5. *On a Stochastic Integral Equation.*

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(Comm. by S. KAKEYA, M.I.A., Feb. 12, 1946.)

In his note "Stochastic Integral"[1] the author has discussed an integral of the type $\int_0^t f(\tau, \omega)\, d_\tau g(\tau, \omega)$, where ω is a variable taking values in a probability field (Ω, P) and $g(t, \omega)$ is a normalized brownian motion on (Ω, P). This note is devoted to the investigation of a stochastic integral equation:

(1) $$x(t, \omega) = c + \int_0^t a(\tau, x(\tau, \omega))\, d\tau + \int_0^t b(\tau, x(\tau, \omega))\, d_\tau g(\tau, \omega),$$

which is closely related to the researches of Markoff process by many authors, especially by S. Bernstein,[2] A. Kolmogoroff,[3] and W. Feller.[4]

Theorem. Let $a(t, x)$ *and* $b(t, x)$ *be continuous in* (t, x) *and satisfy*

(2) $|a(t,x) - a(t,y)| \leqq A|x-y|$, (3) $|b(t,x) - b(t,y)| \leqq B|x-y|$, *where* $0 \leqq t \leqq 1$ *and* $-\infty < x, y < \infty$. *Then the integral cquation (1) has one and only one continuous (in t with P-measure 1) solution.*

Proof. Firstly we shall find a solution by the method of successive approximation. We define $x_k(t, \omega)$ for $k = 0, 1, 2, \ldots$ as follows,

(4) $$x_0(t, \omega) \equiv c,$$

(5) $$x_k(t, \omega) = c + \int_0^t a(\tau, x_{k-1}(\tau, \omega))\, d\tau + \int_0^t b(\tau, x_{k-1}(\tau, \omega))\, d_\tau g(\tau, \omega);$$

the possibility of these definitions can be verified recursively if we make use of the properties of the stochastic integral shown in S.I..

By (5) we have, for $k = 0, 1, 2, \ldots.$

(6) $$x_{k+1}(t, \omega) - x_k(t, \omega) = \int_0^t (a(\tau, x_k(\tau, \omega)) - a(\tau, x_{k-1}(\tau, \omega)))\, d\tau + \int_0^t (b(\tau, x_k(\tau, \omega)) - b(\tau, x_{k-1}(\tau, \omega)))\, d_\tau g(\tau, \omega).$$

Since $a(t, x)$ and $b(t, x)$ are continuous, $|a(t, c)|$ and $|b(t, c)|$ are bounded in $0 \leqq t \leqq 1$ by a finite upper bound, say M. Then we have

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1) These proceedings Vol. XX. No. 8. p. 519. This paper will be cited as S.I. in the following.

2) S. Bernstein: Equations différrentielles stochastiques, Actuarités Scientifiques 738.

3) A. Kolmogoroff: Über die analytischen Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann. 104, p. 415.

4) W. Feller: Zur Theorie der stochastischen Prozesse. (Existenz und Eindeutigkeitssätze.), Math. Ann. 113, p. 113.

(7) $$E(x_1(t,\omega)-x_0(t,\omega))^2 \leqq \left\{\left(E\left(\int_0^t a(\tau,c)d\tau\right)^2\right)^{\frac{1}{2}}+\left(E\left(\int_0^t b(\tau,c)d_\tau g(\tau,\omega)\right)^2\right)^{\frac{1}{2}}\right\}^2$$
$$= \left\{\left|\int_0^t a(\tau,c)d\tau\right| + \left(\int_\Omega \int_0^t b(\tau,c)^2 d\tau P(d\omega)\right)^{\frac{1}{2}}\right\}^2$$
$$\leqq (Mt + Mt^{\frac{1}{2}})^2 \leqq 4M^2 t.$$

In general we can show the following inequalities by mathematical induction for $k = 1, 2, 3, \dots,$

(8) $$E\left(\int_0^t |a(\tau,x_k(\tau,\omega))-a(\tau,x_{k-1}(\tau,\omega))|d\tau\right)^2 \leqq 4A^2(A+B)^{2(k-1)}M^2 \frac{t^{k-1}}{\underline{|k+1}},$$

(9) $$E\left(\int_0^t (b(\tau,x_k(\tau,\omega))-b(\tau,x_{k-1}(\tau,\omega))d_\tau g(\tau,\omega)\right)^2$$
$$\leqq 4B^2(A+B)^{2(k-1)}M^2 \frac{t^{k-1}}{\underline{|k-1}},$$

(10) $$E(x_k(t,\omega)-x_{k-1}(t,\omega))^2 \leqq 4(A+B)^{2(k-1)}M^2 \frac{t^k}{\underline{|k}}$$

By Bienaymé's inequality we can deduce from (8)

