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ON A SUBCLASS OF CERTAIN STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. A certain subclass $T_\Omega(n,p,\lambda,\alpha)$of starlike functions in the unit disk is introduced. The object of the present paper is to derive several interesting properties of functions belonging to the class $T_\Omega(n,p,\lambda,\alpha)$. Coefficient inequalities, distortion theorems and closure theorems of functions belonging to the class $T_\Omega(n,p,\lambda,\alpha)$ are determined. Also we obtain radii of convexity for the class $T_\Omega(n,p,\lambda,\alpha)$. Furthermore, integral operators and modified Hadamard products of several functions belonging to the class $T_\Omega(n,p,\lambda,\alpha)$ are studied here.

1. Introduction

Let A be class of functions $f(z)$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the open unit disk $U = \{z : |z| < 1\}$. For $f(z)$ belong to A, Salagean [5] has introduced the following operator called the Salagean operator:

$$D^0 f(z) = f(z), D^1 f(z) = Df(z) = zf'(z),$$
$$D^n f(z) = D(D^{n-1} f(z)) \ (n \in \mathrm{N} = \{1,2,3,\ldots\}).$$

Note that $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, n \in \mathrm{N}_0 = \{0\} \cup \mathrm{N}$.

Let $T(n,p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \tag{1.1}$$
$$(a_{k+p} \geq 0; p \in \mathrm{N} := \{1,2,3,\ldots\}; n \in \mathrm{N}),$$

which are analytic in the open unit disk $U = \{z \in \mathrm{C} : |z| < 1\}$.

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A function $f(z) \in T(n,p)$ is said to be in the class $T(n,p,\lambda,\alpha)$ if it satisfies the inequality:

$$Re\left\{\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right\} > \alpha \tag{1.2}$$

for some $\alpha \quad (0 \le \alpha < 1)$ and $\lambda \quad (0 \le \lambda \le 1)$, and for all $z \in U$ [2].

We can write the following equalities for the functions $f(z)$ belong to the class $T(n,p)$

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = Df(z) = zf'(z) = z[pz^{p-1} - \sum_{k=n}^{\infty}(k+p)a_{k+p}z^{k+p-1}]$$

$$= pz^p - \sum_{k=n}^{\infty}(k+p)a_{k+p}z^{k+p},$$

$$D^2 f(z) = D(Df(z)) = p^2 z^p - \sum_{k=n}^{\infty}(k+p)^2 a_{k+p}z^{k+p},$$

$$\vdots$$

$$D^\Omega f(z) = D(D^{\Omega-1}f(z)) = p^\Omega z^p - \sum_{k=n}^{\infty}(k+p)^\Omega a_{k+p}z^{k+p}.$$

A function $f(z) \in T(n,p)$ is said to be in the class $T_\Omega(n,p,\lambda,\alpha)$ if it satisfies the inequality:

$$Re\left\{\frac{(1-\lambda)z(D^\Omega f(z))' + \lambda z(D^{\Omega+1}f(z))'}{(1-\lambda)D^\Omega f(z) + \lambda D^{\Omega+1}f(z)}\right\} > \alpha \quad (\Omega \in \mathrm{N}_0) \tag{1.3}$$

for some $\alpha(0 \le \alpha < 1)$ and $\lambda(0 \le \lambda \le 1)$, and for all $z \in U$ [2] .

We note that

$$T_0(n,p,\lambda,\alpha) \equiv T(n,p,\lambda,\alpha),$$
$$T_0(n,1,0,\alpha) \equiv T_\alpha(n),$$
$$T_0(n,1,1,\alpha) \equiv C_\alpha(n),$$
$$T_0(1,1,0,\alpha) \equiv T^*(\alpha),$$
$$T_0(1,1,1,\alpha) \equiv C(\alpha),$$
$$T_0(n,1,\lambda,\alpha) \equiv P(n,\lambda,\alpha),$$

and

$$T_1(n,1,\lambda,\alpha) \equiv C(n,\lambda,\alpha).$$

The classes $T_\alpha(n)$ and $C_\alpha(n)$ were studied earlier by Srivastava et al. [8] , the classes $T^*(\alpha) \equiv T_\alpha(1)$ and $C(\alpha) \equiv C_\alpha(1)$ were studied by Silverman [7], the class $P(n,\lambda,\alpha)$ was studied by Altıntaş [1], the class $T(n,p,\lambda,\alpha)$ were studied by Altıntaş et al. [2], and the class $C(n,\lambda,\alpha)$ were studied by Kamali and Akbulut [4].

