13. *On a Sequence of Fourier Coefficients*

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§ 1. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Let its Fourier series be

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\sum_{n=0}^{\infty}A_n(t). \tag{1.1}$$

Then the conjugate series of (1.1) is

$$\sum_{n=1}^{\infty}(b_n\cos nt-a_n\sin nt)=\sum_{n=1}^{\infty}B_n(t). \tag{1.2}$$

Let $\{p_n\}$ be a sequence such that $P_n=\sum_{k=0}^{n}p_k\neq 0$ for $n=0,1,2,\cdots$. A series $\sum_{n=0}^{\infty}a_n$ with its partial sum s_n is said to be summable (N,p_n) to sum s, if

$$\frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}s_k\to s \quad \text{as} \quad n\to\infty.$$

The $(N,p_n)(C,1)$ method is obtained by superimposing the method (N,p_n) on the Cesàro means of order one.

Throughout this paper, let $\{p_n\}$ be a sequence such that $p_n\geqq 0$, $p_n\downarrow$, $P_n\to\infty$, and we write

$$\psi(t)=f(x+t)-f(x-t)-l,$$
$$\Psi(t)=\int_0^t|\psi(u)|\,du$$

and $\tau=[1/t]$, where $[\lambda]$ is the integral part of λ.

§ 2. Varshney [9] proved that if

$$\Psi(t)=o(t/\log t^{-1}) \quad \text{as} \quad t\to+0, \tag{2.1}$$

then the sequence $\{nB_n(x)\}$ is summable $(N,1/(n+1))(C,1)$ to l/π. This was generalized by Sharma [6], Singhal [8] and Dikshit [1], respectively, as follows.

Theorem A (Sharma [6]). *If*

$$\Psi(t)=o(t) \quad as \quad t\to+0, \tag{2.2}$$

and, for some fixed δ, $0<\delta<1$,

$$\int_t^{\delta}\frac{|\psi(u)|}{u}\log\frac{1}{u}\,du=o(\log t^{-1}) \quad as \quad t\to+0, \tag{2.3}$$

then the sequence $\{nB_n(x)\}$ is summable $(N,1/(n+1))(C,1)$ to l/π.

Remark 1. (2.3) implies (2.2), because

$$\Psi(t)=\int_0^t|\psi(u)|du=\int_0^t\frac{|\psi(u)|}{u}\log\frac{1}{u}\,\frac{u}{\log u^{-1}}\,du$$
$$=-\left[\frac{u}{\log u^{-1}}\int_u^\delta\frac{|\psi(x)|}{x}\log\frac{1}{x}\,dx\right]_0^t$$
$$+\int_0^t\frac{\log u^{-1}+1}{(\log u^{-1})^2}\,du\int_u^\delta\frac{|\psi(x)|}{x}\log\frac{1}{x}\,dx$$
$$=o(t).$$

Therefore, we see that *the assumption* (2.2) *is superfluous.*

Theorem B (Singhal [8]). *Let* $\alpha(t)$ *be a positive non-decreasing function such that* $\log n=O(\alpha(P_n))$, *as* $n\to\infty$, *and* $t/\alpha(t)$ *is non-decreasing. If*

(2.4) $$\Psi(t)=o(t/\alpha(P_\tau))\quad as\quad t\to+0,$$

then the sequence $\{nB_n(x)\}$ *is summable* $(N,p_n)(C,1)$ *to* l/π.

Remark 2. We see, from the proof of Theorem 2 in [3], that *the assumption on monotone of* $\alpha(t)$ *is superfluous.*

Remark 3. Singhal has written his theorem with

$$\Psi(t)=o(t/\log t^{-1})\quad \text{as}\quad t\to+0,$$

instead of (2.4). But he has proved his theorem under the condition (2.4).

Theorem C (Dikshit [1]). *Let* $\beta(t)$ *be a non-negative non-decreasing function such that* $\beta(n)\log n=O(P_n)$, *as* $n\to\infty$. *If*

(2.5) $$\Psi(t)=o(t\beta(1/t)/P_\tau)\quad as\quad t\to+0,$$

then the sequence $\{nB_n(x)\}$ *is summable* $(N,p_n)(C,1)$ *to* l/π.