(11) $$P\left\{\omega; \int_0^1 |a(\tau,x_k(\tau,\omega))-a(\tau,x_{k-1}(\tau,\omega))|d\tau > \frac{1}{2^{k+1}}\right\}$$
$$\leqq 4A^2(A+B)^{2(k-1)}M^2 \frac{2^{k+1}}{\underline{|k+1}}.$$

Since the series whose k-th term in the above right side is evidently convergent, we can conclude by Borel-Cantelli's theorem with P-measure 1 that

$$\int_0^1 |a(\tau,x_k(\tau,\omega))-a(\tau,x_{k-1}(\tau,\omega))|d\tau \leqq \frac{1}{2^{k+1}},$$

a fortiori

(12) $$\sup_{0\leqq t\leqq 1}\left|\int_0^t (a(\tau,x_k(\tau,\omega))-(a\,\tau,x_{k-1}(\tau,\omega)))d\tau\right| \leqq \frac{1}{2^{k+1}}$$

but for finite exceptional values of k.

By a property of stochastic integral we obtain from (9)

(13) $$P\left\{\omega; \sup_{0\leqq t\leqq 1}\left|\int_0^t (b(\tau,x_k(\tau,\omega))-b(\tau,x_{k-1}(\tau,\omega)))d_\tau g(\tau,\omega)\right| > \frac{1}{2^{k+1}}\right\}$$
$$\leqq 4B^2(A+B)^{2(k-1)}M^2 \frac{2^{k+1}}{\underline{|k+1}}$$

and so we can see, by making use of Borel-Cantelli's theorem again, with P-measure 1, that

(14) $$\sup_{0\leqq t\leqq 1}\left|\int_0^t (b(\tau,x_k(\tau,\omega))-b(\tau,x_{k-1}(\tau,\omega)))d_\tau g(\tau,\omega)\right| \leqq \frac{1}{2^{k+1}}$$

but for finite exceptional values of k.

From (6), (12) and (14) we see, with P-measure 1, that

(15) $$\sup_{0\leqq t\leqq 1}|x_k(t,\omega)-x_{k-1}(t,\omega)| \leqq \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} = \frac{1}{2^k}$$

but for finite exceptional values of k. Thus the sequence $x_k(t,\omega)$ is uniformly (in $0\leqq t\leqq 1$) convergent with P-measure 1. We shall denote the limit by

$x(t, \omega)$, which will be shown to be a solution of (1) in the following.

Integrating both sides of (10) from $t = 0$ to $t = 1$, we obtain

(16) $$\int_0^1 \int_\Omega (x_k(t, \omega) - x_{k-1}(t, \omega))^2 P(d\omega)\, dt \leqq 4(A+B)^{2(k-1)} M^2 \frac{1}{\underline{|k+1}}.$$

Therefore $x_k(t, \omega)$ is convergent in the norm of $L_2([0,1] \times \Omega)$ and so the limit $x(t, \omega)$ belongs to $L_2([0,1] \times \Omega)$. Since $x(t, \omega)$ belongs to $\mathbf{G}$[5] by the definition and it is continuous (in t) as the uniform (in t) limit with P-measure 1, $x(t, \omega)$ belongs to $\mathbf{S}^*$.[6] Consequently we have

(17) $x(t, \omega) \varepsilon \bar{\mathbf{S}}$.[7]

Therefore we can apply Theorem 2.2 in S.I. to $x(t, \omega)$.

Now we have

(18) $$E\Big(\int_0^t a(\tau, x(\tau, \omega))\, d\tau - \int_0^t a(\tau, x_k(\tau, \omega))\Big)^2 \leqq t \int_0^t \int_\Omega A^2 (x, (\tau, \omega) - x_k(\tau, \omega))^2 P(d\omega)\, d\tau \leqq A^2 \int_0^1 \int_\Omega (x(\tau, \omega) - x_k(\tau, \omega))^2 P(d\omega)\, d\tau$$

(19) $$E\Big(\int_0^t b(\tau, x(\tau, \omega))\, d_\tau g(\tau, \omega) - \int_0^t b(\tau, x_k(\tau, \omega))\, d_\tau g(\tau, \omega)\Big)^2$$
$$\leqq t \int_0^t \int_\Omega B^2 (x(\tau, \omega) - x_k(\tau, \omega))^2 P(d\omega)\, d\tau$$
$$\leqq B^2 \int_0^1 \int_\Omega (x(\tau, \omega) - x_k(\tau, \omega))^2 P(d\omega)\, d\tau,$$

Taking the $L_2([0,1] \times \Omega)$-limits of both sides of (5) for $k \to \infty$, we obtain

(20) $$x(t, \omega) = c + \int_0^t a(\tau, x(\tau, \omega))\, d\tau + \int_0^t b(\tau, x(\tau, \omega))\, d_\tau g(\tau, \omega)$$

with P-measure 1 for any assigned t. But the above both sides are continuous in t with P-measure 1. Therefore (20) holds for any t with P-measure 1. Thus we have obtained a solution $x(t, \omega)$ of (1) which is continuous in t with P-measure 1.

