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ON A SINGULAR NONLINEAR ELLIPTIC BOUNDARY-VALUE PROBLEM

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ABSTRACT. We consider the singular boundary-value problem $\Delta u + p(x)u^{-\gamma} = 0$ in Ω, $u \mid \partial\Omega = 0$, where $\gamma > 0$. Under the assumption $p(x) > 0$ and certain smoothness assumptions, we show that there exists a solution which is smooth on Ω and continuous on $\overline{\Omega}$.

1. INTRODUCTION

In this paper we consider the singular boundary-value problem

$$\begin{cases} \Delta u(x) + p(x)u(x)^{-\gamma} = 0 & x \in \Omega \\ u \mid \partial\Omega = 0, \end{cases} \tag{1}$$

where Ω is a sufficiently regular bounded domain in $\mathbf{R}^N$, $N \geq 1$, and p is a sufficiently regular function which is positive in $\overline{\Omega}$. In the case $N = 1$, this problem arises in certain problems in fluid mechanics and pseudoplastic flow [6], [7]. The N-dimensional problem (1) has been studied in [1] for general regions and, in [2], under the assumption that Ω is the open unit ball in $\mathbf{R}^N$ and $p(x) = q(|x|)$, where q is a continuous function which is defined continuous and nonnegative on $[0, 1)$. In [1], it is shown that solutions exist if Ω is C^3, and estimates are given for the behavior of the solution as the boundary of Ω is approached. In particular, if $\gamma > 1$, it is shown that solutions fail to be in $C^1(\overline{\Omega})$.

Actually, the authors in [1] prove more general results for the existence of solutions, but the above is the case where behavior near the boundary is studied. In [1], the results are divided into two sections; first, the existence of solutions is proved, by an upper-lower solution method, and later, in an extremely complicated way, using localization near the boundary, the boundary behavior is deduced.

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In this note we show that if Ω has a regular boundary, p is regular on $\overline{\Omega}$, and γ is any positive number, we can give a unified simple proof that there is a unique solution of (1), positive on Ω, which is in $C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$. We emphasize that there is *no* restriction on the shape of Ω. We also give a necessary and sufficient condition that this solution have a finite Dirichlet integral.

This unified proof is made possible by the choice of new upper and lower solutions.

In this sense, we show that equation (1) can have a classical solution but not a weak solution. Finally, we briefly discuss two cases not covered by the results of [1], namely, the case where Ω can have corners, such as a square, and the case where $p(x)$ is not assumed to be strictly positive on $\overline{\Omega}$.

Theorem 1. *Let $\Omega \subset \mathbf{R}^N$, $N \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$ (of class $C^{2+\alpha}$, $0 < \alpha < 1$). If $p \in C^{\alpha}(\overline{\Omega})$, $p(x) > 0$ for all $x \in \overline{\Omega}$ and $\gamma > 0$, then there exists a unique function $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ such that $u(x) > 0$ for all $x \in \Omega$ and u is a solution of (1). If ϕ_1 denotes an eigenfunction corresponding to the smallest eigenvalue λ_1 of the problem $\Delta\phi + \lambda\phi = 0$, $\phi \mid \partial\Omega = 0$ such that $\phi_1(x) > 0$ on Ω and $\gamma > 1$, then there exist positive constants b_1 and b_2 such that $b_1\phi_1(x)^{2/(1+\gamma)} \leq u(x) \leq b_2\phi_1(x)^{2/(1+\gamma)}$ on $\overline{\Omega}$.*

The proof of this theorem is the main result of [1].

Theorem 2. *The solution u of Theorem 1 is in $W^{1,2}$ if and only if $\gamma < 3$. If $\gamma > 1$, then u is not in $C^1(\overline{\Omega})$.*

