On a singular nonlinear semilinear elliptic problem

Junping Shi*
Department of Mathematics, Brigham Young University, Provo,
Utah 84602-6539, U.S.A.

Miaoxin Yao
Department of Mathematics, Brigham Young University, Provo,
Utah 84602-6539, U.S.A., and Department of Mathematics,
Tianjin University, Tainjin 300072, P.R. China

We consider the singular boundary value problem

$$-\Delta u + K(x)u^{-\alpha} = \lambda u^p \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

We study the existence, uniqueness, regularity and the dependency on parameters of the positive solutions under various assumptions.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$, $n \geqq 2$, with $C^{2,\beta}$ boundary $\partial\Omega$, where $\beta \in (0, 1)$. We consider a singular boundary value problem

$$\begin{aligned}
-\Delta u + K(x)u^{-\alpha} &= \lambda u^p &&\text{in } \Omega, \\
u &> 0 &&\text{in } \Omega, \\
u &= 0 &&\text{on } \partial\Omega.
\end{aligned} \qquad (1.1)$$

where $K(x) \in C^{2,\beta}(\bar{\Omega})$, $\alpha, p \in (0, 1)$ and λ is a real parameter. Such singular elliptic problems arise in the contexts of chemical heterogeneous catalysts, nonNewtonian fluids and also the theory of heat conduction in electrically conducting materials, see [3, 5, 8, 9] for a detailed discussion.

Obviously (1.1) cannot have a solution $u \in C^2(\bar{\Omega})$ if $K(x)$ is not vanishing near $\partial\Omega$. However, under various appropriate assumptions on $K(x)$, we will obtain classical solutions of (1.1) for λ belonging to a certain range, and we will also obtain some uniqueness criteria. Here a *classical solution* is a solution u of (1.1) which belongs to $C^2(\Omega) \cap C(\bar{\Omega})$ with $u > 0$ in Ω. We also study the boundary behaviour of solutions of (1.1), and we will show that the solution u of (1.1) lies in a certain Hölder class.

The special case when $K(x)$ is negative and $\lambda = 0$ has been studied by several authors. The existence and uniqueness of the solution were established by Crandall, Rabinowitz and Tartar [6], Del Pino [7], Gomes [10] and Lazer and McKenna

* *Current Address:* Department of Mathematics, Tulane University, New Orleans, Louisiana 70118, U.S.A.

[**13**], while in [**6**, **7**, **12**, **13**], the regularity and the boundary behaviour of solutions were investigated.

For $\lambda \neq 0$, if $K(x) \equiv 1$, Zhang [**17**] proved that (1.1) has a positive solution u_λ when λ is large enough; if $K(x) \equiv -1$, M. M. Coclite and G. Palmieri [**4**] proved that if $0 < p < 1$, then (1.1) has at least one solution u_λ for all $\lambda \geqq 0$ and $p \geqq 1$, there exists $\tilde{\lambda} > 0$ such that (1.1) has a solution u_λ for $\lambda \in [0, \tilde{\lambda})$ and no solution for $\lambda > \tilde{\lambda}$.

If $K(x) \equiv 0$, it is well known that there exists a unique solution u_λ of (1.1) in $C^{2,\beta}(\Omega) \cap C^2(\bar{\Omega})$ if and only if $\lambda > 0$. Thus throughout this paper we assume that $K(x) \not\equiv 0$ in $\bar{\Omega}$.

The main results of this paper are stated in the following theroems, where

$$K^* = \max_{x \in \bar{\Omega}} K(x),$$

$$K_* = \min_{x \in \bar{\Omega}} K(x),$$

$$E = \{u \in C^{2,\beta}(\Omega) \cap C^0(\bar{\Omega}) : u^{-\alpha} \in L^1(\Omega)\},$$

$$d(x) = \text{dist}\,(x, \partial\Omega).$$

THEOREM 1.1. *Let $K^* < 0$ [respectively $K^* = 0$]; then:*
  (i) *(1.1) has one and only one solution $u_\lambda \in E$ for any $\lambda \in \mathbf{R}$ [respectively $\lambda \geqq 0$];*
  (ii) *u_λ is increasing with respect to λ;*
  (iii) *$c_1 d(x) \leqq u_\lambda(x) \leqq c_2 d(x)$ for any $x \in \bar{\Omega}$, and some $c_1, c_2 > 0$ independent of x;*
  (iv) *$u_\lambda \in C^{1,\gamma}(\bar{\Omega})$, where $\gamma = 1 - \alpha$.*

THEOREM 1.2. *Let $K_* > 0$; then:*
  (i) *there exists a $\lambda_* > 0$ such that (1.1) has at least one classical solution u_λ for $\lambda > \lambda_*$, and (1.1) has no classical solution for $\lambda < \lambda_*$;*
  (ii) *for $\lambda > \lambda_*$, (1.1) has a maximal solution $\bar{u}_\lambda \in E$ and $\bar{u}_\lambda$ is increasing with respect to λ;*
  (iii) *for $\lambda > \lambda_*$, $c_1 d(x) \leqq \bar{u}_\lambda(x) \leqq c_2 d(x)$ for any $x \in \bar{\Omega}$, and some $c_1, c_2 > 0$ independent of x;*
  (iv) *$\bar{u}_\lambda \in C^{1,\gamma}(\bar{\Omega})$, where $\gamma = 1 - \alpha$.*