Remark 4. If Theorem C holds, then Theorem B also holds and conversely, when $\beta(1/t)/\beta(\tau)=O(1)$ as $t\to+0$, if Theorem B holds, then Theorem C also holds. (See [3], Theorem 2.)

Theorem D (Dikshit [1]). *Let* $\gamma(t)$ *be a non-negative function such that* (i) $\{\gamma(t)/(t\log t)\}$ *is monotonic,* (ii) $\gamma(t)=O(\log t)$, $t\to\infty$ *and* (iii)

(2.6) $$\sum_{k=2}^{n}\gamma(k)P_k/(k\log k)=O(P_n),\quad as\quad n\to\infty.$$

If

(2.7) $$\Psi(t)=o(t\gamma(1/t)/\log t^{-1})\quad as\quad t\to+0,$$

then the sequence $\{nB_n(x)\}$ *is summable* $(N,p_n)(C,1)$ *to* l/π.

Remark 5. After the proof of Theorem 2 below, we see that *the condition* (ii) *is superfluous.* And Theorem in which $\gamma(t)=1$ is due to Singh [7].

The main result of this paper is the following theorem which includes all the results mentioned above.

Theorem 1. *Let* $p(t)$ *be monotone non-increasing and positive for* $t\geqq 0$. *Let* $p_n=p(n)$ *and let*

(2.8) $$P(t)=\int_0^t p(u)du\to\infty\quad as\quad t\to+\infty.$$

If, for some fixed $\delta, 0<\delta<1$,

$$\int_{1/n}^{\delta} \Psi(t)\frac{d}{dt}\frac{P(1/t)}{t}dt=o(P_n) \quad as \quad n\to\infty, \tag{2.9}$$

then the sequence $\{nB_n(x)\}$ *is summable* $(N,p_n)(C,1)$ *to* l/π.

Theorem 2. *If Theorem* 1 *holds, then Theorems A, B, C and D also hold.*

Corollary 1. *Let* $\alpha(t)$ *be a positive function such that*

$$\int_{1/n}^{\delta}\frac{P_\tau}{\alpha(P_\tau)}\frac{1}{t}dt=O(P_n), \qquad 0<\delta<1.$$

If

$$\Psi(t)=o(t/\alpha(P_\tau)) \quad as \quad t\to+0,$$

then the sequence $\{nB_n(x)\}$ *is summable* $(N,p_n)(C,1)$ *to* l/π.
This is contained in the result of Theorem 1 (see [3], pp. 305–306), but this includes Theorem B.

Corollary 2. *Let* $\beta(t)$ *be a positive integrable function such that*

$$\int_{\eta}^{n}u^{-1}\beta(u)du=O(P_n) \quad as \quad n\to\infty, \quad for\ any\ fixed\ \eta>0.$$

If

$$\Psi(t)=o(t\beta(1/t)/P_\tau) \quad as \quad t\to+0,$$

then the sequence $\{nB_n(x)\}$ *is summable* $(N,p_n)(C,1)$ *to* l/π.
This is also contained in the result of Theorem 1, but this includes Theorem C (see [2], p. 450). Further we see that this includes Corollary 1 and conversely, when $\beta(1/t)\sim\beta(\tau)$ as $t\to+0$, Corollary 1 includes Corollary 2.

§ 3. Proof of Theorem 1. From the method of Mohanty and Nanda [4], we have

$$\frac{1}{n}\sum_{k=1}^{n}kB_k(x)-\frac{l}{\pi}=\frac{1}{\pi}\int_0^{\delta}\psi(t)\left\{\frac{\sin nt}{nt^2}-\frac{\cos nt}{t}\right\}dt+o(1),$$

by Riemann-Lebesgue's theorem. Since the (N,p_n) method is regular, in order to prove our theorem it is sufficient to show that

$$\begin{aligned}\frac{1}{P_n}\sum_{k=1}^{n}p_{n-k}\frac{1}{\pi}\int_0^{\delta}\psi(t)\left\{\frac{\sin kt}{kt^2}-\frac{\cos kt}{t}\right\}dt \\ =\int_0^{\delta}\psi(t)N_n(t)dt=o(1),\end{aligned} \tag{3.1}$$

where

$$N_n(t)=\frac{1}{\pi P_n}\sum_{k=1}^{n}p_{n-k}\left\{\frac{\sin kt}{kt^2}-\frac{\cos kt}{t}\right\}$$

$$=\begin{cases}O(n) & (0<t\leq 1/n) \quad (3.2)\\ O(P_\tau/tP_n) & (0<t\leq\pi) \quad (3.3)\end{cases}.$$