2. Proof of Theorem 1

As is well known,

$$\nabla\phi_1(x) \neq 0, \quad \forall x \in \partial\Omega.$$

Assume first that $\gamma > 1$. In this case, let $t = 2/(1+\gamma)$ and let $\Psi(x) = b\phi_1(x)^t$, where $b > 0$ is a constant. We have that

$$\Delta\Psi(x) + q(x, b)\Psi(x)^{-\gamma} = 0, \ x \in \Omega,$$

where $q(x, b) = b^{1+\gamma}[t(1-t) \mid \nabla\phi_1(x)|^2 + t\lambda_1\phi_1(x)^2]$. Since $0 < t < 1$, it follows from (2) that we can choose numbers b_1 and b_2 with $0 < b_1 < b_2$ such that

$$q(x, b_1) < p(x) < q(x, b_2), \quad \forall x \in \overline{\Omega}. \tag{3}$$

For $k = 1, 2$, let $u_k(x) = b_k\phi_1(x)^t$. Since for $x \in \Omega$

$$\Delta u_k(x) + p(x)u_k(x)^{-\gamma} = [p(x) - q(x, b_k)]u_k^{-\gamma}(x),$$

it follows from (3) that

$$\Delta u_1(x) + p(x)u_1(x)^{-\gamma} > 0$$

and

$$\Delta u_2(x) + p(x)u_2(x)^{-\gamma} < 0$$

for all $x \in \Omega$.

We claim that if u is continuous on $\overline{\Omega}$, smooth on Ω, and satisfies (1) and $u(x) > 0$ on Ω, then

$$u_1(x) \le u(x) \le u_2(x)$$

for all $x \in \overline{\Omega}$. Indeed, if the first inequality did not hold, then there would exist an x_0 in Ω such that $0 < u(x_0) < u_1(x_0)$ and the minimum of the continuous functions $u - u_1$ on $\overline{\Omega}$ is assumed at x_0. But according to the above, this would imply that

$$\Delta(u - u_1)(x_0) < p(x_0)[u_1(x_0)^{-\gamma} - u(x_0)^{-\gamma}] < 0,$$

which is impossible since x_0 is a point of minimum. This contradiction proves the first inequality, and the proof of the second is similar.

For $0 < \gamma$, let $u_* = \varepsilon\phi_1$, where $\varepsilon > 0$ is a small positive number to be determined. If $\delta > 0$, then, for $x \in \Omega$,

$$\Delta u_*(x) + p(x)[u_*(x) + \delta]^{-\gamma} = p(x)(\varepsilon\phi_1(x) + \delta)^{-\gamma} - \varepsilon\lambda_1\phi_1(x).$$

Therefore, since $\phi_1 \mid \partial\Omega = 0$, we may choose $\varepsilon > 0$ and $\delta_0 > 0$ such that if $0 < \delta \le \delta_0$, then

$$\Delta u_*(x) + p(x)[u_*(x) + \delta]^{-\gamma} > 0, \quad \forall x \in \overline{\Omega}. \tag{4}$$

If $\gamma > 1$, we let $u^*(x) = u_2(x) = b_2\phi_1(x)^t$. Clearly, if δ_0 is as above,

$$\Delta u^*(x) + p(x)[u^*(x) + \delta]^{-\gamma} < 0, \quad \forall x \in \Omega. \tag{5}$$

If $\gamma > 1$, we also suppose that $\varepsilon > 0$ is chosen so that $u_*(x) = \varepsilon\phi_1(x) < u_2(x) = u^*(x)$.

If $0 < \gamma \le 1$, let s be chosen to satisfy the two inequalities

$$0 < s < 1, \quad s(1 + \gamma) < 2. \tag{6}$$

Let $u^*(x) = c\phi_1(x)^s$ where c is a large positive constant to be chosen. For $x \in \Omega$ we have

$$\begin{aligned} &\Delta u^*(x) + p(x)u^*(x)^{-\gamma} \\ &\quad = -\phi_1(x)^{s-2}[|\nabla\phi_1(x)|^2 cs(1 - s) - p(x)c^{-\gamma}\phi_1(x)^{2-(1+\gamma)s}] - c\lambda_1 s\phi_1(x)^s. \end{aligned}$$

Since the inequalities (6) hold, we can choose $c > 0$ so large that $\Delta u^*(x) + p(x)u^*(x)^{-\gamma} < 0$ for all $x \in \Omega$. Therefore, if δ_0 is as above, then (5) holds for $0 < \delta \le \delta_0$.