THEOREM 1.3. *Let $K^* > 0 > K_*$, then:*
  (i) *there exists a $\lambda_* > 0$ such that (1.1) has at least one solution $u_\lambda \in E$ for any $\lambda > \lambda_*$;*
  (ii) *for $\lambda > \lambda_*$, u_λ is increasing with respect to λ;*
  (iii) *for $\lambda > \lambda_*$, $c_1 d(x) \leqq u_\lambda(x) \leqq c_2 d(x)$ for any $x \in \bar{\Omega}$, and some $c_1, c_2 > 0$ independent of x;*
  (iv) *$u_\lambda \in C^{1,\gamma}(\bar{\Omega})$, where $\gamma = 1 - \alpha$.*

REMARK 1.4. (a) In contrast to the case $K(x) < 0$, the uniqueness of the solution does not hold when $K(x) > 0$. In [**14**], the present authors and Ouyang studied the equation (1.1) with $K(x) \equiv 1$, and $\Omega = B^n$, the unit ball in $\mathbf{R}^n$. We showed that (1.1) has at least two solutions for $\lambda > \lambda_*$ and sufficiently close to λ_* by a bifurcation method, and that $\lambda = \lambda_*$ is a turning point on a solution curve.

(b) The bounds of solution in Theorem 1.2 or Theorem 1.3 may not hold for all the solutions of (1.1). In [**14**], the present authors and Ouyang showed that when

$K(x) \equiv 1$, and $\Omega = B^n$, the unit ball in $\mathbf{R}^n$, (1.1) has a solution u_λ such that

$$u_\lambda(x) = \frac{\partial u_\lambda}{\partial n}(x) = 0 \quad \text{for any } x \in \partial\Omega,$$

and

$$c_1 d(x)^{2/(1+\alpha)} \leqq u_\lambda(x) \leqq c_2 d(x)^{2/(1+\alpha)}.$$

It is interesting to know, for the general bounded smooth domain, whether some solutions with vanishing normal derivative exist, and if they exist, whether the normal derivative vanishes at isolated points or on the entire boundary. We conjecture that such a positive solution exists for any bounded smooth domain, and in general, the normal dervative only vanishes at isolated points.

The paper is organised as follows. Some preliminary lemmas are stated and proved in Section 2. In Section 3, we give the proofs of all the theorems.

2. Preliminaries

Let $\Psi_1(x)$ denote the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of the problem

$$-\Delta u = \lambda u \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega.$$

Then, as is well known, $\lambda_1 > 0$ and $\Psi_1 \in C^2(\bar\Omega)$. Moreover, we have the following property of Ψ_1:

LEMMA 2.1 [13]. *Let $s > 0$; then*

$$\int_\Omega [\Psi_1(x)]^{-s} \, ds < \infty \quad \text{if and only if } s < 1.$$

With the regularity theory and this lemma, we have $\Psi_1 \in E$. We will also need the following result from [17], which is proved by using a super-subsolution method.

LEMMA 2.2 [17]. *Suppose that $K(x) \equiv 1$; then there exists a $\tilde\lambda > 0$ such that (1.1) has a solution $u_\lambda \in E$ for all $\lambda > \tilde\lambda$. Moreover, $u_\lambda(x) \geqq C(\lambda)(\Psi_1)^{2/(1+\alpha)}(x)$ for $x \in \bar\Omega$ and $\lambda > \tilde\lambda$.*

Next we establish a comparison lemma, which is proved by using a method motivated by method II in [2, p. 103]; see also [1, Lemma 3.3].

LEMMA 2.3. *Suppose that $f : \Omega \times \mathbf{R}^+ \to \mathbf{R}$ is a continuous function such that $s^{-1}f(x, s)$ is strictly decreasing for $s > 0$ at each $x \in \Omega$. Let $w, v \in C(\bar\Omega) \cap C^2(\Omega)$ satisfy:*
  (a) $\Delta w + f(x, w) \leqq 0 \leqq \Delta v + f(x, v)$ *in Ω;*
  (b) $w, v > 0$ *in Ω and $w \geqq v$ on $\partial\Omega$;*
  (c) $\Delta v \in L^1(\Omega)$.
Then $w \geqq v$ in $\bar\Omega$.

Proof. We argue by contradiction. If $w \geqq v$ is not true, then there exist $\varepsilon_0, \delta_0 > 0$ and a ball $B \subset\subset \Omega$ such that

$$v(x) - w(x) \geqq \varepsilon_0, \quad x \in B \tag{2.1}$$

and

$$\int_{\mathbf{B}} vw \left(\frac{f(x, w)}{w} - \frac{f(x, v)}{v} \right) dx \geqq \delta_0. \qquad (2.2)$$

Let

$$M = \max \{1, \|\Delta v\|_{L^1(\Omega)}\}$$

and

$$\varepsilon = \min \left\{ 1, \varepsilon_0, \frac{\delta_0}{4M} \right\}.$$

Let θ be a smooth function on $\mathbf{R}$ such that $\theta(t) = 0$ if $t \leqq \frac{1}{2}$, $\theta(t) = 1$ if $t \geqq 1$, $\theta(t) \in (0, 1)$ if $t \in (\frac{1}{2}, 1)$, and $\theta'(t) \geqq 0$ for $t \in \mathbf{R}$. Then, for $\varepsilon > 0$, define the function $\theta_\varepsilon(t)$ by

$$\theta_\varepsilon(t) = \theta \left(\frac{t}{\varepsilon} \right), \quad t \in \mathbf{R}.$$

