

ON A SEQUENCE INVOLVING SUMS OF PRIMES

ZHI-WEI SUN

(Received 2 August 2012; accepted 5 October 2012; first published online 18 January 2013)

Abstract

For $n = 1, 2, 3, \dots$ let S_n be the sum of the first n primes. We mainly show that the sequence $a_n = \sqrt[n]{S_n/n}$ ($n = 1, 2, 3, \dots$) is strictly decreasing, and moreover the sequence a_{n+1}/a_n ($n = 10, 11, \dots$) is strictly increasing. We also formulate similar conjectures involving twin primes or partitions of integers.

2010 *Mathematics subject classification*: primary 11A41; secondary 05A17, 05A20, 11B75, 11B83, 11J99, 11P83.

Keywords and phrases: primes, sums of primes, monotonicity, twin primes, partitions of integers.

1. Introduction

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ let p_n denote the n th prime. The unsolved Firoozbakht conjecture (see [R, p. 185]) asserts that

$$\sqrt[n]{p_n} > \sqrt[n+1]{p_{n+1}} \quad \text{for all } n \in \mathbb{Z}^+,$$

that is, the sequence $(\sqrt[n]{p_n})_{n \geq 1}$ is strictly decreasing. This implies the inequality $p_{n+1} - p_n < \log^2 p_n - \log p_n + 1$ for large n , which is even stronger than Cramér's conjecture $p_{n+1} - p_n = O(\log^2 p_n)$. Let P_n be the product of the first n primes. Then $P_n < p_{n+1}^n$ and hence $P_n^{n+1} < P_{n+1}^n$. So the sequence $(\sqrt[n]{P_n})_{n \geq 1}$ is strictly increasing.

Now let us look at a simple example not related to primes.

EXAMPLE 1.1. Let $a_n = \sqrt[n]{n}$ for $n \in \mathbb{Z}^+$. Then the sequence $(a_n)_{n \geq 3}$ is strictly decreasing, and the sequence $(a_{n+1}/a_n)_{n \geq 4}$ is strictly increasing. To see this we investigate the function $f(x) = \log(x^{1/x}) = (\log x)/x$ with $x \geq 3$. As $f'(x) = (1 - \log x)/x^2 < 0$, we have $f(n) > f(n+1)$ for $n = 3, 4, \dots$. Since

$$f''(x) = \frac{2 \log x - 3}{x^3} > 0 \quad \text{for } x \geq 4.5,$$

Supported by the National Natural Science Foundation (grant 11171140) of China and the PAPD of Jiangsu Higher Education Institutions.

© 2013 Australian Mathematical Publishing Association Inc. 0004-9727/2013 \$16.00

the function $f(x)$ is strictly convex over the interval $(4.5, +\infty)$ and so

$$2f(n + 1) < f(n) + f(n + 2) \text{ (that is, } a_{n+1}^2 < a_n a_{n+2}) \text{ for } n = 5, 6, \dots$$

The inequality $a_5^2 < a_4 a_6$ can be verified directly.

A sequence $(a_n)_{n \geq 1}$ of nonnegative real numbers is said to be *log-convex* if $a_{n+1}^2 \leq a_n a_{n+2}$ for all $n = 1, 2, 3, \dots$. Many combinatorial sequences (such as the sequence of Catalan numbers) are log-convex; the reader may consult [LW] for some results on log-convex sequences.

For $n \in \mathbb{Z}^+$ let $S_n = \sum_{k=1}^n p_k$ be the sum of the first n primes. For instance,

$$S_1 = 2, \quad S_2 = 2 + 3 = 5, \quad S_3 = 2 + 3 + 5 = 10, \quad S_4 = 2 + 3 + 5 + 7 = 17.$$

Recently the author [S] conjectured that for any positive integer n the interval (S_n, S_{n+1}) contains a prime. As $S_n < np_{n+1}$ for all $n \in \mathbb{Z}^+$, the sequence $(S_n/n)_{n \geq 1}$ is strictly increasing.

In the next section we will state our theorems involving the sequence $(a_n)_{n \geq 1}$ with $a_n = \sqrt[n]{S_n/n}$, and pose three related conjectures for further research. Section 3 is devoted to our proofs of the theorems.

