Research Article

On a Singular Second-Order Multipoint Boundary Value Problem at Resonance

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The aim of this paper is to derive existence results for a second-order singular multipoint boundary value problem at resonance using coincidence degree arguments.

1. Introduction

In this paper we derive existence results for the second-order singular multipoint boundary value problem of the form

$$x''(t) = f(t, x(t), x'(t)) + g(t), \quad 0 < t < 1,$$

$$x'(0) = 0,$$

$$x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i),$$

(1)

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is Caratheodory's function (i.e., for each $(x, y) \in \mathbb{R}^2$ the function $f(\cdot, x, y)$ is measurable on $[0,1]$; for a.e. $t \in [0,1]$, the function $f(t, \cdot, \cdot)$ is continuous on $\mathbb{R}^2$). Let $\xi_i \in (0,1)$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in (0,1)$ $i = 1, 2, \ldots, m-2$, and $\sum_{i=1}^{m-2} a_i = 1$, where f and g have singularity at $t = 1$.

In [1] Gupta et al. studied the above equation when f and g have no singularity and $\sum_{i=1}^{m-2} a_i \neq 1$. They obtained existence of a $C^1[0,1]$ solution by utilising the Leray-Schauder continuation principle. In [2] Ma and O'Regan derived existence results for the same equation when f and g have a singularity at $t = 1$ and $\sum_{i=1}^{m-2} a_i \neq 1$. They also utilised the Leray-Schauder continuation method. These results correspond to the nonresonance case. The purpose of this article is therefore to derive existence results for (1) when $\sum_{i=1}^{m-2} a_i = 1$ (the resonance case) and when f and g have

a singularity at $t = 1$. We shall employ coincidence degree arguments in obtaining our results. In this case, the methods used in [1, 2] are not valid.

Research on singular differential equations is important because singular differential equations are useful in the modeling of many problems in the physical and engineering sciences; see [3].

In general singular boundary value problems can be difficult to solve because they may blow up near the singularity. The existence and multiplicity of solutions for second-order nonsingular boundary value problems have been extensively studied by many researchers. However to the best of our knowledge the corresponding problem for second-order differential equations at resonance and with a singularity had not received much attention in the literature. For recent results in these directions see [1, 2, 4–9] and references therein.

The rest of this paper is organised as follows. In Section 2, we present some definitions, lemmas, and theorems necessary for obtaining our main results. In Section 3, we derive some lemmas and the main theorem. In what follows we shall utilise the following assumptions:

(A0) For $\xi_i \in (0,1)$, $i = 1, 2, \ldots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $\sum_{i=1}^{m-2} a_i = 1$.

(A1) There exist $a(t), c(t) \in L^1[0,1]$ with $(1-t)a(t), (1-t)c(t), b(t) \in L^1[0,1]$ and $|f(t, x, y)| \leq a(t)|x| + b(t)|y| + c(t)$, a.e., $t \in [0,1]$, $(x, y) \in \mathbb{R}^2$.

(A2) $g : [0, 1] \to \mathbb{R}$ is such that $\int_0^1 (1 - t)|g(t)| < \infty$.

2. Preliminaries

In this section we state some definitions, theorems, and lemmas that will be used in the subsequent section.

Definition 1. Let X and Z be real Banach spaces. One says that the linear operator $L : \text{dom } L \subset X \to Z$ is a Fredholm mapping of index zero if $\text{Ker } L$ and $Z/\text{Im } L$ are of finite dimension, where $\text{Im } L$ denotes the image of L.

As a result of Definition 1, we will require the continuous projections $P : X \to X$, $Q : Z \to Z$ such that $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L$, $X = \text{Ker } L \oplus \text{Ker } P$, $Z = \text{Im } L \oplus \text{Im } Q$, and $L|_{\text{dom } L \cap \text{Ker } P} : \text{dom } L \cap \text{Ker } P \to \text{Im } L$ is an isomorphism.

Definition 2. Let L be a Fredholm mapping of index zero and Ω a bounded open subset of X such that $\text{dom } L \cap \Omega \neq \phi$. The map $N : X \to Z$ is called L-compact on $\overline{\Omega}$, if the map $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)$ is compact, where one denotes by $K_p : \text{Im } L \to \text{dom } L \cap \text{Ker } P$ the generalised inverse of L. In addition N is L-completely continuous if it is L-compact on every bounded $\Omega \subset X$.