Since $0 < s < 1$ and $\phi_1(x) = 0$ for $x \in \partial\Omega$, we can assume c is so large that $\varepsilon\phi_1(x) < c\phi_1(x)^s$. It follows that with both the definitions of $u_*(x)$ and $u^*(x)$ given for $\gamma > 1$ and the definitions of these functions when $0 < \gamma \le 1$, we have

$$0 < u_*(x) < u^*(x), \quad \forall x \in \Omega. \tag{7}$$

Let δ be a fixed number, with $0 < \delta < \delta_0$, and let $k > 0$ be so large that the function $f(x, \xi) = k\xi + p(x)[\delta + \xi]^{-\gamma}$ is strictly increasing in ξ for $0 \le \xi \le M = \max\{u^*(x) | x \in \overline{\Omega}\}$ and $x \in \overline{\Omega}$. Let $w(x)$ be a smooth function such that

$$(8) \qquad \begin{cases} -\Delta w(x) + kw(x) = f(x, u^*(x)), & x \in \Omega \\ w \mid \partial\Omega = 0. \end{cases}$$

Since, according to (6), $-\Delta u^*(x) + ku^*(x) > f(x, u^*(x))$ for all $x \in \Omega$, it follows that $-\Delta(u^* - w)(x) + k(u^*(x) - w(x)) > 0$ for all $x \in \Omega$. Therefore, since $(u^* - w) \mid \partial\Omega = 0$ and $u^* - w \in C(\overline{\Omega}) \cap C^2(\Omega)$, it follows that $u^*(x) - w(x) > 0$ for all $x \in \Omega$. Hence, it follows from (8) that

$$(9) \qquad \begin{cases} -\Delta w(x) + kw(x) > f(x, w(x)), & \forall x \in \Omega \\ w \mid \partial\Omega = 0. \end{cases}$$

According to (5), we have

$$-\Delta u_*(x) + ku_*(x) < f(x, u_*(x)), \quad \forall x \in \Omega$$
$$u_* \mid \partial\Omega = 0.$$

By the same type of argument as given above, it follows that if $v(x)$ is a smooth function such that

$$-\Delta v(x) + kv(x) = f(x, u_*(x)), \quad x \in \Omega$$
$$v \mid \partial\Omega = 0,$$

then $u_*(x) < v(x)$ for $x \in \Omega$, so

$$(10) \qquad \begin{cases} -\Delta v(x) + kv(x) < f(x, v(x)) \\ v \mid \partial\Omega = 0. \end{cases}$$

Since

$$-\Delta(w - v) + k(w - v) = f(x, u^*) - f(x, u_*) > 0,$$

we have

$$(11) \qquad v(x) < w(x), \quad \forall x \in \Omega.$$

Since u and w are smooth on $\overline{\Omega}$, it follows from (9), (10), and (11) and the basic result on the method of subsolutions and supersolutions [4] that there exists a smooth function z defined on $\overline{\Omega}$ such that

$$\begin{cases} -\Delta z + kz = f(x, z) \text{ in } \Omega \\ z \mid \partial\Omega = 0, \end{cases}$$

and $u_*(x) < v(x) \le z(x) \le w(x) < u^*(x)$ for $x \in \Omega$. This means that

$$(12) \qquad \begin{cases} \Delta z(x) + p(x)[z(x) + \delta]^{-\gamma} = 0, & x \in \Omega \\ z \mid \partial\Omega = 0. \end{cases}$$

Let $\{\delta_n\}_1^\infty$ be a sequence of numbers such that $0 < \delta_{n+1} < \delta_n < \delta_0$ for all $n \geq 1$ and, for $n \geq 1$, let $Z_n(x)$ be a smooth positive solution of (12) when $\delta = \delta_n$ such that $u_*(x) < Z_n(x) < u^*(x)$ on Ω. From (12) we have that

$$\Delta Z_n(x)+p(x)[\delta_{n+1}+Z_n(x)]^{-\gamma} > \Delta Z_n(x)+p(x)[\delta_n+Z_n(x)]^{-\gamma} = 0 \text{ for all } x \in \Omega.$$

We claim that $Z_{n+1}(x) > Z_n(x)$ for all $x \in \Omega$. Assuming the contrary, it would follow that, since $(Z_{n+1} - Z_n) \mid \partial\Omega = 0$, there would be a point $x_0 \in \Omega$ where $Z_n - Z_{n+1}$ assumes a nonnegative maximum. But, from the above,

$$\Delta(Z_n - Z_{n+1})(x_0) > p(x_0)([\delta_{n+1} + Z_{n+1}(x_0)]^{-\gamma} - [\delta_{n+1} + Z_n(x_0)]^{-\gamma}) \geq 0,$$

which is a contradiction.