2. Our results and conjectures

THEOREM 2.1. *The sequences $(\sqrt[n]{S_n})_{n \geq 2}$ and $(\sqrt[n]{S_n/n})_{n \geq 1}$ are strictly decreasing.*

REMARK 2.2. Note that S_n/n is just the arithmetic mean of the first n primes. It is interesting to compare Theorem 2.1 with Firoozbakht’s conjecture that $(\sqrt[n]{p_n})_{n \geq 1}$ is strictly decreasing.

For $\alpha > 0$ and $n \in \mathbb{Z}^+$ define

$$S_n^{(\alpha)} = \sum_{k=1}^n p_k^\alpha.$$

We actually obtain the following extension of Theorem 2.1.

THEOREM 2.3. *Let $\alpha \geq 1$ and $n \in \mathbb{Z}^+$ with $n \geq \max\{100, e^{2 \times 1.348^\alpha + 1}\}$. Then*

$$\sqrt[n]{\frac{S_n^{(\alpha)}}{n}} > \sqrt[n+1]{\frac{S_{n+1}^{(\alpha)}}{n+1}} \tag{2.1}$$

and hence

$$\sqrt[n]{S_n^{(\alpha)}} > \sqrt[n+1]{S_{n+1}^{(\alpha)}}. \tag{2.2}$$

REMARK 2.4. In view of Example 1.1, (2.1) implies (2.2) if $n \geq 3$. We conjecture that (2.1) holds for any $\alpha > 0$ and $n \in \mathbb{Z}^+$.

Note that $\lfloor e^{2 \times 1.348^2 + 1} \rfloor = 40$ and we can easily verify that

$$\sqrt[n]{\frac{S_n}{n}} > \sqrt[n+1]{\frac{S_{n+1}}{n+1}} \quad \text{for every } n = 1, \dots, 99.$$

So Theorem 2.1 follows from Theorem 2.3 in the case $\alpha = 1$.

COROLLARY 2.5. *For each $\alpha \in \{2, 3, 4\}$, the sequences*

$$\left(\sqrt[n]{\frac{S_n^{(\alpha)}}{n}} \right)_{n \geq 1} \quad \text{and} \quad \left(\sqrt[n]{S_n^{(\alpha)}} \right)_{n \geq 1}$$

are strictly decreasing.

PROOF. Observe that

$$\lfloor e^{2 \times 1.348^2 + 1} \rfloor = 102, \quad \lfloor e^{2 \times 1.348^3 + 1} \rfloor = 364, \quad \lfloor e^{2 \times 1.348^4 + 1} \rfloor = 2005.$$

In light of Theorem 2.3 and Example 1.1, it suffices to verify that

$$\sqrt[n]{\frac{S_n^{(\alpha)}}{n}} > \sqrt[n+1]{\frac{S_{n+1}^{(2)}}{n+1}}$$

whenever $\alpha \in \{2, 3, 4\}$ and $n \in \{1, \dots, \lfloor e^{2 \times 1.348^\alpha + 1} \rfloor\}$. This can be easily done via computer. □

The following theorem is more sophisticated than Theorem 2.3.

THEOREM 2.6. *Let $\alpha \geq 1$. Then the sequence*

$$\left(\sqrt[n+1]{\frac{S_{n+1}^{(\alpha)}}{(n+1)}} / \sqrt[n]{\frac{S_n^{(\alpha)}}{n}} \right)_{n \geq N(\alpha)}$$

is strictly increasing, where

$$N(\alpha) = \max \left\{ 350000, \lceil e^{((\alpha+1)^2 1.2^{2\alpha+1} + (\alpha+1) 1.2^{\alpha+1}) / \alpha} \rceil \right\}. \tag{2.3}$$