Theorem 3 (see [10]). *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\text{dom } L \setminus \text{Ker } L) \cap \partial\Omega] \times (0, 1)$.*

(ii) *$Nx \notin \text{Im } L$, for every $x \in \text{Ker } L \cap \partial\Omega$.*

(iii) *$\deg(QN|_{\text{Ker } L \cap \partial\Omega}, \Omega \cap \text{Ker } L, 0) \neq 0$,*

with $Q : Z \to Z$ being a continuous projection such that $\text{Ker } Q = \text{Im } L$. Then the equation $Lx = Nx$ has at least one solution in $\text{dom } L \cap \overline{\Omega}$.

In what follows, we shall make use of the following classical spaces, $C[0, 1]$, $C^1[0, 1]$, $L^1[0, 1]$, and $L^\infty[0, 1]$. Let $AC[0, 1]$ denote the space of all absolute continuous functions on $[0, 1]$, $AC^1[0, 1] = \{x \in C^1[0, 1] : x'(t) \in AC[0, 1]\}$, $L^1_{\text{loc}}[0, 1] = \{x : x|_{[0,d]} \in L^1[0, d]$ for every compact interval $[0, d] \subseteq [0, 1)\}$.
$AC_{\text{loc}}[0, 1) = \{x : x|_{[0,d]} \in AC[0, d]\}$.
Let Z be the Banach space defined by

$$Z = \left\{ y \in L^1_{\text{loc}}[0, 1) : (1 - t) y(t) \in L^1[0, 1] \right\}, \quad (2)$$

with the norm

$$\|y\|_Z = \int_0^1 (1 - t) |y(t)| \, dt. \quad (3)$$

Let X be the Banach space

$$X = \left\{ x \in C^1[0, 1) : x \right.$$
$$\left. \in C[0, 1], \lim_{t \to 1^-} (1 - t) x'(t) \text{ exists} \right\}, \quad (4)$$

with the norm

$$\|x\|_X = \max \left\{ \|x\|_\infty, \left\|(1 - t) x'(t)\right\|_\infty \right\}$$
$$\text{where } \|x\|_\infty = \sup_{t \in [0,1]} |x(t)|. \quad (5)$$

We denote the norm in $L^1[0, 1]$ by $\| \cdot \|_1$. We define the linear operator $L : \text{dom } L \subset X \to Z$ by

$$Lx = x''(t), \quad (6)$$

where $\text{dom } L = \{x \in X : x'(0) = 0, \ x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)\}$ and $N : X \to Z$ is defined by

$$Nx = f\left(t, x(t), x'(t)\right) + g(t). \quad (7)$$

Then boundary value problem (1) can be written as

$$Lx = Nx. \quad (8)$$

Lemma 4 (see [2]). *Let $y \in Z$. Then*

(i) *$\int_0^t y(s) ds \in L^1[0, 1]$.*

(ii) *$\lim_{t \to 1^-} (1 - t) \int_0^t y(s) ds = 0$.*

Lemma 5. *If $\sum_{i=1}^{m-2} a_i = 1$ then*

(i) *$\text{Ker } L = \{x \in \text{dom } L : x(t) = c, \ c \in \mathbb{R}, \ t \in [0, 1]\}$;*

(ii) *$\text{Im } L = \{y \in Z : \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \int_0^s y(\tau) d\tau \, ds = 0\}$;*

(iii) *$L : \text{dom } L \subset X \to Z$ is a Fredholm operator of index zero and the continuous operator $Q : Z \to Z$ can be defined by*

$$Qy = \frac{e^t}{h} \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \int_0^s y(\tau) \, d\tau \, ds, \quad (9)$$

where $h = \sum_{i=1}^{m-2} a_i[e + \xi_i - e^{\xi_i} - 1] \neq 0$.