Since $Z_n(x) < Z_{n+1}(x) < u^*(x)$ for all $x \in \overline{\Omega}$, $\lim_{n\to\infty} Z_n(x) \equiv u(x)$ exists for all $x \in \overline{\Omega}$ and

$$u_*(x) \leq u(x) \leq u^*(x) \tag{13}$$

for $x \in \overline{\Omega}$. We claim that $u \in C^{2+\alpha}(\Omega)$ and that

$$\Delta u(x) + p(x)u(x)^{-\gamma} = 0, \qquad \forall x \in \Omega. \tag{14}$$

Although this follows from more or less standard arguments, we sketch the details.

Let $x_0 \in \Omega$ and let $r > 0$ be chosen so that $\overline{B(x_0, r)} \subset \Omega$, where $B(x_0, r)$ denotes the open ball of radius r centered at x_0. Let Ψ be a C^∞ function which is equal to 1 on $\overline{B(x_0, r/2)}$ and equal to 0 off $B(x_0, r)$. We have

$$\Delta(\Psi Z_n) = 2\nabla\Psi \cdot \nabla Z_n + p_n$$

for $n \geq 1$, where p_n is a term whose L^∞ norm is bounded independently of n. Therefore, for $n \geq 1$, we have

$$\Psi Z_n \Delta(\Psi Z_n) = \sum_{j=1}^{N} b_{nj} \frac{\partial(\Psi Z_n)}{\partial x_j} + q_n,$$

where b_{nj}, $j = 1, \ldots, n$, and q_n are terms bounded independently of n for $n \geq 1$. Integrating the above equation, we have that there exist constants $c_1 > 0$ and $c_2 > 0$, independent of n, such that

$$\int_{B(x_0, r)} |\nabla\Psi Z_n|^2\, dx \leq c_1 \left(\int_{B(x_0, r)} |\nabla\Psi Z_n|^2\, dx \right)^{1/2} + c_2.$$

From this, it follows that the $L^2(B(x_0, r))$-norm of $|\nabla\Psi Z_n|$ is bounded independently of n. Hence, the $L^2(B(x_0, r/2))$-norm of $|\nabla Z_n|$ is bounded independently of n. Let Ψ_1 be a C^∞ function which is equal to 1 on $\overline{B(x_0, r/4)}$ and equal to 0 off $B(x_0, r/2)$. We have, for $n \geq 1$, $\Delta(\Psi_1 Z_n) = 2\nabla\Psi_1 \cdot \nabla Z_n + p_{1n}$, where p_{1n} is a term whose $L^\infty(B(x_0, r/2))$-norm is bounded independently of n. From standard elliptic theory, the $W^{2,2}(B(x_0, r/2))$-norm of $\Psi_1 Z_n$ is bounded independently of n and hence, the $W^{2,2}(B(x_0, r/4))$-norm of Z_n is bounded independently of n. Since the $W^{1,2}(B(x_0, r/4))$-norms of the components of ∇Z_n are bounded independently of n, it follows from the Sobolev imbedding theorem that, if $q = 2N/(N-2) > 2$ if $N > 2$ and $q > 2$ is arbitrary if $N \leq 2$, then the $L^q(B(x_0, r/4))$-norm of $|\nabla Z_n|$ is bounded independently of n. If Ψ_2 is a C^∞ function which is equal to 1 on $\overline{B(x_0, r/8)}$ and equal to 0 off $B(x_0, r/4)$, then $\Delta\Psi_2 Z_n = 2\nabla\Psi_2 \cdot \nabla Z_n + p_{2n}$ where p_{2n} is bounded independently of n in $L^\infty(B(x_0, r/4))$. Since the right-hand side of the above equation is bounded in $L^q(B(x_0, r/4))$, independently of n, the $W^{2,q}(B(x_0, r/4))$-norm of $\Psi_2 Z_n$ is also bounded independently of n. Hence, the $W^{2,q}(B(x_0, r/8))$-norm of Z_n is bounded independently of n. Continuing the line of reasoning, after a finite number of steps, we find a number $r_1 > 0$ and $q_1 > N/(1-\alpha)$ such that the $W^{2,q_1}(B(x_0, r_1))$-norm of Z_n is bounded independently of n. Hence, there is a subsequence of $\{Z_n\}_1^\infty$, which we may assume is the sequence itself, which converges in $C^{1+\alpha}\overline{(B(x_0, r_1))}$. If θ is a C^∞ function which is equal to 1 on $\overline{B(x_0, r_1/2)}$ and equal to 0 off $B(x_0, r_1)$, then

$$\Delta(\theta Z_n) = 2\nabla\theta \cdot \nabla Z_n + \widehat{p}_n, \quad \text{where } \widehat{p}_n = \theta\Delta Z_n + Z_n\Delta\theta.$$