It then follows from (a) and the fact that $\theta_\varepsilon(t) \geqq 0$ for $t \in \mathbf{R}$ that

$$(w\Delta v - v\Delta w)\theta_\varepsilon(v - w) \geqq vw \left(\frac{f(x, w)}{w} - \frac{f(x, v)}{v} \right) \theta_\varepsilon(v - w) \quad \text{in } \Omega.$$

On the other hand, by the continuity of w, v and θ_ε, and the fact that $w \geqq v$ on $\partial\Omega$, it is easy to see that there exists a subdomain Ω^* with smooth boundary, such that $B \subset \Omega^* \subset\subset \Omega$ satisfying that $v(x) - w(x) < (\varepsilon/2)$, for all $x \in \Omega \backslash \Omega^*$. Then we have

$$\int_{\Omega^*} (w\Delta v - v\Delta w)\theta_\varepsilon(v - w) \, dx \geqq \int_{\Omega^*} vw \left(\frac{f(x, w)}{w} - \frac{f(x, v)}{v} \right) \theta_\varepsilon(v - w) \, dx.$$

Denote

$$\Theta_\varepsilon(t) = \int_0^t s\theta'_\varepsilon(s) \, ds, \quad t \in \mathbf{R};$$

then it is easy to verify that

$$0 \leqq \Theta_\varepsilon(t) \leqq 2\varepsilon, \ t \in \mathbf{R} \quad \text{and} \quad \Theta_\varepsilon(t) = 0, \quad \text{if } t < \frac{\varepsilon}{2}. \qquad (2.3)$$

Therefore

$$\int_{\Omega^*} (w\Delta v - v\Delta w)\theta_\varepsilon(v - w) \, dx$$

$$= \int_{\partial\Omega^*} w\theta_\varepsilon(v - w) \frac{\partial v}{\partial n} \, ds - \int_{\Omega^*} (\nabla v \cdot \nabla w)\theta_\varepsilon(v - w) \, dx$$

$$- \int_{\Omega^*} w\theta'_\varepsilon(v - w)\nabla v \cdot (\nabla v - \nabla w) \, dx - \int_{\partial\Omega^*} v\theta_\varepsilon(v - w) \frac{\partial w}{\partial n} \, ds$$

$$+ \int_{\Omega^*} (\nabla w \cdot \nabla v)\theta_\varepsilon(v - w) \, dx + \int_{\Omega^*} v\theta'_\varepsilon(v - w)\nabla w \cdot (\nabla v - \nabla w) \, dx$$

$$= \int_{\Omega^*} v\theta'_\varepsilon(v-w)(\nabla w - \nabla v) \cdot (\nabla v - \nabla w) \, dx + \int_{\Omega^*} (v-w)\theta'_\varepsilon(v-w)\nabla v \cdot (\nabla v - \nabla w) \, dx$$

$$\leqq \int_{\Omega^*} \nabla v \cdot \nabla(\Theta_\varepsilon(v-w)) \, dx$$

$$= \int_{\partial\Omega^*} \Theta_\varepsilon(v-w)\frac{\partial v}{\partial n} \, ds - \int_{\Omega^*} \Theta_\varepsilon(v-w)\Delta v \, dx$$

$$\leqq 2\varepsilon \int_{\Omega^*} |\Delta v| \, dx \quad \text{(by (2.3))}$$

$$\leqq 2\varepsilon M < \frac{\delta_0}{2}.$$

But we have

$$\int_{\Omega^*} vw\left(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\right)\theta_\varepsilon(v-w) \, dx \geqq \int_B vw\left(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\right)\theta_\varepsilon(v-w) \, dx$$

$$= \int_B vw\left(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\right) dx \quad \text{(by (2.1))}$$

$$\geqq \delta_0 \quad \text{(by (2.2))},$$

which is a contradiction. Thus the lemma is proved. □

To end this section, we state a lemma which is proved in [**16**]. For the proof, readers are referred to [**16**, Theorem 2].

LEMMA 2.4. *Suppose that function f satisfies*:
(F1) $f : \bar{\Omega} \times (0, +\infty) \to \mathbf{R}$ *is Hölder continuous with exponent $\beta \in (0, 1)$ on each compact subset of $\bar{\Omega} \times (0, +\infty)$*;
(F2)

$$\limsup_{s \to +\infty}\left(s^{-1} \max_{x \in \Omega} f(x,s)\right) < \lambda_1;$$

(F3) *for each $t > 0$, there exists a constant $D(t) > 0$ such that*

$$f(x,r) - f(x,s) \geqq -D(t)(r-s) \quad \text{for } x \in \bar{\Omega} \quad \text{and} \quad r \geqq s \geqq t;$$

(F4) *there exist a $\delta > 0$ and an open subset Ω_0 of Ω such that*

$$\min_{x \in \bar{\Omega}} f(x,s) \geqq 0 \quad \text{for } s \in (0, \delta),$$

and

$$s^{-1}f(x,s) \to +\infty \quad \text{as } s \to 0^+ \quad \text{uniformly for } x \in \Omega_0.$$

Then for any non-negative function $\varphi_0(x) \in C^{2,\beta}(\partial\Omega)$, *the problem*

$$\Delta u + f(x, u) = 0 \quad in \ \Omega,$$
$$u > 0 \quad in \ \Omega, \qquad\qquad (2.4)$$
$$u = \varphi_0 \quad on \ \partial\Omega,$$

has at least one positive solution $u(x)$ *of problem* (2.4) *such that, for any compact subsets* G *of* $\Omega \cup \{x \in \partial\Omega : \varphi_0(x) > 0\}$, $u(x) \in C^{2,\beta}(G) \cap C^0(\bar{\Omega})$.