2. A theorem on coefficient bounds

We begin by proving some sharp coefficient inequalities contained in the following theorem.

THEOREM 1. *A function $f(z) \in T(n,p)$is in the class $T_\Omega(n,p,\lambda,\alpha)$ if and only if*

$$\sum_{k=n}^{\infty}(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)a_{k+p} \leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda) \tag{2.1}$$

$$(0 \leq \alpha < 1; 0 \leq \lambda \leq 1; p \leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)(p \neq 1);\ p \in \mathrm{N}; n \in \mathrm{N}; \Omega \in \mathrm{N}_0).$$

The result is sharp.

Proof. Suppose that $f(z) \in T_\Omega(n,p,\lambda,\alpha)$. Then we find from (1.3) that

$$Re\left\{\frac{(1+\lambda p-\lambda)p^{\Omega+1}z^p - \sum\limits_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega+1}a_{k+p}z^{k+p}}{(1+\lambda p-\lambda)p^{\Omega}z^p - \sum\limits_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}z^{k+p}}\right\} > \alpha$$

$$(0 \leq \alpha < 1; 0 \leq \lambda \leq 1; p \leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)(p \neq 1);\ p \in \mathrm{N}; n \in \mathrm{N}; \Omega \in \mathrm{N}_0; z \in U).$$

If we choose z to be real and let $z \to 1^-$, we get

$$\left\{\frac{(1+\lambda p-\lambda)p^{\Omega+1} - \sum\limits_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega+1}a_{k+p}}{(1+\lambda p-\lambda)p^{\Omega} - \sum\limits_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}}\right\} \geq \alpha$$

$$(0 \leq \alpha < 1; 0 \leq \lambda \leq 1; p \leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)(p \neq 1);\ p \in \mathrm{N}; n \in \mathrm{N}; \Omega \in \mathrm{N}_0)$$

or, equivalently,

$$\sum_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega+1}a_{k+p}$$
$$-\ \alpha\sum_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}$$
$$\leq (1+\lambda p-\lambda)p^{\Omega+1}-\alpha(1+\lambda p-\lambda)p^{\Omega}$$
$$(0\leq\alpha<1;0\leq\lambda\leq 1;p\leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)(p\neq 1);$$
$$p\in\mathrm{N};n\in\mathrm{N};\Omega\in\mathrm{N}_0).$$

Thus, we obtain

$$\sum_{k=n}^{\infty}(k+p-\alpha)(k+p)^{\Omega}(\lambda k+\lambda p+1-\lambda)a_{k+p}$$
$$\leq (p-\alpha)p^{\Omega}(1+\lambda p-\lambda)$$
$$(0\leq\alpha<1;0\leq\lambda\leq 1;p\leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)(p\neq 1);$$
$$p\in\mathrm{N};n\in\mathrm{N};\Omega\in\mathrm{N}_0).$$

Conversely, suppose that the inequality (2.1) holds true and let

$$z\in\partial U=\{z:z\in\mathrm{C}\,|z|=1\}.$$

Then we find from the definition (1.1) that

$$\left|\frac{(1-\lambda)z(D^{\Omega}f(z))'+\lambda z(D^{\Omega+1}f(z))'}{(1-\lambda)D^{\Omega}f(z)+\lambda D^{\Omega+1}f(z)}-p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)\right|$$
$$=\left|\frac{(1+\lambda p-\lambda)p^{\Omega+1}z^{p}-\sum\limits_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega+1}a_{k+p}z^{k+p}}{(1+\lambda p-\lambda)p^{\Omega}z^{p}-\sum\limits_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}z^{k+p}}\right.$$
$$\left.-\ p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)\right|$$
$$\leq\frac{\left|-(1+\lambda p-\lambda)p^{\Omega}\{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)-p\}z^{p}\right|+\left|\sum\limits_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}\{k+p-p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)\}a_{k+p}z^{k+p}\right|}{\left|(1+\lambda p-\lambda)p^{\Omega}z^{p}\right|-\left|\sum\limits_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}z^{k+p}\right|}$$
$$=\frac{(1+\lambda p-\lambda)p^{\Omega}\{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)-p\}+\sum\limits_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}\{k+p-p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)\}a_{k+p}\,|z|^{k}}{(1+\lambda p-\lambda)p^{\Omega}-\sum\limits_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}\,|z|^{k}}$$

$$\leq \frac{\begin{array}{l}(1+\lambda p-\lambda)p^{\Omega}\{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)-p\} \\ +\{p-p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)\}\sum_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}\end{array}}{(1+\lambda p-\lambda)p^{\Omega}-\sum_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}}$$