The right-hand side of the above equation converges in $C^\alpha\overline{(B(x_0, r_1))}$. So, by Schauder theory, $\{\theta Z_n\}_1^\infty$ converges in $C^{2+\alpha}\overline{(B(x_0, r_1))}$ and hence $\{Z_n\}_1^\infty$ converges in $C^{2+\alpha}\overline{(B(x_0, r_1/2))}$. Since $x_0 \in \Omega$ was arbitrary, this shows that $u \in C^{2+\alpha}(\Omega)$. Clearly, (14) holds.

Since $u_*(x) \leq u(x) \leq u^*(x)$ for $x \in \overline{\Omega}$ and $u_* \mid \partial\Omega = u^* \mid \partial\Omega = 0$, if $x_1 \in \partial\Omega$, then $\lim_{x \to x_1} u(x) = 0 = u(x_1)$. Since u is continuous at each interior point of Ω, $u \in C(\overline{\Omega})$.

To prove the uniqueness of u, suppose that $\widehat{u}$ is also a function in $C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ which is positive on Ω such that $\Delta\widehat{u} + p(x)\widehat{u}^{-\gamma} = 0$ on Ω and $\widehat{u} \mid \partial\Omega = 0$. If $\widehat{u} \not\equiv u$, then we may assume that $\widehat{u} - u$ assumes a positive value somewhere in Ω. This implies that $\widehat{u} - u$ attains a positive maximum at a point $x_0 \in \Omega$. But $\Delta(\widehat{u} - u)(x_0) = p(x_0)[u(x_0)^{-\gamma} - \widehat{u}(x_0)^{-\gamma}] > 0$, which is a contradiction. Hence $u \equiv \widehat{u}$. This concludes the proof of Theorem 1.

3. Proof of Theorem 2

To prove this theorem, we use the following:

Lemma.

$$\int_\Omega \phi_1^r \, dx < \infty$$

if and only if $r > -1$.

Proof. Let $x_0 \in \partial\Omega$. By the smoothness of $\partial\Omega$, we may assume that $x_0 = 0$ and that there exists a neighborhood U of x_0 such that if $V = U \cap \Omega$, then V

consists of points $x = (x^1, \dots, x^N)$ such that $|x^j| < r$ for $1 \le j \le N-1$ and $0 < x^N < r$ and $U \cap \partial\Omega$ is the set of points x with $|x^j| < r$ for $1 \le j \le N-1$ and $x^N = 0$. Since $\phi_1(\overline{x}) = 0$ and $\frac{\partial \phi_1}{\partial x^N}(\overline{x}) > 0$ for $\overline{x} \in \partial\Omega$, we may assume that r is so small that there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 x^N < \phi_1(x) < c_2 x^N \tag{15}$$

for $x \in V$. Since ϕ_1 is bounded below by a positive constant on any compact subset of Ω, the assertion of the lemma follows from (15) and a partition-of-unity argument.

In the remainder of this paper, we modify the definition of u_* as follows: If $0 < \gamma \le 1$, we define u_* as before, while if $1 < \gamma$ we set $u_*(x) = u_1(x) = b_1\phi_1(x)^t$. It follows from what was shown above that in either case, if u is the unique solution of (1) positive on Ω, then (13) continues to hold for all $x \in \Omega$.

Suppose first that $1 < \gamma < 3$, so $u_*(x) = b_1\phi_1(x)^t$, where $t = 2/(1+\gamma)$. Let the sequences $\{\delta_n\}_1^\infty$ and $\{Z_n\}_1^\infty$ be as above. Since $u_*(x) \le Z_n(x)$ for $x \in \Omega$ and $n \ge 1$, it follows that

$$\begin{aligned} p(x)Z_n(x)[Z_n(x) + \delta_n]^{-\gamma} &\le p(x)[Z_n(x) + \delta_n]^{1-\gamma} \\ &\le p(x)[u_*(x) + \delta_n]^{1-\gamma} < Mu_*(x)^{1-\gamma}, \end{aligned}$$

for all $x \in \Omega$, where M is the maximum of $p(x)$ on $\overline{\Omega}$. If $r = 2(1-\gamma)/(1+\gamma)$, then $r > -1$ so, by the lemma,

$$\int_\Omega u_*(x)^{1-\gamma} dx < \infty.$$

Since for $n \ge 1$,

$$\int_\Omega |\nabla Z_n|^2 \, dx = \int_\Omega p(x)Z_n(x)[Z_n(x) + \delta_n]^{-\gamma} dx.$$