3. Proofs of main theorems

In this section, we will always assume that $f_\lambda(x, u) = \lambda u^p - K(x) u^{-\alpha}$.

Proof of Theorem 1.1. (i) (*Existence*) Let $\varphi_0(x) \equiv 0$; then we can apply Lemma 2.4 to obtain the existence of a solution $u_\lambda(x) \in C^{2,\beta}(\Omega) \cap C^0(\bar{\Omega})$ for any $\lambda \in \mathbf{R}$ (respectively $\lambda \geqq 0$).

(ii) (*Uniqueness*) If $\lambda \leqq 0$, $f_\lambda(x, u)$ is decreasing in u on $(0, \infty)$. Thus, by a standard argument of the maximum principle, we obtain the uniqueness of u_λ for $\lambda \leqq 0$.

If $\lambda > 0$, we have that $u^{-1} f_\lambda(x, u)$ is strictly decreasing in u for $u > 0$ and $x \in \bar{\Omega}$. Hence, if u_1 and u_2 are two solutions to (1.1), then $u_1 \equiv u_2$ by Lemma 2.3 (noting that $\Delta u_i \in L^1(\Omega)$ by (iii) below.) Therefore the uniqueness of u_λ for $\lambda > 0$ is also proved.

(iii) (*Dependence on* λ) First, we assume that $\lambda_1 < \lambda_2 \leqq 0$ and u_{λ_1}, u_{λ_2} are the corresponding unique solutions to (1.1). We prove by contradiction that $u_{\lambda_1}(x) \leqq u_{\lambda_2}(x)$. Suppose not; then

$$A = \{x \in \Omega : u_{\lambda_1}(x) > u_{\lambda_2}(x)\} \neq \varnothing.$$

The function $w(x) = u_{\lambda_1}(x) - u_{\lambda_2}(x)$ satisfies

$$\Delta w(x) = \lambda_2 u_{\lambda_2}^p - \lambda_1 u_{\lambda_1}^p + K(x) u_{\lambda_1}^{-\alpha} - K(x) u_{\lambda_2}^{-\alpha}$$
$$\geqq 0, \quad x \in A,$$

and

$$w(x) = 0 \quad x \in \partial A.$$

By the maximum principle, $w(x) < 0$ in A, which is a contradiction.

Secondly, we assume that $0 \leqq \lambda_1 < \lambda_2$ and u_{λ_1}, u_{λ_2} are the corresponding unique solutions to (1.1). Since $u_\lambda(x) \geqq c_1 \Psi_1(x)$, it is easy to see that $\Delta u_{\lambda_1} \in L^1(\Omega)$ by Lemma 2.1,

$$\Delta u_{\lambda_2} - K(x) u_{\lambda_2}^{-\alpha} + \lambda_2 u_{\lambda_2}^p = 0 < \Delta u_{\lambda_1} - K(x) u_{\lambda_1}^{-\alpha} + \lambda_2 u_{\lambda_1}^p$$

for $x \in \Omega$, and $u_{\lambda_1}(x) = u_{\lambda_2}(x)$ on $\partial\Omega$. Therefore, by Lemma 2.3,

$$u_{\lambda_1}(x) \leqq u_{\lambda_2}(x) \quad in \ \bar{\Omega}.$$

Moreover, by the maximum principle, we have $u_{\lambda_1}(x) < u_{\lambda_2}(x)$, $x \in \Omega$, for both cases. So u_λ is increasing with respect to λ.

(iv) (*Bounds of solutions*) First, let $v(x) = c\zeta(x)$, where $\zeta(x)$ is the solution of the

problem:

$$\Delta \zeta = K(x) \quad x \in \Omega,$$

$$\zeta(x) = 0 \quad x \in \partial\Omega.$$

Then, for $\lambda < 0$ when $K^* < 0$, or $\lambda = 0$ when $K^* = 0$, we have

$$
\begin{aligned}
-\Delta v - f_\lambda(x, v) &= -\Delta v + K(x)v^{-\alpha} - \lambda v^p \\
&= -cK(x) + K(x)(c\zeta)^{-\alpha} - \lambda(c\zeta)^p \\
&\leqq -cK(x) + \frac{K(x)}{2}c^{-\alpha}\zeta^{-\alpha} + \frac{K^*}{2}c^{-\alpha}\zeta^{-\alpha} - \lambda c^p \zeta^p \\
&= K(x)c^{-\alpha}\zeta^{-\alpha}\left(\frac{1}{2} - c^{1+\alpha}\zeta^\alpha\right) + c^{-\alpha}\zeta^{-\alpha}\left(\frac{K^*}{2} - \lambda c^{p+\alpha}\zeta^{p+\alpha}\right) \\
&\leqq 0
\end{aligned}
\tag{3.1}
$$

for $c > 0$ small enough. Fixing $c_0 > 0$ such that (3.1) holds, we claim that $u_\lambda(x) \geqq c_0 \zeta(x)$. Suppose not; then $A = \{x \in \Omega : u_\lambda(x) < c_0 \zeta(x)\}$ is not empty, and we have

$$\Delta[u_\lambda(x) - c_0\zeta(x)] \leqq -[f_\lambda(x, u_\lambda) - f_\lambda(x, c_0\zeta)] \leqq 0, \quad x \in A,$$

and

$$u_\lambda(x) - c_0\zeta(x) = 0, \quad x \in \partial A,$$

which gives $u_\lambda(x) \geqq c_0 \zeta(x)$, $x \in A$, a contradiction. Therefore, $u_\lambda(x) \geqq c_0 \zeta(x) \geqq c_1 d(x)$ for some $c_1 > 0$. For any $\lambda \geqq 0$, we fix $\lambda_1 < 0$ when $K^* < 0$, or $\lambda_1 = 0$ when $K^* = 0$; then

$$u_\lambda(x) \geqq u_{\lambda_1}(x) \geqq c_1 d(x),$$

since u_λ is increasing with respect to λ. Therefore, for any $\lambda \in \mathbf{R}$ when $K^* < 0$ or any $\lambda \geqq 0$ when $K^* = 0$, we have $u_\lambda(x) \geqq c_1 d(x)$. In particular, $u_\lambda^{-\alpha} \in L^1(\Omega)$ by Lemma 2.1 and hence $u_\lambda \in E$.