$$+\frac{\sum_{k=n}^{\infty}k(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}}{(1+\lambda p-\lambda)p^{\Omega}-\sum_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}}$$

$$\leq \frac{\{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)-p\}\{(1+\lambda p-\lambda)p^{\Omega}-\sum_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}\}}{(1+\lambda p-\lambda)p^{\Omega}-\sum_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}}$$

$$+\frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)-\sum_{k=n}^{\infty}(p-\alpha)(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}}{(1+\lambda p-\lambda)p^{\Omega}-\sum_{k=n}^{\infty}(\lambda k+\lambda p+1-\lambda)(k+p)^{\Omega}a_{k+p}}$$

$$= p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)-p+p-\alpha$$

$$= p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)-\alpha$$

$$(0\leq\alpha<1; 0\leq\lambda\leq 1; p\leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)(p\neq 1);$$
$$p\in \mathrm{N}; n\in \mathrm{N}; \Omega\in \mathrm{N}_0),$$

provided that the inequality (2.1) is satisfied. Hence, by the maximum modulus theorem, we have

$$f(z)\in T_{\Omega}(n,p,\lambda,\alpha).$$

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the extremal function being

$$f(z)=z^p-\frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p+1-\lambda)}z^{k+p} \tag{2.2}$$
$$(k\geq n; p,n\in \mathrm{N}, \Omega\in \mathrm{N}_0).$$

COROLLARY 1. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $T_{\Omega}(n,p,\lambda,\alpha)$. *Then*

$$a_{k+p}\leq\frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p+1-\lambda)}(k\geq n). \tag{2.3}$$

The equality in (2.3) is attained for the function $f(z)$ given by (2.2).

3. Distortion theorems

THEOREM 2. *Let the function $f(z)$ defined by (1.1) be in the class $T_\Omega(n,p,\lambda,\alpha)$. Then we have*

$$|f(z)| \le |z|^p + \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^\Omega(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)}|z|^{p+n} \tag{3.1}$$

and

$$|f(z)| \ge |z|^p - \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^\Omega(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)}|z|^{p+n} \tag{3.2}$$

for $z \in U$. Then equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^\Omega(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)}z^{p+n}. \tag{3.3}$$

Proof. Note that

$$\begin{aligned}
&(n+p)^\Omega(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)\sum_{k=n}^{\infty} a_{k+p}\\
&\le \sum_{k=n}^{\infty}(k+p)^\Omega(k+p-\alpha)(\lambda k+\lambda p+1-\lambda)a_{k+p}\\
&\le p^\Omega(p-\alpha)(1+\lambda p-\lambda),
\end{aligned}$$

this last inequality following from Theorem 1. Thus

$$\begin{aligned}
|f(z)| \le\ & |z|^p + \sum_{k=n}^{\infty}|a_{k+p}|\,|z|^{k+p}\\
\le\ & |z|^p + |z|^{n+p}\sum_{k=n}^{\infty} a_{k+p}\\
\le\ & |z|^p + \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^\Omega(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)}|z|^{p+n}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|f(z)| \ge\ & |z|^p - |z|^{n+p}\sum_{k=n}^{\infty} a_{k+p}\\
\ge\ & |z|^p - \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^\Omega(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)}|z|^{p+n}.
\end{aligned}$$

THEOREM 3. *If $f(z) \in T_\Omega(n,p,\lambda,\alpha)$, then*

$$|z|^{p-1}\{1 - \frac{p^{\Omega-1}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)}|z|^n\}$$
$$\leq \frac{1}{p}|f'(z)|$$
$$\leq |z|^{p-1}\{1 + \frac{p^{\Omega-1}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)}|z|^n\}.$$

Proof. We have

$$\begin{aligned} |f'(z)| &\leq p|z|^{p-1} + \sum_{k=n}^{\infty}(k+p)a_{k+p}|z|^{k+p-1} \\ &\leq p|z|^{p-1} + |z|^{n+p-1}\sum_{k=n}^{\infty}(k+p)a_{k+p}. \end{aligned} \tag{3.4}$$

In view of Theorem 1, we have

$$\sum_{k=n}^{\infty}(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)a_{k+p}$$
$$\leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)$$

and then

$$(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)(n+p)^{\Omega-1}\sum_{k=n}^{\infty}(k+p)a_{k+p}$$
$$\leq \sum_{k=n}^{\infty}(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)a_{k+p}$$
$$\leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)$$

or

$$\begin{aligned} &\sum_{k=n}^{\infty}(k+p)a_{k+p} \\ &\leq \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)}. \end{aligned} \tag{3.5}$$

A substitution of (3.5) in to (3.4) yields the right-hand inequality.