It follows that the $W^{1,2}$-norm of Z_n is bounded independently of n. Therefore some subsequence of $\{Z_n\}_1^\infty$ converges weakly in $W^{1,2}(\Omega)$ to a function $\widehat{Z}$ in $W^{1,2}$. Since $\{Z_n\}_1^\infty$ converges pointwise to u in Ω it is easy to see that $\widehat{Z} = u$. Hence $u \in W^{1,2}(\Omega)$.

If $0 < \gamma < 1$, then if

$$\begin{aligned} x \in \Omega p(x)Z_n(x)[Z_n(x) + \delta_n]^{-\gamma} &\le p(x)[Z_n(x) + \delta_n]^{1-\gamma} \\ &\le p(x)[u^*(x) + \delta_n]^{1-\gamma} \end{aligned}$$

where $u^*(x) = c\phi_1(x)^s$ and s is a positive number satisfying the inequalities (6). The above argument shows that the sequence $\{Z_n\}_1^\infty$ is bounded in $W^{1,2}(\Omega)$, and it follows that $u \in W^{1,2}(\Omega)$.

Suppose now that $\gamma \ge 3$. In this case $u^*(x) = b_2\phi_1(x)^t$ where $t = 2/(1+\gamma)$ so $t(1-\gamma) \le -1$. Since $u(x) \ge u^*(x)$ for $x \in \Omega$ and $p(x) \ge m > 0$ for all

$x \in \Omega$, it follows from the lemma that

$$\int_\Omega p(x)u(x)^{1-\gamma}dx = \infty. \tag{16}$$

Suppose, contrary to the assertion of the theorem, that $u \in W^{1,2}(\Omega)$. Since $u \in C(\overline{\Omega})$ and $u \mid \partial\Omega = 0$, it follows that $u \in W_0^{1,2}(\Omega)$ [3, p. 147]. It follows that there exists a sequence C^∞ functions $\{w_n\}_1^\infty$ having compact supports contained in Ω such that $w_n \to u$ in $W^{1,2}(\Omega)$ as $n \to \infty$. If for each n we set $w_n^+ = \max(w_n, 0)$, then $w_n^+ \in W_0^{1,2}(\Omega)$, $\nabla w_n^+ = \nabla w_n$ where $w_n > 0$, and $\nabla w_n^+ = 0$ where $w_n < 0$ [5]. From this it follows readily that $\{w_n^+\}_1^\infty$ converges to u in $W^{1,2}$. For $n \geq 1$, $w_n^+(x)p(x)u(x)^{-\gamma} \geq 0$ for all $x \in \Omega$, and some subsequence of $\{w_n^+\}_1^\infty$ converges to u almost everywhere on Ω. Therefore, if we replace $\{w_n^+\}_1^\infty$ by this subsequence it follows by (16) and Fatou's Lemma that

$$\lim_{n\to\infty} \int_\Omega w_n^+ p u^{-\gamma} dx = \infty.$$

Since $\Delta u = -p(x)u^{-\gamma}$ on Ω and $w_n^+ \in W_0^{1,2}(\Omega)$ for $n \geq 1$ it follows that for $n \geq 1$

$$\int_\Omega \nabla u \cdot \nabla w_n^+ dx = -\int_\Omega w_n^+ \Delta u dx = \int_\Omega w_n^+ p u^{-\gamma} dx.$$

Hence

$$\int_\Omega |\nabla u|^2 dx = \lim_{n\to\infty} \int_\Omega \nabla u \cdot \nabla w_n^+ dx = \infty,$$

which contradicts the assumption that $u \in W^{1,2}(\Omega)$.