(For these estimations, see Singh's [7], p. 441 or Singhal's [8], pp. 273–274.) Now we write

$$\int_0^{\delta}\psi(t)N_n(t)dt=\left(\int_0^{1/n}+\int_{1/n}^{\delta}\right)\psi(t)N_n(t)dt=I_1+I_2, \text{ say}.$$

From Rajagopal's lemma ([5], Lemma (a)), (2.8) and (2.9) together imply

$$\Psi(t)=o(t) \quad \text{as} \quad t\to+0.$$

Hence, by (3.2), we have

$$I_1=O(n)\left(\int_0^{1/n}|\psi(t)|\,dt\right)=o(1).$$

Next, by (3.3) and (2.9), we have

$$\begin{aligned} I_2&=O\left(\int_{1/n}^{\delta}|\psi(t)|\frac{P_\tau}{tP_n}\,dt\right)=O\left(\int_{1/n}^{\delta}|\psi(t)|\,\frac{P(1/t)}{tP_n}\,dt\right)\\ &=O\left(\left[\Psi(t)\frac{P(1/t)}{tP_n}\right]_{1/n}^{\delta}\right)+O\left(\frac{1}{P_n}\int_{1/n}^{\delta}\Psi(t)\left|\frac{d}{dt}\,\frac{P(1/t)}{t}\right|dt\right)\\ &=o(1). \end{aligned}$$

Therefore we have (3.1) and the proof of Theorem 1 is complete.

§ 4. Proof of Theorem 2. For the proof, since, by Remark 4, Theorem C includes Theorem B, it is sufficient to prove that the conditions of Theorems A, C and D, respectively, imply the one of Theorem 1. We define a function $p(t)$ by

$$p(t)=p_n \quad \text{for } n\leqq t<n+1, \quad n=0,1,2,\cdots.$$

Further define a function $P(t)$ as in (2.8). Then the function $p(t)$ is monotone non-increasing and positive for $t\geqq 0$. And $P(t)\to\infty$ as $t\to\infty$. Furthermore, we get, $P_\tau\sim P(1/t)$ and $1/tp(1/t)\sim P(1/t)$ as $t\to+0$. Here, we want to show that the condition (2.9) is satisfied under the assumption.

Concerning Theorem A, since $(P(1/t)/t)'=O(P(1/t)/t^2)$ $=O((\log t^{-1})/t^2)$, we have

$$\begin{aligned} \int_{1/n}^{\delta}\Psi(t)\frac{d}{dt}\,\frac{P(1/t)}{t}\,dt&=O\left(\int_{1/n}^{\delta}\Psi(t)\,\frac{\log t^{-1}}{t^2}\,dt\right)\\ &=O\left(\int_{1/n}^{\delta}\frac{\log t^{-1}}{t^2}\,dt\int_0^t|\psi(u)|\,du\right)\\ &=O\left(\int_{1/n}^{\delta}|\psi(u)|\,du\int_u^{\delta}\frac{\log t^{-1}}{t^2}\,dt+\int_0^{1/n}|\psi(u)|\,du\int_{1/n}^{\delta}\frac{\log t^{-1}}{t^2}\,dt\right)\\ &=O\left(\int_{1/n}^{\delta}|\psi(u)|\log\frac{1}{u}\,du\int_u^{\delta}\frac{1}{t^2}\,dt\right)+O\left(\int_0^{1/n}|\psi(u)|\,du(\log n)\int_{1/n}^{\delta}\frac{1}{t^2}\,dt\right)\\ &=O\left(\int_{1/n}^{\delta}\frac{|\psi(u)|}{u}\log\frac{1}{u}\,du\right)+O(\Psi(1/n)n\log n)\\ &=o(\log n)\\ &=o(P_n), \end{aligned}$$

from (2.3) and (2.2). Thus the condition (2.9) is satisfied.