Next, we prove that $u_\lambda(x) \leqq c_2 d(x)$. Since u_λ is increasing with respect to λ, it suffices to prove the case $\lambda > 0$. Our following method is similar to and motivated by the proof in [**12**, pp. 1024–5]. Let $B_r(x)$ denote the ball in $\mathbf{R}^n$ with radius r and centred at the x, and w_K be the unique solution of

$$\Delta w + w^{-\alpha} + \lambda w^p = 0 \quad \text{in } B_K(0) \backslash B_1(0),$$

$$w > 0 \quad \text{in } B_K(0) \backslash B_1(0),$$

$$w = 0 \quad \text{on } \partial(B_K(0) \backslash B_1(0)),$$

where $K > 1$ will be determined later. By the uniqueness of the solution, we know that w_K is radially symmetric and satisfies

$$
w'' + \frac{n-1}{r}w' + w^{-\alpha} + \lambda w^p = 0 \quad \text{in } (1, K),
$$
$$
w(1) = w(K) = 0.
\tag{3.2}
$$

Since $w_K(x) \geqq c_1 d(x)$, then we can obtain that both $w_K'(1)$ and $w_K'(K)$ are finite by integrating (3.2). Consequently,

$$w_K(x) \leqq C \min\{K - |x|, |x| - 1\},$$

for any $x \in B_K(0) \backslash B_1(0)$ and some $C > 0$. By the smoothness of $\partial\Omega$, there exists $\delta > 0$, $K > 0$ such that for any $x_0 \in \Omega_\delta = \{x \in \Omega : d(x) \leqq \delta\}$, we have $\Omega \subset (B_{2K\delta}(y) \backslash B_\delta(y))$, and

$$d(x_0) = |x_0 - y| - \delta,$$

for some $y \notin \Omega$. Let

$$v(x) = c_0 w_K\left(\frac{x - y}{\delta}\right), \quad x \in \Omega;$$

then $v(x)$ satisfies

$$\Delta v - K_* v^{-\alpha} + \lambda v^p \leqq 0, \quad x \in \Omega,$$

provided $c_0 > \Lambda$, where Λ depends on α, δ, K (not on x_0). On the other hand, we have

$$\Delta u_\lambda - K_* u_\lambda^{-\alpha} + \lambda u_\lambda^p \geqq 0, \quad x \in \Omega,$$

and $\Delta u_\lambda \in L^1(\Omega)$. Since $u^{-1}(\lambda u^p - K_* u^{-\alpha})$ is strictly decreasing in u for $u > 0$ and $x \in \bar\Omega$, then by Lemma 2.3,

$$u_\lambda(x) \leqq v(x), \quad x \in \Omega.$$

In particular,

$$u_\lambda(x_0) \leqq c_0 w_K\left(\frac{x_0 - y}{\delta}\right) \leqq c_2 d(x_0),$$

for any $x_0 \in \Omega_\delta$ and some $c_2 > 0$ independent of x_0. Therefore, $u_\lambda(x) \leqq c_2 d(x)$ for $x \in \Omega$.

(v) (*Regularity*) By Green's formula, we have

$$u_\lambda(x) = \int_\Omega G(x, y)[K(y)u^{-\alpha}(y) - \lambda u^p(y)] \, dy, \quad x \in \Omega.$$

$$\nabla u_\lambda(x) = \int_\Omega G_x(x, y)[K(y)u^{-\alpha}(y) - \lambda u^p(y)] \, dy, \quad x \in \Omega.$$

If $x_1, x_2 \in \Omega$, then

$$|\nabla u_\lambda(x_1) - \nabla u_\lambda(x_2)| \leqq \int_\Omega |G_x(x_1, y) - G_x(x_2, y)| \cdot |K(y)u^{-\alpha}(y) - \lambda u^p(y)| \, dy$$

$$\leqq \int_\Omega |G_x(x_1, y) - G_x(x_2, y)| \cdot |K(y)u^{-\alpha}(y)| \, dy$$

$$+ |\lambda| \int_\Omega |G_x(x_1, y) - G_x(x_2, y)| \cdot u^p(y) \, dy$$

$$\equiv I + II.$$

Since $c_1 d(x) \leqq u_\lambda(x) \leqq c_2 d(x)$, we have

$$I \leqq c d^{1-\alpha}(x_1, x_2),$$

by the proof of [**12**, Theorem 1]. On the other hand, $u_\lambda \in C^0(\bar{\Omega})$; then

$$II \leqq c d(x_1, x_2),$$

by the standard regularity theory (see for example [**11**]). Therefore, $u_\lambda \in C^{1,\gamma}(\bar{\Omega})$, where $\gamma = 1 - \alpha$. This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. (i) (*Existence*) By Lemma 2.2, for $\mu > \tilde{\lambda}$, there is a solution $v_\mu \in E$ to the problem