To prove the final statement, we note that if $x_0 \in \partial\Omega$ and $\vec{n}$ denotes the inner normal to $\partial\Omega$ at x_0, then $\phi_1(x_0) = 0$, and

$$\lim_{s\to 0+} \frac{\phi_1(x_0 + s\vec{n})}{s} = \lim_{s\to 0+} \frac{\phi_1(x_0 + s\vec{n}) - \phi_1(x_0)}{s} = \nabla\phi_1(x_0) \cdot \vec{n} > 0.$$

If $\gamma > 1$, then $t = 2/(1+\gamma) < 1$ and, as shown above, for $x \in \Omega$, $u(x) \geq b_1\phi_1(x)^t$, where $b_1 > 0$. Since $u(x_0) = 0$, it follows that, for $s > 0$,

$$\frac{u(x_0 + s\vec{n}) - u(x_0)}{s} \geq b_1\phi_1(x_0 + s\vec{n})^{t-1} \frac{\phi_1(x_0 + s\vec{n})}{s}.$$

Therefore

$$\lim_{s\to 0+} \frac{u(x_0 + s\vec{n}) - u(x_0)}{s} = +\infty,$$

so u is not in $C^1(\overline{\Omega})$. This proves the theorem.

4. Remarks and generalizations

In this section, we collect some obvious generalizations, where our method of proof gives additional information. All of our results can be written in terms of a more general nonlinearity $f(x, u)$ with the appropriate abstract hypotheses on f, but we leave this as an exercise for the reader.

(i) In case Ω and p are radially symmetric, our proof shows that u is radially symmetric.

(ii) We do not know if it is always the case that u does not belong to $C^1(\Omega)$ if $\gamma = 1$. The following simple example shows that, in general, u does not belong to $C^1(\overline{\Omega})$ when $\gamma = 1$.

Let $N = 1$, $\gamma = 1$, $\Omega = (0, 1)$, and $\gamma = 1$. In this case,

$$u''(x) + u(x)^{-1} = 0$$

for $0 < x < 1$, $u(0) = u(1) = 0$, and $u(x) > 0$ for $0 < x < 1$.

It follows that

$$u'(x)^2/2 + \log u(x) = c,$$

where c is a constant. Since $\log u(x) \to -\infty$ as $x \to 0$ or $x \to 1$, it follows that $u' \to \infty$ as $x \to 0$ or $x \to 1$. Hence u does not belong to $C^1(\overline{\Omega})$.

(iii) A careful examination of our proof shows that additional results are available in the case where $p(x)$ is not bounded away from zero.

If, instead of $p(x) > c_3 > 0$ uniformly on Ω, we assume that $p(x)\phi_1^{-\delta} \geq c_3 > 0$ uniformly on Ω, where δ satisfies $0 < \delta < \gamma + 1$, then instead of $u_1(x) = b_1\phi_1(x)^t$, we choose $u_1(x) = b_1\phi_1(x)^{(2+\delta)/(1+\gamma)}$, then we still have $u_1(x) \leq u_2(x)$, and

$$\Delta u_1(x) + p(x)u_1(x)^{-\gamma} > 0.$$

Thus we can show that $b_1\phi_1(x)^{(2+\delta)/(1+\gamma)} \leq u(x) \leq b_2\phi_1(x)^{2/(1+\gamma)}$ on $\overline{\Omega}$ for b_1 small and b_2 large.

(iv) In the case where the region Ω has corners, our method of proof still gives some information. If one assumes that the region is a square in the plane, one can show that if there exist constants c_1 and c_2 such that, near the boundary,

$$c_1 < p(x)/(|\nabla\phi_1|)^2 < c_2$$

where, as before, ϕ_1 is the first eigenfunction of the Laplacian for this region, then the conclusion of Theorem 1 applies.

We cannot give good boundary estimates in the case where the function $p(x)$ does not vanish at the corners.

(v) We can also give regularity results in the case where $p(x)$ goes to infinity at the boundary, at least in the case where the rate of growth is not too great.

If there exists a δ so that $s = (2 - \delta)/(\gamma + 1)$ is less than one, and the function $p(x)$ satisfies $c_4 \geq p(x)\phi_1^{\delta} \geq c_3 > 0$ uniformly on Ω for some positive constants c_3 and c_4, then, by choosing $u_2(x) = b_2\phi_1(x)^{(2-\delta)/(1+\gamma)}$, we can conclude that $b_1\phi_1(x)^{2/(1+\gamma)} \leq u(x) \leq b_2\phi_1(x)^{(2-\delta)/(1+\gamma)}$ on $\overline{\Omega}$, for b_1 small and b_2 large.

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