(iv) *The linear operator $K_p : \text{Im } L :\to \text{dom } L \cap \text{Ker } P$ can be defined as*

$$K_p y = \int_0^t \int_0^s y(\tau) \, d\tau \, ds. \quad (10)$$

(v) *$\|K_p y\|_X \leq \|y\|_Z$ for all $y \in Z$.*

Proof. (i) It is obvious that

$$\text{Ker } L = \{x \in \text{dom } L : x(t) = c, \ c \in \mathbb{R}\}. \quad (11)$$

(ii) We show that

$$\text{Im } L = \left\{ y \in Z : \sum_{i=1}^{m-2} a_i \int_{\xi_i}^1 \int_0^s y(\tau) \, d\tau \, ds = 0 \right\}. \quad (12)$$

To do this, we consider the problem

$$x''(t) = y(t), \quad (13)$$

and we show that (13) has a solution $x(t)$ satisfying $x'(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ if and only if

$$\sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \int_{0}^{s} y(\tau) \, d\tau \, ds = 0. \tag{14}$$

Suppose (13) has a solution $x(t)$ satisfying $x'(0) = 0$, $x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i)$; then we obtain from (13) that

$$x(t) = x(0) + \int_{0}^{t} \int_{0}^{s} y(\tau) \, d\tau \, ds, \tag{15}$$

and applying the boundary conditions we get

$$\sum_{i=1}^{m-2} a_i \int_{0}^{\xi_i} \int_{0}^{s} y(\tau) \, d\tau \, ds = \int_{0}^{1} \int_{0}^{s} y(\tau) \, d\tau \, ds, \tag{16}$$

since $\sum_{i=1}^{m-2} a_i = 1$, and using (i) of Lemma 4 we get

$$\sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \int_{0}^{s} y(\tau) \, d\tau \, ds = 0. \tag{17}$$

On the other hand if (14) holds, let $x_0 \in \mathbb{R}$; then $x(t) = x_0 + \int_{0}^{t} \int_{0}^{s} y(\tau) d\tau \, ds$, where $y \in Z$ and $x'(t) \in AC_{\text{loc}}[0,1)$. Then from Lemma 4 $\int_{0}^{t} y(\tau) d\tau \in L^1[0,1]$ and $\lim_{t \to 1^-}(1-t)x'(t) = \lim_{t \to 1^-}(1-t)\int_{0}^{t} y(\tau) d\tau = 0$. Hence

$$x''(t) = y(t). \tag{18}$$

(iii) For $y \in Z$, we define the projection Qy as

$$Qy = \frac{e^t}{h} \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \int_{0}^{s} y(\tau) \, d\tau \, ds, \quad t \in [0,1], \tag{19}$$

where $h = \sum_{i=1}^{m-2} a_i[e + \xi_i - e^{\xi_i} - 1] \neq 0$.

We show that $Q : Z \to Z$ is well defined and bounded.

$$|Qy(t)| \leq \frac{|e^t|}{|h|} \sum_{i=1}^{m-2} |a_i| \int_{0}^{1} (1-s) |y(s)| \, ds$$

$$= \frac{1}{|h|} \sum_{i=1}^{m-2} |a_i| \|y\|_Z |e^t|,$$

$$\|Qy\|_Z \leq \int_{0}^{1} (1-t) |Qy(t)| \, dt \tag{20}$$

$$\leq \frac{1}{|h|} \sum_{i=1}^{m-2} |a_i| \|y\|_Z \int_{0}^{1} (1-t) |e^t| \, dt$$

$$= \frac{1}{|h|} \sum_{i=1}^{m-2} |a_i| \|y\|_Z \|e^t\|_Z.$$

In addition it is easily verified that

$$Q^2 y = Qy, \quad y \in Z. \tag{21}$$

We therefore conclude that $Q : Z \to Z$ is a projection. If $y \in \text{Im } L$, then from (14) $Qy(t) = 0$. Hence $\text{Im } L \subseteq \text{Ker } Q$. Let $y_1 = y - Qy$; that is, $y_1 \in \text{Ker } Q$. Then

$$\sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \int_{0}^{s} y_1(\tau) \, d\tau \, ds$$

$$= \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \int_{0}^{s} y(\tau) \, d\tau \, ds - \frac{1}{h} \sum_{i=1}^{m-2} \int_{\xi_i}^{1} \int_{0}^{1} y(\tau) \, d\tau \, ds \tag{22}$$

$$\cdot h = 0.$$

Thus, $y_1 \in \text{Im } L$ and therefore $\text{Ker } Q \subseteq \text{Im } L$ and hence $Z = \text{Im } L + \text{Im } Q = \text{Im } L + \mathbb{R}$. It follows that since $\text{Im } L \cap \mathbb{R} = \{0\}$, then $Z = \text{Im } L \oplus \text{Im } Q$. Therefore

$$\dim \text{Ker } L = \dim \text{Im } Q = \dim \mathbb{R} = \text{codim } \text{Im } L = 1. \tag{23}$$

This implies that L is Fredholm mapping of index zero.