COROLLARY 2.7. *All the sequences*

$$\begin{aligned} & \left(\sqrt[n+1]{\frac{S_{n+1}}{(n+1)}} / \sqrt[n]{\frac{S_n}{n}} \right)_{n \geq 10}, \quad \left(\sqrt[n+1]{S_{n+1}} / \sqrt[n]{S_n} \right)_{n \geq 5}, \\ & \left(\sqrt[n+1]{\frac{S_{n+1}^{(2)}}{(n+1)}} / \sqrt[n]{\frac{S_n^{(2)}}{n}} \right)_{n \geq 13}, \quad \left(\sqrt[n+1]{S_{n+1}^{(2)}} / \sqrt[n]{S_n^{(2)}} \right)_{n \geq 10}, \\ & \left(\sqrt[n+1]{\frac{S_{n+1}^{(3)}}{(n+1)}} / \sqrt[n]{\frac{S_n^{(3)}}{n}} \right)_{n \geq 17}, \quad \left(\sqrt[n+1]{S_{n+1}^{(3)}} / \sqrt[n]{S_n^{(3)}} \right)_{n \geq 10}, \\ & \left(\sqrt[n+1]{\frac{S_{n+1}^{(4)}}{(n+1)}} / \sqrt[n]{\frac{S_n^{(4)}}{n}} \right)_{n \geq 35}, \quad \left(\sqrt[n+1]{S_{n+1}^{(4)}} / \sqrt[n]{S_n^{(4)}} \right)_{n \geq 17} \end{aligned}$$

are strictly increasing.

PROOF. For $N(\alpha)$ given by (2.3), via computation

$$N(1) = 350000, \quad N(2) = 974267, \quad N(3) = 3163983273$$

and

$$N(4) = 2271069361863763.$$

Via computer we can verify that

$$\frac{\sqrt[n+1]{S_{n+1}^{(\alpha)} / (n+1)}}{\sqrt[n]{S_n^{(\alpha)} / n}} < \frac{\sqrt[n+2]{S_{n+2}^{(\alpha)} / (n+2)}}{\sqrt[n+1]{S_{n+1}^{(\alpha)} / (n+1)}}$$

for all $\alpha \in \{1, 2, 3, 4\}$ and $n = N_0(\alpha), \dots, N(\alpha) - 1$, where

$$N_0(1) = 10, \quad N_0(2) = 13, \quad N_0(3) = 17, \quad N_0(4) = 35.$$

Combining this with Theorem 2.6, we obtain that

$$\left(\sqrt[n+1]{S_{n+1} / (n+1)} / \sqrt[n]{S_n / n} \right)_{n \geq N_0(\alpha)}$$

is strictly increasing for each $\alpha = 1, 2, 3, 4$. Recall that $(\sqrt[n+1]{n+1} / \sqrt[n]{n})_{n \geq 4}$ is strictly increasing by Example 1.1. So $(\sqrt[n+1]{S_{n+1} / (n+1)} / \sqrt[n]{S_n / n})_{n \geq N_0(\alpha)}$ is strictly increasing for any $\alpha \in \{1, 2, 3, 4\}$. It remains to check that

$$\frac{\sqrt[n+1]{S_{n+1}^{(\alpha)}}}{\sqrt[n]{S_n^{(\alpha)}}} < \frac{\sqrt[n+2]{S_{n+2}^{(\alpha)}}}{\sqrt[n+1]{S_{n+1}^{(\alpha)}}}$$

for all $\alpha \in \{1, 2, 3, 4\}$ and $n = n_0(\alpha), \dots, N_0(\alpha) - 1$, where $n_0(1) = 5, n_0(2) = n_0(3) = 10$, and $n_0(4) = 17$. This can be easily done via computer. □

We conclude this section by posing three conjectures.

CONJECTURE 2.8. The two constants

$$s_1 = \sum_{n=1}^{\infty} \frac{1}{S_n} \quad \text{and} \quad s_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{S_n}$$

are both transcendental numbers.

REMARK 2.9. Our computation shows that $s_1 \approx 1.023476$ and $s_2 \approx -0.3624545778$.

If p and $p + 2$ are both primes, then they are called twin primes. The famous twin prime conjecture states that there are infinitely many twin primes.