Next we shall the uniqueness of the solution of (1). Let $y(t, \omega)$ and $z(t, \omega)$ be a continuous (in t with P-measure 1) solution of (1):

(21) $$y(t, \omega) = c + \int_0^t a(\tau, y(\tau, \omega))\, d\tau + \int_0^t b(\tau, y(\tau, \omega))\, d_\tau g(\tau, \omega),$$

(22) $$z(t, \omega) = c + \int_0^t a(\tau, z(\tau, \omega))\, d\tau + \int_0^t b(\tau, z(\tau, \omega))\, d_\tau g(\tau, \omega).$$

In the case that $|a(t,x)|$ and $|b(t,x)|$ are bounded by an upper bound G, we have

(23) $$E(y(t, \omega) - z(t, \omega))^2 \leqq \Big(\Big(E\Big(\int_0^t G\, d\tau\Big)^2\Big)^{\frac{1}{2}} + \Big(E\Big(\int_0^t G^2\, d\tau\Big)^2\Big)^{\frac{1}{2}}\Big)^2$$

But we have

(24) $$E(y(t, \omega) - z(t, \omega))^2 \leqq (A+B)^2 \int_0^t E(y(t, \omega) - z(t, \omega))^2\, d\tau$$

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5) Cf. S.I. 2.

6) Cf. S.I. 2.

7) Cf. S.I. Foot Note 4).

Therefore we have

(25) $E(y(t,\omega)-z(t,\omega))^2 \leqq 4G^2(A+B)^{2k}\frac{t^k}{\underline{|k}}.$

Thus we obtain, as k tends to ∞, $E(y(t,\omega)-z(t,\omega))^2=0$, and so $y(t,\omega)=z(t,\omega)$ with P-measure 1 for any t. By the continuity (in t) of $y(t,\omega)$ and $z(t,\omega)$, $y(t,\omega)=z(t,\omega)$ holds for any t with P-measure 1

In the general case we obtain, by the assumption,

(26) $|a(t,y)| \leqq |a(t,c)| + |a(t,y)-a(t,c)| \leqq M + A|y-c|,$

(27) $|b(t,y)| \leqq |b(t,c)| + |b(t,y)-b(t,c)| \leqq M + B|y-c|.$

Put

(28) $\Omega_k = \{\omega;\ \sup_{0\leqq t\leqq 1}|y(t,\omega)-c|<K\} \wedge \{\omega;\ \sup_{0\leqq t\leqq 1}|z(t,\omega)-c|<K\}$

Ω_k increases with K and tends to a set Ω^* of P-measure 1 on account of the continuity of $y(t,\omega)$ and $z(t,\omega)$.

Now we have on Ω_k

(29) $|a(t,y(t,\omega))|,\quad |b(t,y(t,\omega))|,\quad |a(t,z(t,\omega))|,\quad |b(t,z(t,\omega))|$
$< M+(A+B)K.$

Denote the right side by G and define $a_G(t,x)$ and $b_G(t,x)$ as follows.

(30) $a_G(t,x)=G,$ when $a(t,x)\geqq G,$
$a_G(t,x)=a(t,x),$ when $|a(t,x)|<G,$
$a_G(t,x)=-G,$ when $a(t,x)\leqq -G,$
$b_G(t,x)=G,$ when $b(t,x)\geqq G,$
$b_G(t,x)=b(t,x),$ when $|b(t,x)|<G,$
$b_G(t,x)=-G,$ when $b(t,x)\leqq -G.$

By (29) we have on Ω_k

(31) $a_G(t,y(t,\omega))=a(t,y(t,\omega)),\quad a_G(t,z(t,\omega))=a(t,z(t,\omega)),$
$b_G(t,y(t,\omega))=b(t,y(t,\omega)),\quad b_G(t,z(t,\omega))=b(t,z(t,\omega)).$

and so on Ω_k both $y(t,\omega)$ and $z(t,\omega)$ satisfy a stochastic integral equation:

(32) $x(t,\omega)=c+\int_0^t a_G(\tau,x(\tau,\omega))\,d\tau+\int_0^t b_G(\tau,x(\tau,\omega))\,d_\tau g(\tau,\omega),$

whicn has a unique continuous (in t with P-measure 1) solution by the argument in the above special case. Thus we have $y(t,\omega)=z(t,\omega)=x_k(t,\omega)$ for any t and for any K and so $y(t,\omega)=z(t,\omega)$ for any t on Ω^*. Q.E.D.

Added in proof: The author has published a detailed investigation of the same subject in a more general case. Cf. On stochastic processes (II)—A stochastic differential equation—forth-coming to the Japanese Journal of Mathematics.