distribution of the radiation is given as a function of γ_0 in Fig. 5. The integration with respect to time has been carried out without assuming $|\gamma^2 \dot{v} z/v| \ll 1$. In Fig. 6, the angle for which the radiation is a maximum is shown as a function of γ_0.

The simplicity of the semi-relativistic treatment, and its good agreement with the fully relativistic treatment in the description of some components of the burst structure (see Ref. 4)), make this approximation scheme particularly useful in the analysis of a large variety of astrophysical processes in relativistic regimes.

§ 4. The observability of the radiation burst

It is well known that the metric perturbation $h_{\mu\nu}$ has a freedom of gauge transformations

$$h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} , \qquad (4\cdot1)$$

where ξ_μ are arbitrary functions, and $h_{\mu\nu}$ itself does not have any physical significance in general. However under a certain situation the direct observation of $h_{\mu\nu}$ is possible.

On flat space-time background or in a region where the background curvature changes sufficiently little, the source free part of $h_{\mu\nu}$ can be put into the TT-gauge[8)]

$$(h_{0\mu})_{TT} = 0 , \qquad (h_\alpha{}^\alpha)_{TT} = 0 , \qquad (h_{ij,j})_{TT} = 0 . \qquad (4\cdot2)$$

Then we can construct a locally flat cartesian coordinate system along the trajectory of an observer, O, which differs from TT-coordinates by $O(h_{\mu\nu})$. In this coordinate system the Jacobi equation gives

$$\left(\frac{d^2x^i}{dt^2}\right)_{GW}=-(R^i_{0j0})x^j_{GW}=\frac{1}{2}\frac{\partial^2}{\partial t^2}(h_j{}^i)x^j_{TT}\,, \tag{4·3}$$

where x^i are the coordinates of another observer in the locally flat coordinate system of O, and the subscript GW is used in order to stress that only the relative acceleration due to the gravitational wave is considered. If $(\partial/\partial t)(h_{ij})_{TT}=0$ holds initially at $t=t_0$ and no other force is present, Eq. (4·3) can easily be integrated to give

$$x^i(t)=\left\{\delta_{ij}+\frac{1}{2}[(h_{ij}(t)_{TT}-(h_{ij}(t_0))_{TT}]\right\}x^j(t_0). \tag{4·4}$$

Therefore the TT-components of the metric perturbation are physically observable. Furthermore, it is clear from Eq. (4·4) that if the initial and the final values of $(h_{ij})_{TT}$ are different the relative distance between these two observers will change permanently. In recent years several kinds of experiments have been proposed in which the direct observation of the TT-components is possible. The laboratory type free-mass detector[9] measures the change of the relative distance between suspended masses by means of a Michelson-Morley type laser interferometer. This is a simple application of Eq. (4·4). The Doppler tracking of spacecraft by radio waves suggested by Estabrook and Wahlquist[10] also gives direct information of $(h_{ij})_{TT}$, which is based on the theory of the propagation of electromagnetic wave through the gravitational field. The frequency shift due to the gravitational wave is given by

$$\left(\frac{\Delta\nu}{\nu}\right)_{GW}=[h^E/(1-\cos\theta)-2h^T\cos\theta/\sin^2\theta-h^R/(1+\cos\theta)]/2\,, \tag{4·5}$$

where θ is the angle between the propagation direction of the gravitational wave and the direction of the radio beam, and $h=n^in^j(h_{ij})_{TT}$, $\boldsymbol{n}$ being the unit vector in the direction of the radio beam. The indices E, T and R of h mean that the value of h is evaluated at the coordinates (t, x^i) of the emitter, the transponder and the receiver of a particular radio beam, respectively.

Mashhoon[11] pointed out that there is a further shift due to the relative motion of the emitter, the transponder and the receiver, which is a consequence of Eq. (4·3). He investigated the perturbation of a binary star system and gave the Doppler shift formula due to the tidal force (coming from the Riemann tensor) induced by a monochromatic gravitational wave as well as the equations governing the perturbed orbital motion of the system.

The case in which the initial and the final values of $(h_{ij})_{TT}$ are different is particularly interesting for the detection experiments mentioned above. The radiation induced by a particle thrusted into a black hole with a finite kinetic energy at infinity, examined in the previous section, is a good example of this case.

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Appendix

From Eq. (2·7), the "field strength" of the gravitational potential is

$$\psi_{\mu\nu,\rho}=\left\{-\frac{4m}{R(l_\alpha u^\alpha)^3}[(l_\alpha u^\alpha)(\dot{u}_\mu u_\nu+\dot{u}_\nu u_\mu)-u_\mu u_\nu(l_\alpha \dot{u}^\alpha)]l_\rho \right.$$
$$\left.+\frac{4m}{R^2(l_\alpha u^\alpha)^3}[u_\mu u_\nu(l_\rho+l^\alpha u_\alpha u_\rho)]\right\}_{\rm ret}, \qquad (\mathrm{A}\cdot 1)$$

where $\dot{u}^\mu=(d/ds)u^\mu$. The first and the second term in the curly brackets are the radiation and the velocity field,[12] respectively.