CONJECTURE 2.10. (i) If $\{t_1, t_1 + 2\}, \dots, \{t_n, t_n + 2\}$ are the first n pairs of twin primes, then the first prime t_{n+1} in the next pair of twin primes is smaller than $t_n^{1+1/n}$, that is, $\sqrt[n]{t_n} > \sqrt[n+1]{t_{n+1}}$.

(ii) The sequence $(\sqrt[n+1]{T_{n+1}}/\sqrt[n]{T_n})_{n \geq 9}$ is strictly increasing with limit 1, where $T_n = \sum_{k=1}^n t_k$.

REMARK 2.11. Via Mathematica the author has verified that $\sqrt[n]{t_n} > \sqrt[n+1]{t_{n+1}}$ for all $n = 1, \dots, 500000$, and $\sqrt[n+1]{T_{n+1}}/\sqrt[n]{T_n} < \sqrt[n+2]{T_{n+2}}/\sqrt[n+1]{T_{n+1}}$ for all $n = 9, \dots, 500000$. Note that $t_{500000} = 115438667$.

Recall that a partition of a positive integer n is a way of writing n as a sum of positive integers with the order of addends ignored. Also, a *strict partition* of $n \in \mathbb{Z}^+$ is a way of writing n as a sum of *distinct* positive integers with the order of addends ignored. For $n = 1, 2, 3, \dots$ we denote by $p(n)$ and $p_*(n)$ the number of partitions of n and the number of strict partitions of n respectively. It is known that

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n} \quad \text{and} \quad p_*(n) \sim \frac{e^{\pi\sqrt{n/3}}}{4(3n^3)^{1/4}} \quad \text{as } n \rightarrow +\infty$$

(see [HR] and [AS, p. 826]) and hence $\lim_{n \rightarrow \infty} \sqrt[n]{p(n)} = \lim_{n \rightarrow \infty} \sqrt[n]{p_*(n)} = 1$. Here we formulate a conjecture similar to Conjecture 2.10.

CONJECTURE 2.12. For $n \in \mathbb{Z}^+$ let

$$q(n) = \frac{p(n)}{n}, \quad q_*(n) = \frac{p_*(n)}{n}, \quad r(n) = \sqrt[n]{q(n)} \quad \text{and} \quad r_*(n) = \sqrt[n]{q_*(n)}.$$

Then the sequences $(q(n+1)/q(n))_{n \geq 31}$ and $(q_*(n+1)/q_*(n))_{n \geq 44}$ are strictly decreasing, and the sequences $(r(n+1)/r(n))_{n \geq 60}$ and $(r_*(n+1)/r_*(n))_{n \geq 120}$ are strictly increasing.

REMARK 2.13. Via Mathematica we have verified the conjecture for n up to 10^5 . In light of Example 1.1, Conjecture 2.12 implies that all the sequences

$$\left(\frac{p(n+1)}{p(n)}\right)_{n \geq 25}, \quad \left(\frac{p_*(n+1)}{p_*(n)}\right)_{n \geq 32}, \quad \left(\sqrt[n]{p(n)}\right)_{n \geq 6}, \quad \left(\sqrt[n]{p_*(n)}\right)_{n \geq 9}$$

are strictly decreasing, and that the sequences $(\sqrt[n+1]{p(n+1)}/\sqrt[n]{p(n)})_{n \geq 26}$ and $(\sqrt[n+1]{p_*(n+1)}/\sqrt[n]{p_*(n)})_{n \geq 45}$ are strictly increasing. The fact that $(p(n+1)/p(n))_{n \geq 25}$ is strictly decreasing was conjectured by Chen [C] and proved by Janoski [J, pp. 7–23].