(iv) We define $P : X \to X$ by

$$Px = x(0), \tag{24}$$

and clearly P is continuous and linear and $P^2 x = P(Px) = Px(0) = x(0) = Px$ and $\text{Ker } P = \{x \in X : x(0) = 0\}$. We now show that the generalised inverse $K_p : \text{Im } L \to \text{dom } L \cap \text{Ker } P$ of L is given by

$$K_p y = \int_{0}^{t} \int_{0}^{s} y(\tau) \, d\tau \, ds. \tag{25}$$

For $y \in \text{Im } L$ we have

$$(LK_p) y(t) = \left[(K_p y)(t) \right]'' = y(t) \tag{26}$$

and for $x \in \text{dom } L \cap \text{Ker } P$ we know that

$$(K_p L) x(t) = \int_{0}^{t} \int_{0}^{s} x''(\tau) \, d\tau \, ds$$

$$= \int_{0}^{t} (t-s) x''(s) \, ds \tag{27}$$

$$= x(t) - x'(0) t - x(0) = x(t)$$

since $x \in \text{dom } L \cap \text{Ker } P$, $x(0) = 0$, and $Px = 0$.

This shows that $K_p = (L|_{\text{dom } L \cap \text{Ker } P})^{-1}$.

(v)

$$\|K_p y\|_{\infty} \leq \max_{t \in [0,1]} \int_{0}^{t} (t-s) |y(s)| \, ds$$

$$\leq \int_{0}^{1} (1-s) |y(s)| \, ds \leq \|y\|_Z, \tag{28}$$

$$\left\| (1-t) (K_p y)' \right\|_{\infty} \leq \max_{t \in [0,1]} \int_{0}^{t} (1-s) |y(s)| \, ds$$

$$\leq \|y\|_Z.$$

We conclude that

$$\left\| K_p y \right\|_X \leq \left\| y \right\|_Z. \tag{29}$$

$\square$

Lemma 6. *The operator $N : X \to Z$ defined by*

$$Nx(t) = f\left(t, x(t), x'(t)\right) + g(t), \quad t \in (0, 1) \tag{30}$$

is L-completely continuous.

Proof. Suppose Ω is an open bounded subset of X. Let $R_1 = \sup\{\|x\|_X : x \in \Omega\}$. From condition (A1) and each $x_n \in \Omega$ we have

$$\left| Nx_n(t) \right| \leq \left| f(t), x_n(t), x'_n(t) \right| + \left| g(t) \right|$$

$$\leq a(t) R_1 + b(t) \frac{R_1}{1-t} + r(t) + \left| g(t) \right| \tag{31}$$

$$= \varphi(t).$$

We can deduce from (A1) and (A2) that $\varphi(t) \in Z$:

$$\left| QNx_n(t) \right| = \left| \frac{e^t}{h} \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{1} \int_0^s Nx_n(\tau) \, d\tau \, ds \right|$$

$$\leq \frac{1}{|h|} \sum_{i=1}^{m-2} |a_i| \int_0^1 (1-s) \left| Nx_n(x) \right| ds \left| e^t \right|$$

$$\leq \frac{1}{|h|} \sum_{i=1}^{m-2} |a_i| \int_0^1 (1-s) \left| \varphi(s) \right| ds \left| e^t \right|$$

$$\leq \frac{1}{|h|} \sum_{i=1}^{m-2} |a_i| \left\| \varphi \right\|_Z \left| e^t \right|, \quad t \in (0, 1), \tag{32}$$

$$\left\| QNx_n \right\|_Z \leq \frac{1}{|h|} \sum_{i=1}^{m-2} |a_i| \left\| \varphi \right\|_Z \int_0^1 (1-t) \left| e^t \right| dt$$

$$= \frac{1}{|h|} \sum_{i=1}^{m-2} |a_i| \left\| \varphi \right\|_Z \left\| e^t \right\|_Z.$$

This shows that $QN(\overline{\Omega})$ is bounded in Z and QN is continuous by using the Lebesgue Dominated Convergence Theorem. Next we show that $K_{P,Q}N(\overline{\Omega}) = K_P(I - Q)N(\overline{\Omega})$ is compact.