First, let us discuss the radiation field. The TT-part of the potential and its derivatives are given, respectively, for large R

$$\psi_{ij}^{TT}=-4m\sum_P\left[\frac{u_k u_l}{R l_\alpha u^\alpha}\overset{P}{\varepsilon}{}^{kl}\overset{P}{\varepsilon}_{ij}\right]_{\rm ret}$$
$$=4m\sum_P\left[\frac{\gamma v_k v_l \overset{P}{\varepsilon}{}^{kl}}{(1-\boldsymbol{v}\cdot\boldsymbol{n})R}\overset{P}{\varepsilon}_{ij}\right]_{\rm ret} \qquad (\mathrm{A}\cdot 2)$$

and

$$\psi_{ij,\alpha}^{TT}=-4m\sum_P\left[\left\{\frac{a_k(\delta_l{}^k+\gamma^2 v^k v_l)}{R(1-\boldsymbol{v}\cdot\boldsymbol{n})^2}+\frac{(a_l-2a_{[l}v_{k]}n^k)}{R(1-\boldsymbol{v}\cdot\boldsymbol{n})^3}\right\}\right.$$
$$\left.\times(\overset{P}{\varepsilon}{}^{lm}v_m)\overset{P}{\varepsilon}_{ij}l_\alpha\right]_{\rm ret}, \qquad (\mathrm{A}\cdot 3)$$

where $v_i=u_i/u_0=\gamma^{-1}u_i$, $a_i=dv_i/dt$ and $\overset{P}{\varepsilon}_{ij}(P=1,2)$ are two independent polarization tensors and are normalized so that $\overset{1}{\varepsilon}{}^{ij}\overset{1}{\varepsilon}_{ij}=\overset{2}{\varepsilon}{}^{ij}\overset{2}{\varepsilon}_{ij}=1$. One of the most convenient choices is

$$\overset{1}{\varepsilon}_{ij}dx^i \otimes dx^j = (e^{\hat\theta} \otimes e^{\hat\theta} - e^{\hat\phi} \otimes e^{\hat\phi})/\sqrt{2}\,,$$

$$\overset{2}{\varepsilon}_{ij}dx^i \otimes dx^j = (e^{\hat\theta} \otimes e^{\hat\phi} + e^{\hat\phi} \otimes e^{\hat\theta})/\sqrt{2}\,, \qquad (\mathrm{A}\cdot 4)$$

where ($e^{\hat\theta}$, $e^{\hat\phi}$) are the orthonormal diad basis introduced in § 2. Now from Eq. (A·2), we see that ψ_{ij}^{TT} does not vanish even for a constant velocity. Thus, the net change of velocity from the initial state to the final state of the particle will entail a net change in the value of ψ_{ij}^{TT}. It is well known that this same feature occurs in the theory of electromagnetic fields and is due to the emission of soft photons (gravitons in our case), which causes (infrared) divergence of the potential.[13]

Next, let us consider the velocity field,

$$\overset{L}{\psi}_{\mu\nu,\rho} = \left\{ \frac{4m}{R^2(u_\alpha l^\alpha)^3} [u_\mu u_\nu (l_\rho + l_\alpha u^\alpha u_\rho)] \right\}_{\mathrm{ret}} . \qquad (\mathrm{A}\cdot 5)$$

The explicit form of the potential $\overset{L}{\psi}_{\mu\nu}$ will not be needed. Since $\overset{L}{\psi}{}^{\mu\nu}{}_{,\nu} = 0$ from Eq. (A·5), the potential $\overset{L}{\psi}{}^{\mu\nu}$ satisfies the Lorentz gauge condition and there is no further freedom of the gauge transformation which might change the form of (A·5). A hypothetical observer at rest in the global Lorentz frame $\{x^\mu\}$ will experience acceleration

$$\frac{d^2x^i}{dt^2} \simeq -\Gamma^i_{00} = \delta^{ij}\left(\overset{L}{\psi}_{00,j} + \frac{1}{2}\overset{L}{\psi}_{\alpha,j} - 2\overset{L}{\psi}_{j0,0} \right)/2$$

$$= -\left\{ \frac{2m}{R^2(l_\alpha u^\alpha)^2} \left[\left(\gamma^2 - \frac{1}{2}\right) s_i + 2\gamma^2 v_i s_0 \right] \right\}_{\mathrm{ret}} , \qquad (\mathrm{A}\cdot 6)$$