3. Proofs of Theorems 2.3 and 2.6

LEMMA 3.1. Let $\alpha \geq 1$ and $n \in \{2, 3, \dots\}$. Then

$$S_n^{(\alpha)} > 2^\alpha + \frac{n^{\alpha+1} \log^\alpha n}{\alpha + 1} \left(1 - \frac{\alpha}{(\alpha + 1) \log n}\right). \tag{3.1}$$

PROOF. It is known that $p_k \geq k \log k$ for $k = 2, 3, \dots$ (see [Ro] and [RS, (3.12)]). Thus

$$S_n^{(\alpha)} - 2^\alpha = \sum_{k=2}^n p_k^\alpha \geq \sum_{k=2}^n (k \log k)^\alpha > \sum_{k=2}^n \int_{k-1}^k (x \log x)^\alpha dx = \int_1^n (x \log x)^\alpha dx.$$

Using integration by parts,

$$\begin{aligned} \int_1^n (x \log x)^\alpha dx &= \frac{x^{\alpha+1}}{\alpha+1} \log^\alpha x \Big|_{x=1}^n - \int_1^n \left(\frac{x^{\alpha+1}}{\alpha+1} \cdot \frac{\alpha(\log x)^{\alpha-1}}{x} \right) dx \\ &= \frac{n^{\alpha+1}}{\alpha+1} \log^\alpha n - \frac{\alpha}{\alpha+1} \int_1^n x^\alpha (\log x)^{\alpha-1} dx \\ &\geq \frac{n^{\alpha+1}}{\alpha+1} \log^\alpha n - \frac{\alpha}{\alpha+1} \int_1^n x^\alpha (\log n)^{\alpha-1} dx \\ &\geq \frac{n^{\alpha+1}}{\alpha+1} \log^\alpha n - \frac{\alpha n^{\alpha+1}}{(\alpha+1)^2} (\log n)^{\alpha-1}. \end{aligned}$$

Therefore (3.1) holds. \square

LEMMA 3.2. Let $\alpha \geq 1$ and $n \in \mathbb{Z}^+$ with $n \geq 55$. Then

$$\log S_n^{(\alpha)} > (\alpha + 1) \log n. \quad (3.2)$$

PROOF. Note that $54 < e^4 < 55 \leq n$. As $\log^\alpha n > 4^\alpha = (2^\alpha)^2 \geq (\alpha + 1)^2$, by Lemma 3.1

$$S_n^{(\alpha)} > \frac{n^{\alpha+1} \log^\alpha n}{\alpha+1} \left(1 - \frac{\alpha}{\alpha+1} \right) = \frac{n^{\alpha+1}}{(\alpha+1)^2} \log^\alpha n \geq n^{\alpha+1}$$

and hence (3.2) follows. \square

PROOF OF THEOREM 2.3. It is known that

$$p_m < m(\log m + \log \log m)$$

for any $m \geq 6$ (see [RS, (3.13)] and [D, Lemma 1]). If $m \geq 101$, then

$$\frac{\log \log m}{\log m} \leq \frac{\log \log 101}{\log 101} < 0.3314$$

and hence $p_m < 1.3314m \log m$. As $n + 1 \leq 1.01n$,

$$\frac{\log(n+1)}{\log n} = 1 + \frac{\log((n+1)/n)}{\log n} \leq 1 + \frac{\log 1.01}{\log n} \leq 1 + \frac{\log 1.01}{\log 100} < 1.0022.$$

Therefore

$$p_{n+1} < 1.3314(n+1) \log(n+1) < 1.3314 \times 1.01n \times 1.0022 \log n < 1.348n \log n.$$

Combining Lemmas 3.1 and 3.2, we see that

$$\begin{aligned}
 & S_n^{(\alpha)} \left(\frac{n+1}{n^{1+1/n}} \sqrt[n]{S_n^{(\alpha)}} - 1 \right) \\
 &= S_n^{(\alpha)} (e^{(\log S_n^{(\alpha)})/n + \log(n+1) - (1+1/n)\log n} - 1) \\
 &\geq S_n^{(\alpha)} (e^{(\log S_n^{(\alpha)} - \log n)/n} - 1) \geq S_n^{(\alpha)} (e^{\alpha \log n/n} - 1) \\
 &> \frac{n^{\alpha+1} \log^\alpha n}{\alpha+1} \left(1 - \frac{\alpha}{(\alpha+1)\log n} \right) \frac{\alpha \log n}{n} \\
 &= \frac{\alpha}{\alpha+1} (n \log n)^\alpha \left(\log n - \frac{\alpha}{\alpha+1} \right) \\
 &> \frac{(n \log n)^\alpha}{2} (\log n - 1).
 \end{aligned}$$