$$-\Delta v + v^{-\alpha} = \mu v^p \quad \text{in } \Omega,$$

$$v > 0 \quad \text{in } \Omega,$$

$$v = 0 \quad \text{on } \partial\Omega.$$

Let

$$\lambda = \mu(K^*)^{(1-p)/(1+\alpha)}, \quad \underline{u}_\lambda = (K^*)^{1/(1+\alpha)} v_\mu;$$

then $\underline{u}_\lambda \in E$ is a solution of

$$-\Delta v + K^* v^{-\alpha} = \lambda v^p \quad \text{in } \Omega,$$

$$v > 0 \quad \text{in } \Omega,$$

$$v = 0 \quad \text{on } \partial\Omega.$$

Now we consider an approximate problem $P_k(\lambda)$ of (1.1) as follows,

$$-\Delta v + K(x)v^{-\alpha} = \lambda v^p \quad \text{in } \Omega,$$

$$v = \frac{1}{k} \quad \text{on } \partial\Omega,$$

where $k = 1, 2, \ldots$. It is easy to verify that $v_k(x) = \underline{u}_\lambda(x) + (1/k)$ is a subsolution to $P_k(\lambda)$. On the other hand, by Lemma 2.4, there exists a solution $w \in C^{2,\beta}(\bar{\Omega})$ to the problem

$$-\Delta w = \lambda w^p \quad \text{in } \Omega,$$

$$w > 0 \quad \text{in } \Omega,$$

$$w = 1 \quad \text{on } \partial\Omega.$$

It is clear that w is a supersolution of $P_1(\lambda)$. Moreover, since

$$\Delta v_1 + \lambda v_1^p \geqq 0 \geqq \Delta w + \lambda w^p \quad \text{in } \Omega,$$

$$v_1 = w \quad \text{on } \partial\Omega,$$

and $\Delta v_1 \in L^1(\Omega)$, it follows from Lemma 2.3 that

$$1 \leqq v_1(x) \leqq w(x) \quad \text{for } x \in \bar{\Omega}.$$

Thus, by the standard super- and subsolution argument, there exists a solution $u_\lambda^{(1)} \in C^{2,\beta}(\bar{\Omega})$ of $P_1(\lambda)$ such that $v_1(x) \leqq u_\lambda^{(1)} \leqq w(x)$. Similarly, taking $u_\lambda^{(1)}$ and v_2 as a

pair of super- and subsolutions for $P_2(\lambda)$, we conclude that there exists a solution $u_\lambda^{(2)} \in C^{2,\beta}(\bar{\Omega})$ of $P_2(\lambda)$ such that $v_2(x) \leqq u_\lambda^{(2)}(x) \leqq u_\lambda^{(1)}(x)$.

Repeating the above arguments, we obtain a sequence $\{u_\lambda^{(i)}\}$ which is decreasing in i, for $i = 1, 2, \ldots$, and is uniformly bounded from below by $\underline{u}_\lambda$ in $\bar{\Omega}$. Thus, as in the proof of Lemma 2.4 (see [16]), let

$$u_\lambda(x) = \lim_{i \to \infty} u_\lambda^{(i)}(x), \quad x \in \bar{\Omega};$$

then by a standard argument using the Schauder-type estimates and the regularity theory (see [11]), we conclude that u_λ is a solution of (1.1) in E if $\lambda > \tilde{\lambda}(K^*)^{(1-p)/(1+\alpha)}$.

(ii) (*Existence of the maximal solution*) We observe the problem

$$-\Delta w = \lambda w^p \quad \text{in } \Omega,$$
$$w > 0 \qquad \text{in } \Omega, \tag{3.3}$$
$$w = 0 \qquad \text{on } \partial\Omega,$$

has a unique solution w_λ for any $\lambda > 0$ by Lemmas 2.3 and 2.4. We claim that for any classical solution u_λ of (1.1) we have

$$u_\lambda \leqq w_\lambda.$$

If $u_\lambda \in E$, then $\Delta u_\lambda \in L^1(\Omega)$. By Lemma 2.3, we have $u_\lambda \leqq w_\lambda$. If $u_\lambda \notin E$, Lemma 2.3 is not applicable. But we can still use the proof of lemma 2.3 to prove it. In fact, if $S = \{u_\lambda > w_\lambda\}$ is not empty, then for any $x \in S$, $u_\lambda(x) > w_\lambda(x) \geqq cd(x)$, for some $c > 0$ independent of x, since $w_\lambda \in C^{2,\beta}(\bar{\Omega})$ and $\partial w_\lambda/\partial n < 0$ for any $x \in \partial\Omega$. On the other hand, u_λ satisfies equation (1.1), then $|\Delta u_\lambda(x)| \leqq cd(x)^{-\alpha}$. Therefore $\Delta u_\lambda \in L^1(S)$. Now let $f(x, u) = \lambda u^p$, $w = w_\lambda$ and $v = u_\lambda$; we can repeat the proof of Lemma 2.3 word by word, only replacing the definition of M by

$$M = \min\{1, \|\Delta u_\lambda\|_{L^1(S)}\}.$$

Note that $\Theta(v - w) = 0$ if $x \notin S$, so

$$\int_{\Omega^*} \Theta(v - w)\Delta v \, dx \leqq 2\varepsilon \int_S |\Delta v| \, dx,$$

in the proof of Lemma 2.3. The other part of the proof remains the same. So we still have a contradiction, and the claim is proved.