where $s_\mu = -u_\mu - l_\mu/(l_\alpha u^\alpha)$. In order to compare this formula with the Newtonian force we introduce a vector $r^i = [R(n^i - v^i)]_{\mathrm{ret}}$. This spatial vector connects the point where the particle would have been at the time of observation, had its velocity been constant, to the point of the observer. Then Eq. (A·6) is rewritten as

$$\frac{d^2x^i}{dt^2} = -\frac{m}{r^2} \frac{(2\gamma^2-1)n'^i - \gamma^2(2\gamma^2+1)(\boldsymbol{n}'\cdot\boldsymbol{v})v^i}{[1-\gamma^2(\boldsymbol{n}'\cdot\boldsymbol{v})^2]^{3/2}} , \qquad (\mathrm{A}\cdot 7)$$

where $r = |\boldsymbol{r}| = R|n - v|$ and $\boldsymbol{n}' = \boldsymbol{r}/r$. Equation (A·7), for $v = 0$, gives the acceleration obtained from the traditional Newtonian gravitational force. In the general case, in which $\boldsymbol{v} \neq 0$, the acceleration of the observer has an additional contribution, which is parallel to the velocity. If the particle moves along a straight line passing through the spatial position of the observer, we have the acceleration

$$\frac{d^2x^i}{dt^2} = -[m(3-2\gamma^2)/\gamma]n'^i/r^2 . \qquad (\mathrm{A}\cdot 8)$$

This equation shows that the acceleration relative to the global Lorentz frame changes in sign if $\gamma^2 \gtrless 3/2$.

This feature can also be seen, from a different point of view, from the direct analysis of the radial geodesics equation in the Schwarzschild geometry. We have

$$\frac{d^2 r}{ds^2} = -(2M/r)^2/4M \,, \tag{A·9}$$

where s is the proper time of the particle. This acceleration does not show any peculiar feature but is actually exactly equal to the Newtonian acceleration. However if one rewrites Eq. (A·9) in terms of the Schwarzschild time t, one obtains

$$\begin{aligned}\frac{d^2 r}{dt^2} &= \left(\frac{ds}{dt}\right)^2 \left\{\frac{d^2 r}{ds^2} - \left(\frac{ds}{dt}\right)\left(\frac{d^2 t}{ds^2}\right)\frac{dr}{ds}\right\} \\ &= \left(\frac{ds}{dt}\right)^2 \left\{\frac{d^2 r}{ds^2} + \frac{2M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2\right\}.\end{aligned} \tag{A·10}$$

The factor $(ds/dt)^2$ is the well-known red shift correction factor. The second term in the curly brackets is due to the change of the red shift factor with distance and is always positive in contrast with the first term which is always negative. From Eq. (A·10) or (2·3), it follows that if the particle is falling into the black hole with initial relativistic factor $\gamma_0 > (3/2)^{1/2}$ at infinity, the second term becomes larger than the Newtonian term, Eq. (A·9). Thus, even though the particle is attracted by the black hole an observer at infinity will see the particle decelerating (see Fig. 3).

References

1) See, e.g., E. Amaldi and G. Pizzella, in *Relativity, Quanta and Cosmology,* ed., M. Pantaleo and F. de Finis (Johnson Reprint Cooperation, N.Y.—San Francisco—London, 1979), p. 9.
2) See, e.g., M. Rees, R. Ruffini and J. A. Wheeler, *Black Holes, Gravitational Waves and Cosmology* (Gordon and Breach, N.Y.—London, 1974).
3) R. Ruffini and J. A. Wheeler, in *Proceedings of the Cortona Symposium on Weak Interactions,* ed., L. Radicati (1971).
4) V. Ferrari and R. Ruffini, in preparation.
5) T. Regge and J. A. Wheeler, Phys. Rev. **108** (1957), 1063.
6) F. J. Zerilli, Phys. Rev. **D2** (1970), 2141.
7) M. Davis, R. Ruffini and J. Tiomno, Phys. Rev. **D5** (1972), 2933.
8) See, e.g., C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
9) G. Moss, L. Miller and R. Forward, Appl. Optics **10** (1971), 2495.
10) F. Estabrook and H. Wahlquist, Gen. Rel. Grav. **6** (1975), 439.
11) B. Mashhoon, Astrophys. J. **227** (1979), 1019.
12) See, e.g., J. Jackson, *Classical Electrodynamics, 2nd Ed.* (John Wiley & Sons, New York, 1975), p. 657.
13) See, e.g., J. Bjorken and S. Drell, *Relativistic Quantum Fields* McGraw-Hill, New York, 1965), p. 202.