As $(\log n - 1)/2 \geq 1.348^\alpha$, from the above

$$(n+1) \left(\frac{S_n^{(\alpha)}}{n} \right)^{1+1/n} - S_n^{(\alpha)} > (1.348n \log n)^\alpha > p_{n+1}^\alpha$$

and hence

$$\left(\frac{S_n^{(\alpha)}}{n} \right)^{(n+1)/n} > \frac{S_{n+1}^{(\alpha)}}{n+1}$$

which yields (2.1). As mentioned in Remark 2.4, (2.2) follows from (2.1). This concludes the proof. \square

PROOF OF THEOREM 2.6. Fix an integer $n \geq N(\alpha)$. For any integer $m \geq 350001$,

$$\frac{\log \log m}{\log m} \leq \frac{\log \log 350001}{\log 350001} < 0.1996$$

and hence

$$p_m < m(\log m) \left(1 + \frac{\log \log m}{\log m} \right) < 1.1996m \log m.$$

As $n \geq 350000$,

$$\frac{\log(n+1)}{\log n} = 1 + \frac{\log(1+1/n)}{\log n} \leq \frac{\log 350001}{\log 350000} < 1 + 10^{-6}.$$

Therefore

$$\begin{aligned}
 p_{n+1} &< 1.1996(n+1) \log(n+1) \\
 &< 1.1996 \times \frac{350001}{350000} n \times (1 + 10^{-6}) \log n < 1.2n \log n.
 \end{aligned}$$

Since $\log n \geq \log 350000 > 1/0.078335$, Lemma 3.1 implies that

$$S_n^{(\alpha)} > \frac{n^{\alpha+1} \log^\alpha n}{\alpha+1} (1 - 0.078335) > \frac{n^{\alpha+1} \log^\alpha n}{1.085(\alpha+1)}.$$

Therefore

$$q_n^{(\alpha)} := \frac{P_{n+1}^\alpha}{S_n^{(\alpha)}} < \frac{c_\alpha}{n}, \tag{3.3}$$

where $c_\alpha = 1.085(\alpha + 1)1.2^\alpha$.

By calculus,

$$x - \frac{x^2}{2} < \log(1 + x) < x \quad \text{for } x > 0$$

and

$$-x - x^2 < \log(1 - x) < -x \quad \text{for } 0 < x < 0.5.$$

Thus

$$\log \frac{S_{n+1}^{(\alpha)}/(n+1)}{S_n^{(\alpha)}/n} = \log\left(1 - \frac{1}{n+1}\right) + \log(1 + q_n^{(\alpha)}) < -\frac{1}{n+1} + q_n^{(\alpha)}$$

and

$$\begin{aligned} \log \frac{S_{n+2}^{(\alpha)}/(n+2)}{S_n^{(\alpha)}/n} &> \log\left(1 - \frac{2}{n+2}\right) + \log(1 + 2q_n^{(\alpha)}) \\ &> -\frac{2}{n+2} - \frac{4}{(n+2)^2} + 2q_n^{(\alpha)} - 2(q_n^{(\alpha)})^2. \end{aligned}$$

Hence

$$\begin{aligned} D_n^{(\alpha)} &:= \frac{2}{n+1} \log \frac{S_{n+1}^{(\alpha)}}{n+1} - \frac{1}{n} \log \frac{S_n^{(\alpha)}}{n} - \frac{1}{n+2} \log \frac{S_{n+2}^{(\alpha)}}{n+2} \\ &< \frac{2}{n+1} \left(\log \frac{S_n^{(\alpha)}}{n} - \frac{1}{n+1} + q_n^{(\alpha)} \right) - \frac{1}{n} \log \frac{S_n^{(\alpha)}}{n} \\ &\quad - \frac{1}{n+2} \left(\log \frac{S_n^{(\alpha)}}{n} - \frac{2}{n+2} - \frac{4}{(n+2)^2} + 2q_n^{(\alpha)} - 2(q_n^{(\alpha)})^2 \right) \\ &= \frac{-2 \log(S_n^{(\alpha)}/n)}{n(n+1)(n+2)} - \frac{2}{(n+1)^2} + \frac{2}{(n+2)^2} + \frac{4}{(n+2)^3} + \frac{2q_n^{(\alpha)}}{(n+1)(n+2)} + \frac{2(q_n^{(\alpha)})^2}{n+2}. \end{aligned}$$