Let $\Omega_j = \{x \in \Omega : d(x) > (1/j)\}$, $j = 1, 2, \ldots$, and w_j be the solution of

$$-\Delta v + K(x)w_{j-1}^{-\alpha} = \lambda w_{j-1}^p \quad \text{in } \Omega_j,$$
$$v = w_{j-1} \qquad \text{in } \bar{\Omega} \backslash \Omega_j,$$

for $j = 1, 2, 3, \ldots$, with $w_0 = w_\lambda$ defined in (3.3). Let u_λ be a classical solution of (1.1). By the maximum principle, we have

$$u_\lambda(x) \leqq w_{j+1}(x) \leqq w_j(x) \leqq w_0(x), \quad x \in \bar{\Omega}.$$

Furthermore, for any compact subset $G \subset\subset \Omega$, $w_j \in C^{2,\beta}(G)$ for j large enough, and $\{w_j\}$ is bounded from below by u_λ. Thus, similar to (i), the function

$$\bar{u}_\lambda(x) = \lim_{j \to \infty} w_j(x)$$

is a solution of (1.1), and for any u_λ, $\bar{u}_\lambda \geqq u_\lambda$. Therefore $\bar{u}_\lambda$ is the maximal solution of (1.1). By (i), we have proved that for $\lambda > \tilde{\lambda}(K^*)^{(1-p)/(1+\alpha)}$, (1.1) has a maximal solution $\bar{u}_\lambda$.

(iii) (*Nonexistence when λ is small*) Since $K_* > 0$, then there exists a $k(\lambda) > 0$ such that $f_\lambda(x, u) \leqq k(\lambda)u$ for $x \in \bar{\Omega}$ and $u > 0$. Moreover, $k(\lambda)$ can be chosen such that $k(\lambda) \to 0$ if $\lambda \to 0$. Suppose that u_λ is a solution of (1.1); then

$$\lambda_1(\Omega) \int_\Omega u_\lambda^2(x)\, dx \leqq \int_\Omega |\nabla u_\lambda(x)|^2\, dx = \int_\Omega u_\lambda(x) f_\lambda(x, u_\lambda)\, dx \leqq k(\lambda) \int_\Omega u_\lambda^2(x)\, dx,$$

where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ in Ω. Therefore, (1.1) has a classical solution only if $\lambda > \lambda_*$, for some positive constant λ_*.

(iv) (*Dependence on λ*) Let

$$H = \{\mu > 0 : (1.1) \text{ has a classical solution with } \lambda = \mu\},$$

and $\lambda_* = \inf\{\mu > 0 : \mu \in H\}$. By (i), $H \neq \varnothing$, and by (iii), $\lambda_* > 0$. Let $\lambda_1 \in H$, and $\bar{u}_{\lambda_1}$ be the corresponding maximal solution of (1.1) for $\lambda = \lambda_1$. Then for any $\lambda_2 > \lambda_1$, $\Delta \bar{u}_{\lambda_1} + \lambda_2 \bar{u}_{\lambda_1}^p \geqq 0$ in Ω. By Lemma 2.3, $\bar{u}_{\lambda_1} \leqq w_{\lambda_2}$ in $\bar{\Omega}$. By the same iteration scheme as in (ii), just replacing u_λ by $\bar{u}_{\lambda_1}$, one can prove that there is a solution u_{λ_2} of (1.1) with $\lambda = \lambda_2$ such that $\bar{u}_{\lambda_1} \leqq u_{\lambda_2} \leqq w_{\lambda_2}$. Therefore, $\lambda_2 \in H$, and $H \supset (\lambda_*, \infty)$. Moreover, by (ii), for any $\lambda_2 > \lambda_1 > \lambda_*$, $\bar{u}_{\lambda_2} \geqq \bar{u}_{\lambda_1}$. By the maximum principle, we have $\bar{u}_{\lambda_1} < \bar{u}_{\lambda_2}$.

(v) (*Bounds of solutions and regularity*) From the above proof, we have

$$\underline{u}_\lambda(x) \leqq u_\lambda(x) \leqq \bar{u}_\lambda(x) \leqq w_\lambda(x),$$

for $\lambda > \lambda_*$. Then from Lemma 2.2 and that $w_\lambda(x) \leqq c_2 d(x)$, we obtain the bounds for w_λ

$$c_1 d(x)^{2/(1+\alpha)} \leqq u_\lambda(x) \leqq c_2 d(x),$$

and $\bar{u}_\lambda(x) \leqq c_2 d(x)$. To prove the lower bound for $\bar{u}_\lambda$, let $\lambda_* < \lambda_2 < \lambda_1$, $\bar{u}_{\lambda_1}$ and $\bar{u}_{\lambda_2}$ be the corresponding maximal solutions of (1.1) for $\lambda = \lambda_1$ and λ_2. Then we have

$$\Delta(\bar{u}_{\lambda_1} - \bar{u}_{\lambda_2}) + \lambda_1(\bar{u}_{\lambda_1}^p - \bar{u}_{\lambda_2}^p) - K(x)(\bar{u}_{\lambda_1}^{-\alpha} - \bar{u}_{\lambda_2}^{-\alpha}) + (\lambda_1 - \lambda_2)\bar{u}_{\lambda_2}^p = 0.$$