By using (31) we derive

$$\left| K_P N_n(t) \right| \leq \int_0^t (t-s) \left| Nx_n(s) \right| ds$$

$$\leq \int_0^1 (1-s) \left| \varphi(s) \right| ds = \left\| \varphi \right\|_Z,$$

$$\left| Nx_n(t) - QNx_n(t) \right| \leq \left| Nx_n(t) \right| + \left| QNx_n(t) \right|$$

$$\leq \left| \varphi(t) \right| + \frac{1}{|h|} \sum_{i=1}^{m-2} |a_i| \left\| \varphi \right\|_Z \left| e^t \right|$$

$$= \alpha_r(t),$$

$$\left| K_{P,Q}Nx_n(t) \right| = \left| K_p\left(Nx_n - QNx_n \right)(t) \right|$$

$$\leq K_P \alpha_r(t) \leq \left\| \alpha_r \right\|_Z. \tag{33}$$

This indicates that the sequence $\{K_{P,Q}Nx_n(\overline{\Omega})\}$ is uniformly bounded in $C[0, 1]$. Also for $t \in [0, 1)$

$$\left| (1-t)\left(K_{P,Q}Nx_n \right)'(t) \right|$$

$$= \left| (1-t) \int_0^t \left(Nx_n(s) - QNx_n(s) \right) ds \right| \tag{34}$$

$$\leq \int_0^1 (1-s) \alpha_r(s) \, ds \leq \left\| \alpha_r \right\|_Z.$$

Hence the sequence $K_{P,Q}Nx_n(t)$ is bounded in $C[0, 1]$ and $\lim_{t \to 1^-}(1-t)(K_{P,Q}Nx_n)'(t) = 0$. Thus $K_{P,Q}Nx_n(t)$ is bounded in X.

Next we show that the sequence $\{K_{P,Q}Nx_n(t)\}$ is equicontinuous. Let $t_1, t_2 \in [0, 1], t_1 < t_2$; then

$$\left| K_{P,Q}x_n(t_2) - K_{P,Q}x_n(t_1) \right| = \left| \int_{t_1}^{t_2} \left(K_{P,Q}Nx_n \right)'(s) \, ds \right|$$

$$\leq \int_{t_1}^{t_2} \int_0^s \left| \left(Nx_n(\tau) - QNx_n(\tau) \right) \right| d\tau \, ds \tag{35}$$

$$\leq \int_{t_1}^{t_2} \int_0^s \alpha_r(\tau) \, d\tau \, ds,$$

for every $t_1, t_2 \in [0, 1]$. By (i) of Lemma 4 $\int_0^s \alpha_r(\tau) d\tau \in L^1[0, 1]$. Thus the sequence $\{K_{P,Q}Nx_n(t)\}$ is equicontinuous on $[0, 1]$ and by Arzela-Ascoli Theorem is convergent. Next we prove that the sequence $\{(1 - t)(K_{P,Q}Nx_n)'\}$ is also equicontinuous on $[0, 1]$. We have for $t \in [0, 1]$

$$\left| \left[(1-t)\left(K_{P,Q}Nx_n \right)'(t) \right]' \right|$$

$$= \left| -\left(K_{P,Q}Nx_n \right)'(t) + (1-t)\left(K_{P,Q}Nx_n \right)''(t) \right|$$

$$\leq \left| \int_0^t \left(Nx_n(s) - QNx_n(s) \right) ds \right| + \left| (1-t)Nx_n(t) \right| \tag{36}$$

$$\leq \int_0^t \alpha_r(s) \, ds + (1-t)\varphi(t) = \psi(t).$$

Using (i) of Lemma 4 and the fact that $\alpha_r(t)$ and $\varphi(t)$ are in Z we conclude that $\psi(t) \in L^1[0,1]$. Therefore

$$
\left| (1 - t_2) (K_{P,Q} N x_n)' (t_2) - (1 - t_1) (K_{P,Q} N x_n)' (t_1) \right|
$$
$$
= \left| \int_{t_1}^{t_2} \left[(1 - s) (K_{P,Q} N x_n)' (s) \right]' ds \right| \tag{37}
$$
$$
\leq \int_{t_1}^{t_2} \left| \left[(1 - s) (K_{P,Q} N x_n)' (s) \right]' \right| ds \leq \int_{t_1}^{t_2} \psi(s)\, ds.
$$

The sequence $\{(1 - t)(K_{P,Q} N x_n)'(t)\}$ is therefore equicontinuous on $[0,1)$ and therefore converges to some $(1 - t)(K_{P,Q} N x_0)'(t) \in C[0,1]$ with $\lim_{t \to 1^-}(1 - t)[(1 - t)(K_{P,Q} N x_n)'(t)] = 0$, $t \in [0,1)$.