Combining this with (3.2) and (3.3) and noting that $(350001/350000)n^2 \geq n(n+1)$,

$$\begin{aligned} D_n^{(\alpha)} &< \frac{-2\alpha \log n}{n(n+1)(n+2)} - \frac{2(2n+3)}{(n+1)^2(n+2)^2} + \frac{4}{(n+2)^3} \\ &\quad + \frac{2c_\alpha}{n(n+1)(n+2)} + \frac{2c_\alpha^2}{n^2(n+2)} \\ &< \frac{-2\alpha \log n}{n(n+1)(n+2)} - \frac{4}{(n+1)(n+2)^2} + \frac{4}{(n+1)(n+2)^2} \\ &\quad + \frac{2c_\alpha + 2(350001/350000)c_\alpha^2}{n(n+1)(n+2)} \\ &= \frac{2((350001/350000)c_\alpha^2 + c_\alpha - \alpha \log n)}{n(n+1)(n+2)}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{350001}{350000}c_\alpha^2 + c_\alpha &= \frac{350001}{350000} \times 1.085^2(\alpha + 1)^2 1.2^{2\alpha} + 1.085(\alpha + 1)1.2^\alpha \\ &< 1.2(\alpha + 1)^2 1.2^{2\alpha} + 1.2(\alpha + 1)1.2^\alpha \leq \alpha \log N(\alpha) \leq \alpha \log n. \end{aligned}$$

So we have $D_n^{(\alpha)} < 0$ and hence

$$\frac{\sqrt[n+1]{S_{n+1}^{(\alpha)}/(n+1)}}{\sqrt[n]{S_n^{(\alpha)}/n}} < \frac{\sqrt[n+2]{S_{n+2}^{(\alpha)}/(n+2)}}{\sqrt[n+1]{S_{n+1}^{(\alpha)}/(n+1)}}$$

as desired. \square

Acknowledgement

This work was done during the author's visit to the University of Illinois at Urbana-Champaign, and the author wishes to thank Professor Bruce Berndt for his kind invitation and hospitality.

References

- [AS] M. Abramowitz and I. A. Stegun (eds.) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th printing (Dover, New York, 1972).
- [C] W. Y. C. Chen, Recent developments on log-concavity and q -log-concavity of combinatorial polynomials. Paper presented at the 22nd Int. Conf. on Formal Power Series and Algebraic Combin. (San Francisco, 2010).
- [D] P. Dusart, 'The k th prime is greater than $k(\log k + \log \log k - 1)$ for $k \geq 2$ ', *Math. Comp.* **68** (1999), 411–415.
- [HR] G. H. Hardy and S. Ramanujan, 'Asymptotic formulae in combinatorial analysis', *Proc. Lond. Math. Soc.* **17** (1918), 75–115.
- [J] J. E. Janoski, *A Collection of Problems in Combinatorics*, PhD Thesis, Clemson University, May 2012.
- [LW] L. L. Liu and Y. Wang, 'On the log-convexity of combinatorial sequences', *Adv. in Appl. Math.* **39** (2007), 453–476.
- [R] P. Ribenboim, *The Little Book of Bigger Primes*, 2nd edn (Springer, New York, 2004).
- [Ro] J. B. Rosser, 'The n th prime is greater than $n \log n$ ', *Proc. Lond. Math. Soc.* **45** (1939), 21–44.
- [RS] J. B. Rosser and L. Schoenfeld, 'Approximate formulas for some functions of prime numbers', *Illinois J. Math.* **6** (1962), 64–94.
- [S] Z.-W. Sun, 'On functions taking only prime values', Preprint, 2012, arXiv:1202.6589.

ZHI-WEI SUN, Department of Mathematics, Nanjing University,
Nanjing 210093, PR China
e-mail: zwsun@nju.edu.cn