Next, concerning Theorem C, we have, from (2.5),

$$\begin{aligned} \int_{1/n}^{\delta}\Psi(t)\frac{d}{dt}\,\frac{P(1/t)}{t}\,dt&=O\left(\int_{1/n}^{\delta}\Psi(t)\frac{P(1/t)}{t^2}\,dt\right)\\ &=o\left(\int_{1/n}^{\delta}\frac{t\beta(1/t)}{P_\tau}\,\frac{P(1/t)}{t^2}\,dt\right) \end{aligned}$$

$$=o\left(\int_{1/n}^{\delta}\frac{1}{t}\beta\left(\frac{1}{t}\right)dt\right)$$
$$=o(\beta(n)\log n)$$
$$=o(P_n),$$

since the function $\beta(t)$ is non-decreasing. Thus the condition (2.9) is satisfied.

Lastly, concerning Theorem D, we have, from (2.7) and (2.6),

$$\int_{1/n}^{\delta}\Psi(t)\frac{d}{dt}\frac{P(1/t)}{t}dt=O\left(\int_{1/n}^{\delta}\Psi(t)\frac{P(1/t)}{t^2}dt\right)$$
$$=o\left(\int_{1/n}^{\delta}\frac{t\gamma(1/t)}{\log t^{-1}}\frac{P(1/t)}{t^2}dt\right)$$
$$=o\left(\int_{1/n}^{\delta}\frac{\gamma(1/t)P(1/t)}{t\log t^{-1}}dt\right)$$
$$=o\left(\int_{1/\delta}^{n}\frac{\gamma(u)P(u)}{u\log u}du\right)$$
$$=o\left(\sum_{k=2}^{n}\frac{\gamma(k)P_k}{k\log k}\right)$$
$$=o(P_n),$$

since the function $\gamma(t)/(t\log t)$ is monotone. Thus the condition (2.9) is satisfied.

Therefore, the proof of Theorem 2 is complete.

§ 5. The proof of Theorem 2 together with known theorems ([5], Theorem 1 and [3], Theorem 1) shows that the following theorems are obtained. In the rest of this paper, we use the following notations.

$$g(t)=\frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\},\qquad G(t)=\int_0^t|g(u)|\,du,$$

$$h(t)=\frac{1}{2}\{f(x+t)-f(x-t)\},\qquad H(t)=\int_0^t|h(u)|\,du.$$

Theorem 3. *Under the conditions of Theorem A, if we replace* $\psi(t)$ *by* $g(t)$ *and* $\Psi(t)$ *by* $G(t)$, *then the series* $\sum_{n=0}^{\infty}A_n(x)$ *is summable* $(N,1/(n+1))$ *to* $f(x)$.

Theorem 4. *Under the conditions of Theorem A, if we replace* $\psi(t)$ *by* $h(t)$ *and* $\Psi(t)$ *by* $H(t)$, *then the series* $\sum_{n=1}^{\infty}B_n(x)$ *is summable* $(N,1/(n+1))$ *to*

$$\bar{f}(x)=\frac{1}{\pi}\int_0^{\pi}h(u)\cot\frac{u}{2}du \tag{5.1}$$

provided that this integral exists as a Cauchy integral at origin.

Theorem 5. *Under the conditions of Theorem D, if we replace* $\Psi(t)$ *by* $G(t)$, *then the series* $\sum_{n=0}^{\infty}A_n(x)$ *is summable* (N,p_n) *to* $f(x)$.

Theorem 6. *Under the conditions of Theorem D, if we replace* $\Psi(t)$

by $H(t)$, then the series $\sum_{n=1}^{\infty} B_n(x)$ is summable (N, p_n) to $\bar{f}(x)$ provided that the integral in (5.1) *exists as a Cauchy integral at origin.*

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