On the other hand,

$$\begin{aligned}|f'(z)| &\geq p\,|z|^{p-1} - \sum_{k=n}^{\infty}(k+p)a_{k+p}\,|z|^{k+p-1}\\ &\geq p\,|z|^{p-1} - |z|^{n+p-1}\sum_{k=n}^{\infty}(k+p)a_{k+p}\\ &\geq p\,|z|^{p-1} - |z|^{n+p-1}\sum_{k=n}^{\infty}(k+p)a_{k+p}\\ &\geq p\,|z|^{p-1} - |z|^{n+p-1}\,\frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}(n+p-\alpha)(\lambda p+\lambda n+1-\lambda)}\end{aligned}$$

or

$$\frac{|f'(z)|}{p} \geq |z|^{p-1}\,\{1 - \frac{p^{\Omega-1}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega-1}(n+p-\alpha)(\lambda p+\lambda n-\lambda+1)}\,|z|^{n}\}.$$

COROLLARY 2. *Let the function $f(z)$ defined by (1.1) be in the class $T_{\Omega}(n,p,\lambda,\alpha)$. Then the unit disk U is mapped onto a domain that contains the disk*

$$|w| < 1 - \left(\frac{p}{n+p}\right)^{\Omega}\left(\frac{p-\alpha}{n+p-\alpha}\right)\left(\frac{1+\lambda p-\lambda}{\lambda p+\lambda n+1-\lambda}\right). \tag{3.6}$$

The result is sharp with the extremal function given by (3.3).

4. Closure theorems

Let the functions $f_j(z)$ be defined, for $j = 1, 2, \ldots, m$ by

$$f_j(z) = z^p - \sum_{k=n}^{\infty} a_{k+p,j}z^{k+p}(a_{k+p,j} \geq 0) \tag{4.1}$$

for $z \in U$ [3].

We shall prove the following results for the closure of functions in the class $T_{\Omega}(n,p,\lambda,\alpha)$.

THEOREM 4. *Let the functions $f_j(z)$ defined by (4.1) be in the class $T_{\Omega}(n,p,\lambda,\alpha)$ for every $j = 1, 2, \ldots, m$. Then the functions $h(z)$ defined by*

$$h(z) = \sum_{j=1}^{m} c_j f_j(z)\,(c_j \geq 0)$$

is also in the same class $T_\Omega(n,p,\lambda,\alpha)$, *where*

$$\sum_{j=1}^{m} c_j = 1.$$

Proof. According to the definition of $h(z)$, we can write

$$\begin{aligned} h(z) &= \sum_{j=1}^{m} c_j[z^p - \sum_{k=n}^{\infty} a_{k+p,j} z^{k+p}] \\ &= (\sum_{j=1}^{m} c_j) z^p - \sum_{k=n}^{\infty} (\sum_{j=1}^{m} c_j a_{k+p,j}) z^{k+p} \\ &= z^p - \sum_{k=n}^{\infty} (\sum_{j=1}^{m} c_j a_{k+p,j}) z^{k+p}. \end{aligned}$$

Further, since $f_j(z)$are in $T_\Omega(n,p,\lambda,\alpha)$ for every $j = 1, 2, \ldots, m$ we get

$$\begin{aligned} &\sum_{k=n}^{\infty} (k+p)^\Omega (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1) a_{k+p,j} \\ &\leq p^\Omega (p-\alpha)(1+\lambda p - \lambda) \end{aligned}$$

for every $j = 1, 2, \ldots, m$. Hence we can see that

$$\begin{aligned} &\sum_{k=n}^{\infty} (k+p)^\Omega (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)(\sum_{j=1}^{m} c_j a_{k+p,j}) \\ &= \sum_{k=n}^{\infty} (k+p)^\Omega (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1) \\ &\times (c_1 a_{k+p,1} + c_2 a_{k+p,2} + \ldots + c_m a_{k+p,m}) \\ &= c_1 \sum_{k=n}^{\infty} (k+p)^\Omega (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1) a_{k+p,1} \\ &+ c_2 \sum_{k=n}^{\infty} (k+p)^\Omega (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1) a_{k+p,2} + \cdots \\ &+ c_m \sum_{k=n}^{\infty} (k+p)^\Omega (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1) a_{k+p,m} \end{aligned}$$

$$\leq c_1[p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)]+c_2[p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)]+\cdots$$
$$+\ c_m[p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)]$$
$$=(c_1+c_2+\ldots+c_m)[p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)]$$
$$=(\sum_{j=1}^{m}c_j)p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)$$
$$=p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)$$

which implies that $h(z)$ in $T_{\Omega}(n,p,\lambda,\alpha)$. Thus we have the theorem.