Since $\bar{u}_{\lambda_1}(x) \geqq \bar{u}_{\lambda_2}(x)$ by the monotonicity of $\bar{u}_\lambda$, then $\Delta(\bar{u}_{\lambda_1} - \bar{u}_{\lambda_2}) < 0$ for any $x \in \Omega$. Also, $v = \bar{u}_{\lambda_1} - \bar{u}_{\lambda_2}$ is continuous up to $\partial\Omega$, and $\partial\Omega$ satisfies an interior sphere condition at any $x_0 \in \partial\Omega$. Therefore, by the remark following the proof of the Strong Maximum Principle in [11, p. 34], we have, for any $x_0 \in \Omega$,

$$\liminf_{x \to x_0} \frac{v(x)}{|x - x_0|} = c(x_0) > 0.$$

where the angle between the vector $x - x_0$ and the normal at x_0 is less than $(\pi/2) - \delta$ for some fixed $\delta > 0$. Moreover, from the proof in [12, p. 34], we can see that $c(x_0)$ can be chosen continuously depending on x_0, hence there exists a $c(\Omega) > 0$, such that for any $x \in \Omega_\delta = \{x \in \Omega : d(x) \leqq \delta\}$,

$$v(x) \geqq c(\Omega)|x - x_0| \geqq c(\Omega)d(x)$$

for some $x_0 \in \partial\Omega$. Therefore $\bar{u}_{\lambda_1}(x) - \bar{u}_{\lambda_2}(x) \geqq c(\Omega)d(x)$. In particular, for any $\lambda > \lambda_*$, $\bar{u}_\lambda(x) \geqq c_1 d(x)$. As long as the bounds of $\bar{u}_\lambda$ are established, the regularity can be

proved in the same way as that of Theorem 1.1. We omit the details here. Notice that $\bar{u}_\lambda \in C^{1,\beta}(\bar\Omega)$ and our above arguments imply that $\partial \bar{u}_\lambda / \partial n$ exists and is negative for any $x_0 \in \partial \Omega$. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. (i) (*Existence*) We first prove that the problem $P_k(\lambda)$ has a solution. Since $K^* > 0 > K_*$, then by Theorem 1.2, there exists a $\lambda_* > 0$, such that for $\lambda > \lambda_*$, the problem

$$-\Delta v + K^* v^{-\alpha} = \lambda v^p \quad \text{in } \Omega,$$

$$v > 0 \quad \text{in } \Omega,$$

$$v = 0 \quad \text{on } \partial \Omega.$$

has a maximal solution $v_\lambda \in E$. Moreover, $v_k = v_\lambda + (1/k)$ is a subsolution of $P_k(\lambda)$. On the other hand, the problem

$$-\Delta w + K_* w^{-\alpha} = \lambda w^p \quad \text{in } \Omega,$$

$$w > 0 \quad \text{in } \Omega,$$

$$w = \frac{1}{k} \quad \text{on } \partial \Omega,$$

has a solution w_k for any $k \in \mathbf{N}$, by Lemma 2.4. Also, w_k is a supersolution of $P_k(\lambda)$. Since

$$\Delta w_k + \lambda w_k^p \leqq 0 \leqq \Delta v_k + \lambda v_k^p,$$

and $v_k = w_k$ on $\partial \Omega$, $\Delta v_k \in L^1(\Omega)$, then by Lemma 2.3,

$$v_k \leqq w_k \quad \text{in } \bar\Omega.$$

Therefore, by the standard super- and subsolution argument, there exists a minimal solution $u_\lambda^{(1)}$ of $P_1(\lambda)$, satisfying $v_1 \leqq u_\lambda^{(1)} \leqq w_1$. Similarly, taking $u_\lambda^{(1)}$ and v_2 as a pair of super- and subsolutions for $P_2(\lambda)$, we conclude that there exists a minimal solution $u_\lambda^{(2)} \in C^{2,\beta}(\bar\Omega)$ of $P_2(\lambda)$ such that $v_2(x) \leqq u_\lambda^{(2)}(x) \leqq u_\lambda^{(1)}(x)$.

Repeating the above arguments, we obtain a sequence $\{u_\lambda^{(i)}\}$ which is decreasing in i, for $i = 1, 2, \ldots$. Therefore, similar to the proof of Theorem 1.2(i), we obtain a solution $u_\lambda(x) = \lim_{i \to \infty} u_\lambda^{(i)}(x)$, and $v_\lambda \leqq u_\lambda \leqq w_1$.

(ii) (*Dependence on* λ) Let $\lambda_* < \lambda_1 < \lambda_2$, $\bar{u}_{\lambda_1}$ and $\bar{u}_{\lambda_2}$ be the corresponding solutions of (1.1) for $\lambda = \lambda_1$ and λ_2 which we obtain in (i). We observe that for any $k \geqq 1$, $u_{\lambda_2}^{(k)}$ is a supersolution of $P_k(\lambda_1)$, and

$$u_{\lambda_2}^{(k)} \geqq v_{\lambda_2} + \frac{1}{k} \geqq v_{\lambda_1} + \frac{1}{k} \quad \text{in } \bar\Omega.$$

Therefore $u_{\lambda_2}^{(k)} \geqq u_{\lambda_1}^{(k)}$, since $u_{\lambda_1}^{(k)}$ is the minimal solution of $P_k(\lambda_1)$ which satisfies $u_{\lambda_1}^{(k)} \geqq v_{\lambda_1} + (1/k)$. Therefore we must have $u_{\lambda_1} \leqq u_{\lambda_2}$.

(iii) (*Bounds of solutions*) This should come directly from the fact that

$$v_\lambda \leqq u_\lambda \leqq w_1$$

and the bounds of the solutions in Theorems 1.1 and 1.2. $\quad \square$

Acknowledgment

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