We then conclude that $K_{P,Q}$ is relatively compact and since $QN(\overline{\Omega})$ is bounded we conclude from Definition 2 that N is L-compact on every bounded subset Ω of X and hence N is L-completely continuous. $\square$

3. Main Result

In this section we will state and prove the main existence results for problem (1).

Theorem 7. *Assume that the following conditions are satisfied:*

(H1) *There exists a positive constant B_1 such that, for each $x \in \operatorname{dom} L$, if $|x(t)| > B_1$ for all $t \in [0,1]$ then*

$$
QNx(t)
$$
$$
= \frac{e^t}{h} \sum_{i=1}^{m=2} a_i \int_{\xi_i}^1 \int_0^s \left[f\left(\tau, x(\tau), x'(\tau) \right) + g(\tau) \right] d\tau\, ds \tag{38}
$$
$$
\neq 0.
$$

(H2) *There exists a positive constant B_2 such that for $c \in \mathbb{R}$ and $|c| > B_2$ either (i) $QN(c) \geq 0$ or (ii) $QN(c) \leq 0$.*

Then (1) has at least one solution in X provided

$$
\|a\|_Z + \|b\|_1 < \frac{1}{2}. \tag{39}
$$

To prove Theorem 7, we first establish some lemmas.

Lemma 8. *Let $\Omega_1 = \{x \in \operatorname{dom} L \setminus \operatorname{Ker} L : Lx = \lambda N x,\ \lambda \in (0,1)\}$ then Ω_1 is bounded in X.*

Proof. Let $x \in \Omega_1$. We let $Lx = \lambda Nx$, $0 < \lambda < 1$. Since $\lambda \neq 0$ it is clear that $Nx \in \operatorname{Im} L = \operatorname{Ker} Q$; hence $QNx = 0$ for all

$t \in [0,1]$. Therefore by assumption (H1) there exist $t_0 \in [0,1]$ such that $|x(t_0)| < B_1$. Now

$$
\int_0^t \int_0^s x''(\tau)\, d\tau\, ds = \int_0^{t_0} (t_0 - s) x''(s)\, ds \tag{40}
$$
$$
= x(t_0) - x(0)
$$

$$
\|Px\|_X = |x(0)|
$$
$$
= \left| x(t_0) + \int_0^{t_0} (t_0 - s) x''(s)\, ds \right| \tag{41}
$$
$$
\leq |x(t_0)| + \int_0^1 (1 - s) |x''(s)|\, ds
$$
$$
\leq B_1 + \|Lx\|_Z \leq B_1 + \|Nx\|_Z.
$$

We note that $(I - P)x \in \operatorname{dom} L \cap \operatorname{Ker} P$:

$$
\|(I - P)x\|_X = \|K_P L (I - P) x\|_X \leq \|K_P Lx\|_X \tag{42}
$$
$$
\leq \|Lx\|_Z < \|Nx\|_Z.
$$

From (41) and (42) we get

$$
\|x\|_X = \|Px + (I - P)x\|_X \leq \|Px\|_X + \|(I - P)x\|_X \tag{43}
$$
$$
< B_1 + 2\|Nx\|_Z.
$$

From the definition of N we obtain

$$
\|Nx\|_Z = \|(1 - t)(Nx)(t)\|_1 = \int_0^1 (1 - t)
$$
$$
\cdot \left[f\left(t, x(t), x'(t) \right) + g(t) \right] dt
$$
$$
\leq \int_0^1 \left[(1 - t) a(t) |x(t)| + |b(t)| (1 - t) x'(t) \right. \tag{44}
$$
$$
\left. + (1 - t) |r(t)| + (1 - t) |g(t)| \right] dt \leq \|a\|_Z
$$
$$
\cdot \|x\|_\infty + \|b\|_1 \left\| (1 - t) x' \right\|_\infty + \|r\|_Z + \|g\|_Z.
$$