COROLLARY 3. *Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by*

$$g(z)=z^p-\sum_{k=n}^{\infty}b_{k+p}z^{k+p}(b_{k+p}\geq p\in \mathrm{N}n\in \mathrm{N})$$

be in the same class $T_{\Omega}(n,p,\lambda,\alpha)$. Then the function $h(z)$ defined by

$$h(z)=(1-\gamma)f(z)+\gamma g(z)$$
$$=z^p-\sum_{k=n}^{\infty}c_{k+p}z^{k+p}$$
$$(c_{k+p}\geq 0; 0\leq\gamma\leq 1; p\in \mathrm{N}; n\in \mathrm{N})$$

is also in the class $T_{\Omega}(n,p,\lambda,\alpha)$.

Proof. Suppose that each of the functions $f(z)$ and $g(z)$ is in the class $T_{\Omega}(n,p,\lambda,\alpha)$. Then making use of (2.1), we see that

$$\sum_{k=n}^{\infty}(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)c_{k+p}$$
$$=(1-\gamma)\sum_{k=n}^{\infty}(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)a_{k+p}$$
$$+\ \gamma\sum_{k=n}^{\infty}(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)b_{k+p}$$
$$\leq(1-\gamma)p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)+\gamma(p-\alpha)p^{\Omega}(1+\lambda p-\lambda)$$
$$=p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)$$
$$(0\leq\alpha<1,0\leq\lambda\leq 1,p\leq p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)(p\neq 1);$$
$$p\in \mathrm{N},n\in \mathrm{N};\Omega\in \mathrm{N}_0),$$

which of the completes the proof of Corollary 3. □

As a consequence of Corollary 3, there exists the extreme points of the class $T_\Omega(n,p,\lambda,\alpha)$.

THEOREM 5. *Let* $f_{n-1}(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^\Omega(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)} z^{k+p}, \quad (k \geq n)$$

for $0 \leq \alpha < 1, 0 \leq \lambda \leq 1$ *and* $n \in \mathrm{N}$. *Then* $f(z)$ *is in the class* $T_\Omega(n,p,\lambda,\alpha)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=n-1}^{\infty} \eta_k f_k(z)$$

where $\eta_k \geq 0, (k \geq n-1)$ *and* $\sum_{k=n-1}^{\infty} \eta_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=n-1}^{\infty} \eta_k f_k(z).$$

Then

$$\begin{aligned} f(z) &= \sum_{k=n-1}^{\infty} \eta_k f_k(z) = \eta_{n-1} f_{n-1}(z) + \sum_{k=n}^{\infty} \eta_k f_k(z) \\ &= \eta_{n-1} z^p + \sum_{k=n}^{\infty} \eta_k [z^p - \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^\Omega(p+k-\alpha)(\lambda k+\lambda p-\lambda+1)}] z^{k+p} \\ &= (\sum_{k=n-1}^{\infty} \eta_k) z^p - \sum_{k=n}^{\infty} \eta_k \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^\Omega(p+k-\alpha)(\lambda k+\lambda p-\lambda+1)} z^{k+p} \\ &= z^p - \sum_{k=n}^{\infty} \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^\Omega(p+k-\alpha)(\lambda k+\lambda p-\lambda+1)} \eta_k z^{k+p}. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{k=n}^{\infty} \eta_k \left[\frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^\Omega(p+k-\alpha)(\lambda k+\lambda p-\lambda+1)}\right] \\ &\qquad \times \left[\frac{(k+p)^\Omega(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)}{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}\right] \\ &= \sum_{k=n}^{\infty} \eta_k = \sum_{k=n-1}^{\infty} \eta_k - \eta_{n-1} = 1 - \eta_{n-1} \leq 1, \end{aligned}$$

so by Theorem 1, $f(z) \in T_\Omega(n,p,\lambda,\alpha)$.

Conversely, suppose $f(z) \in T_\Omega(n,p,\lambda,\alpha)$. Since

$$a_{k+p} \leq \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^\Omega(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)} \quad (k=n,n+1,\ldots),$$

we may set

$$\eta_k = \frac{(k+p)^\Omega(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)}{p^\Omega(p-\alpha)(1+\lambda p-\lambda)} a_{k+p}$$

and

$$\eta_{n-1} = 1 - \sum_{k=n}^{\infty} \eta_k.$$

Then

$$\begin{aligned}
f(z) &= z^p - \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \\
&= z^p - \sum_{k=n}^{\infty} \frac{p^\Omega(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^\Omega(p+k-\alpha)(\lambda k+\lambda p-\lambda+1)} \eta_k z^{k+p} \\
&= z^p - \sum_{k=n}^{\infty} \eta_k [z^p - f_k(z)] \\
&= z^p - \sum_{k=n}^{\infty} \eta_k z^p + \sum_{k=n}^{\infty} \eta_k f_k(z) \\
&= (1 - \sum_{k=n}^{\infty} \eta_k) z^p + \sum_{k=n}^{\infty} \eta_k f_k(z) \\
&= \eta_{n-1} z^p + \sum_{k=n}^{\infty} \eta_k f_k(z) \\
&= \eta_{n-1} f_{n-1}(z) + \sum_{k=n}^{\infty} \eta_k f_k(z) \\
&= \sum_{k=n-1}^{\infty} \eta_k f_k(z).
\end{aligned}$$

This completes the proof. □

5. Integral operators

THEOREM 6. *Let the function $f(z)$ defined by (1.1) be in the class $T_\Omega(n,p,\lambda,\alpha)$and let c be real number such that $c > -p$. Then the function $F(z)$defined by*

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \tag{5.1}$$

also belongs to the class $T_\Omega(n,p,\lambda,\alpha)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z^p - \sum_{k=n}^{\infty} b_{k+p}z^{k+p},$$

where

$$b_{k+p} = \left(\frac{c+p}{c+p+k}\right)a_{k+p}.$$

Therefore,

$$\begin{aligned}&\sum_{k=n}^{\infty}(k+p)^\Omega(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)b_{k+p}\\ &= \sum_{k=n}^{\infty}(k+p)^\Omega(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)\left(\frac{c+p}{c+p+k}\right)a_{k+p}\\ &\le \sum_{k=n}^{\infty}(k+p)^\Omega(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)a_{k+p}\\ &\le p^\Omega(p-\alpha)(1+\lambda p-\lambda),\end{aligned}$$

since $f(z) \in T_\Omega(n,p,\lambda,\alpha)$. Hence, by Theorem 1, $F(z) \in T_\Omega(n,p,\lambda,\alpha)$.

THEOREM 7. *Let c be real number such that $c > -p$. If $F(z) \in T_\Omega(n,p,\lambda,\alpha)$, then the function $f(z)$ defined by (5.1) is $p-$valent in $|z| < R_p^*$, where*

$$\begin{aligned}R_p^* = \inf_k\Bigg\{&\left(\frac{k+p}{p}\right)^{\Omega-1}\left(\frac{c+p}{c+p+k}\right)\left(\frac{k+p-\alpha}{p-\alpha}\right)\\ &\left(\frac{\lambda k+\lambda p-\lambda+1}{\lambda p-\lambda+1}\right)\Bigg\}^{\frac{1}{k}}\ (k\ge n).\end{aligned} \tag{5.2}$$

The result is sharp.

Proof. Let

$$F(z) = z^p - \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (a_{k+p} \geq 0).$$

It follow from (5.1) that

$$\begin{aligned} f(z) &= \frac{z^{1-c}(z^c F(z))'}{(c+p)} \quad (c > -p) \\ &= z^p - \sum_{k=n}^{\infty} \left(\frac{c+k+p}{c+p}\right) a_{k+p} z^{k+p}. \end{aligned}$$

To prove the result, it suffices to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p \text{ for } |z| < R_p^*.$$

Now

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|\frac{pz^{p-1} - \sum_{k=n}^{\infty} (k+p)\left(\frac{c+k+p}{c+p}\right) a_{k+p} z^{k+p-1}}{z^{p-1}} - p\right|$$

$$\left|-\sum_{k=n}^{\infty} (k+p)\left(\frac{c+p+k}{c+p}\right) a_{k+p} z^k\right| \leq \sum_{k=n}^{\infty} (k+p)\left(\frac{c+p+k}{c+p}\right) a_{k+p} |z|^k.$$

Thus $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p$ if

$$\sum_{k=n}^{\infty} \left(\frac{k+p}{p}\right)\left(\frac{c+p+k}{c+p}\right) a_{k+p} |z|^k \leq 1. \tag{5.3}$$

But Theorem 1 confirms that

$$\sum_{k=n}^{\infty} \frac{(k+p)^{\Omega}(k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)}{p^{\Omega}(p-\alpha)(1+\lambda p - \lambda)} a_{k+p} \leq 1. \tag{5.4}$$