From (43) and (44) we get

$$
\|x\|_X < B_1
$$
$$
+ 2 \left[\|a\|_Z \|x\|_X + \|b\|_1 \|x\|_X + \|r\|_Z + \|g\|_Z \right]. \tag{45}
$$

Since $1 - 2[\|a\|_Z + \|b\|_1] > 0$ we obtain that

$$
\|x\|_X < \frac{B_1 + 2\|r\|_Z}{1 - 2\left[\|a\|_Z + \|b\|_1 \right]} + \frac{2\|g\|_Z}{1 - 2\left[\|a\|_Z + \|b\|_1 \right]}. \tag{46}
$$

Therefore Ω_1 is bounded in X. $\square$

Lemma 9. *The set $\Omega_2 = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}$ is a bounded subset of X.*

Proof. Let $x \in \Omega_2$ with $x(t) = c$, $c \in \mathbb{R}$. Then $QN(c) = 0$ implies $N(c) \in \operatorname{Im} L = \operatorname{Ker} Q$. We therefore derive from (H2) that

$$
\|x\|_X = |c| = \max \{|c|, \|(1 - t) 0\|\} = |c| < B_2. \tag{47}
$$
$\square$

Lemma 10. *The sets $\Omega_3^+ = \{x \in \operatorname{Ker} L : \lambda x + (1 - \lambda)QNx = 0, \ \lambda \in [0, 1]\}$ and $\Omega_3^- = \{x \in \operatorname{Ker} L : -\lambda x + (1 - \lambda)QNx = 0, \ \lambda \in [0, 1]\}$ are bounded in X provided (H2)(i) and (H2)(ii) are satisfied simultaneously.*

Proof. If $QN(c) \geq 0$ then, for $x \in \Omega_3^+$ with $x(t) = c, c \in \mathbb{R}$, we have

$$\lambda c = -(1 - \lambda) QN(c). \tag{48}$$

If $\lambda = 0$, it follows from (48) that $N(c) \in \operatorname{Ker} Q = \operatorname{Im} L$; that is, $N(c) \in \Omega_2$, and therefore by Lemma 9. we have $\|x\|_X \leq B_2$. However if $\lambda \in (0, 1)$ and $\|c\| > B_2$ then using assumption (H2)(i) we obtain the contradiction

$$\lambda c^2 = -(1 - \lambda) cQN(c) \leq 0. \tag{49}$$

Thus $\|x\|_X = |c| < B_2$. Hence Ω_3^+ is bounded in X. We can use the same argument to prove that Ω_3^- is also bounded in X. $\qquad\square$

Proof of Theorem 7. We show that the conditions of Theorem 3 are satisfied where Ω is an open and bounded set such that $\bigcup_{i=1}^{3} \Omega_i \subset \Omega$. It is easily seen that conditions (i) and (ii) of Theorem 3 are satisfied by using Lemmas 8 and 9. To verify the third condition we set $H(x, \lambda) = \pm\lambda x + (1-\lambda)QNx$. We choose the isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ defined by $J(c) = c, c \in \mathbb{R}$. By Lemma 10, we derive that $H(x, \lambda) \neq 0$ for all $(x, \lambda) \in (\operatorname{Ker} L \cap \partial\Omega) \times [0, 1]$. Hence

$$\deg\left(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0\right)$$
$$= \deg\left(\pm J, \Omega \cap \operatorname{Ker} L, 0\right) \neq 0. \tag{50}$$

Therefore problem (1) has at least one solution in X. $\qquad\square$

Conflicts of Interest

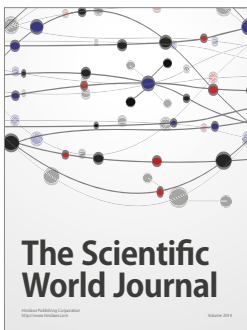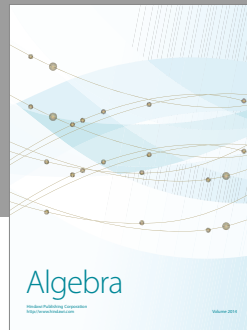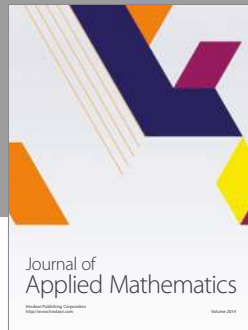The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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