Thus (5.3) will be satisfied if

$$\begin{aligned} &\left(\frac{k+p}{p}\right)\left(\frac{c+p+k}{c+p}\right)|z|^k \\ &\leq \frac{(k+p)^{\Omega}}{p^{\Omega}}\left(\frac{k+p-\alpha}{p-\alpha}\right)\left(\frac{\lambda k + \lambda p - \lambda + 1}{1+\lambda p - \lambda}\right) \quad (k \geq n), \end{aligned}$$

or if

$$
(5.5)\quad |z| \le \left\{ \left(\frac{k+p}{p}\right)^{\Omega-1} \left(\frac{c+p}{c+p+k}\right) \left(\frac{k+p-\alpha}{p-\alpha}\right) \left(\frac{\lambda k+\lambda p-\lambda+1}{\lambda p-\lambda+1}\right) \right\}^{\frac{1}{k}} (k \ge n).
$$

The required result follows now from (5.5). The result is sharp for the function

$$
(5.6)\quad f(z) = z^p - \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)} \times \left(\frac{c+p+k}{c+p}\right) z^{k+p} \quad (k \ge n).
$$

THEOREM 8. *Let the function $f(z)$ defined by (1.1) be in the class $T_{\Omega}(n,p,\lambda,\alpha)$. Then $f(z)$ is convex of order $q(0 \le q < 1)$ in $|z| < r$, where*

$$
r = \inf_k \left\{ \left(\frac{k+p}{p}\right)^{\Omega-1} \left(\frac{p-q}{p-\alpha}\right) \left(\frac{k+p-\alpha}{k+p-q}\right) \left(\frac{\lambda k+\lambda p-\lambda+1}{\lambda p-\lambda+1}\right) \right\}^{\frac{1}{k}} (k \ge n).
$$

Proof. We must show that

$$
\left| \frac{zf''(z)}{f'(z)} + 1 - p \right| < p - q
$$

$(0 \le q < 1)$ for $|z| < r$. We have

$$
\left| \frac{zf''(z)}{f'(z)} + 1 - p \right| = \left| \frac{zf''(z) + (1-p)f'(z)}{f'(z)} \right|
$$

$$
= \left| \frac{p(p-1)z^{p-1} - \sum\limits_{k=n}^{\infty} (k+p)(k+p-1)a_{k+p}z^{k+p-1}}{pz^{p-1} - \sum\limits_{k=n}^{\infty} (k+p)a_{k+p}z^{k+p-1}} - \frac{(p-1)pz^{p-1} + \sum\limits_{k=n}^{\infty} (p-1)(k+p)a_{k+p}z^{k+p-1}}{pz^{p-1} - \sum\limits_{k=n}^{\infty} (k+p)a_{k+p}z^{k+p-1}} \right|
$$

$$= \left| \frac{-\sum_{k=n}^{\infty} (k+p)k a_{k+p} z^{k+p-1}}{p z^{p-1} - \sum_{k=n}^{\infty} (k+p) a_{k+p} z^{k+p-1}} \right|$$

$$\leq \frac{\sum_{k=n}^{\infty} k(k+p) a_{k+p} |z|^k}{p - \sum_{k=n}^{\infty} (k+p) a_{k+p} |z|^k}.$$

Thus $\left|\frac{zf''(z)}{f'(z)} + 1 - p\right| < p - q$ if

$$\sum_{k=n}^{\infty} \frac{(k+p)(k+p-q)}{p(p-q)} a_{k+p} |z|^k \leq 1. \tag{5.7}$$

But Theorem 1 confirms that

$$\sum_{k=n}^{\infty} \frac{(k+p)^{\Omega}(k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)}{p^{\Omega}(p-\alpha)(1+\lambda p - \lambda)} a_{k+p} \leq 1.$$

Hence (5.7) will be true if

$$\frac{(k+p)(k+p-q)}{p(p-q)} |z|^k \leq \frac{(k+p)^{\Omega}(k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)}{p^{\Omega}(p-\alpha)(1+\lambda p - \lambda)}$$

or if

$$|z|^k \leq \frac{(k+p)^{\Omega}(k+p-\alpha)(\lambda k + \lambda p - \lambda + 1) p(p-q)}{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)(k+p-q)(k+p)},$$

$$|z| \leq \left\{ \left(\frac{k+p}{p}\right)^{\Omega-1} \left(\frac{p-q}{p-\alpha}\right) \left(\frac{k+p-\alpha}{k+p-q}\right) \left(\frac{\lambda k + \lambda p - \lambda + 1}{1 + \lambda p - \lambda}\right) \right\}^{\frac{1}{k}} (k \geq n).$$

6. Modified Hadamard products

Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by

$$g(z) = z^p - \sum_{k=n}^{\infty} b_{k+p} z^{k+p} \quad (b_{k+p} \geq 0; p \in \mathrm{N}, n \in \mathrm{N})$$

be in the same class $T_\Omega(n,p,\lambda,\alpha)$. We define the modified Hadamard product of the functions $f(z)$ and $g(z)$ by

$$f * g(z) = z^p - \sum_{k=n}^{\infty} a_{k+p} b_{k+p} z^{k+p}.$$

THEOREM 9. *If each of the functions $f(z)$ and $g(z)$ is in the class $T_\Omega(n,p,\lambda,\alpha)$, then*

$$f * g(z) \in T_\Omega(n,p,\lambda,\delta),$$

where

(6.1)

$$\delta \leq p - n \times \frac{p^\Omega (p-\alpha)^2 (1+\lambda p - \lambda)}{(n+p)^\Omega (n+p-\alpha)^2 (\lambda n + \lambda p - \lambda + 1) - p^\Omega (p-\alpha)^2 (1+\lambda p - \lambda)} \quad (p \in \mathrm{N}, n \in \mathrm{N}).$$

The result is sharp for the functions $f(z)$ and $g(z)$ given by

$$f(z) = g(z) = z^p - \frac{p^\Omega (p-\alpha)(1+\lambda p - \lambda)}{(n+p)^\Omega (n+p-\alpha)(\lambda n + \lambda p - \lambda + 1)} z^{n+p} \quad (p \in \mathrm{N}, n \in \mathrm{N}).$$

Proof. Employing the technique used earlier by Schild and Silverman [6], we need to find the largest δ such that

$$\sum_{k=n}^{\infty} \frac{(k+p)^\Omega (k+p-\delta)(\lambda k + \lambda p - \lambda + 1)}{p^\Omega (p-\delta)(1+\lambda p - \lambda)} a_{k+p} b_{k+p} \leq 1.$$

Since

$$\sum_{k=n}^{\infty} \frac{(k+p)^\Omega (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)}{p^\Omega (p-\alpha)(1+\lambda p - \lambda)} a_{k+p} \leq 1$$

and

$$\sum_{k=n}^{\infty} \frac{(k+p)^\Omega (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)}{p^\Omega (p-\alpha)(1+\lambda p - \lambda)} b_{k+p} \leq 1,$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=n}^{\infty} \frac{(k+p)^\Omega (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)}{p^\Omega (p-\alpha)(1+\lambda p - \lambda)} \sqrt{a_{k+p} b_{k+p}} \leq 1.$$

Thus it is sufficient to show that

$$\frac{(k+p)^{\Omega}(k+p-\delta)(\lambda k+\lambda p-\lambda+1)}{p^{\Omega}(p-\delta)(1+\lambda p-\lambda)}a_{k+p}b_{k+p} \leq \frac{(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)}{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}\sqrt{a_{k+p}b_{k+p}},$$

that is, that

$$\sqrt{a_{k+p}b_{k+p}} \leq \frac{(k+p-\alpha)}{(k+p-\delta)}.\frac{(p-\delta)}{(p-\alpha)}.$$

Not that

$$\sqrt{a_{k+p}b_{k+p}} \leq \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)} \quad (k \geq n).$$

Consequently, we need only to prove that

$$\frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(k+p)^{\Omega}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)} \leq \frac{k+p-\alpha}{k+p-\delta}\frac{p-\delta}{p-\alpha} \quad (k \geq n),$$

or, equivalently, that

$$\delta \leq p - \frac{p^{\Omega}(p-\alpha)^2(1+\lambda p-\lambda)}{(k+p)^{\Omega}(k+p-\alpha)^2(\lambda k+\lambda p-\lambda+1)-p^{\Omega}(p-\alpha)^2(1+\lambda p-\lambda)}k \quad (k \geq n).$$

Since

$$\psi(k) = p - \frac{p^{\Omega}(p-\alpha)^2(1+\lambda p-\lambda)}{(k+p)^{\Omega}(k+p-\alpha)^2(\lambda k+\lambda p-\lambda+1)-p^{\Omega}(p-\alpha)^2(1+\lambda p-\lambda)}k \quad (k \geq n). \tag{6.2}$$

is an increasing function of k $(k \geq n)$, letting $k = n$ (6.2), we obtain

$$\delta \leq \psi(n) = p - \frac{p^{\Omega}(p-\alpha)^2(1+\lambda p-\lambda)}{(n+p)^{\Omega}(n+p-\alpha)^2(\lambda n+\lambda p-\lambda+1)-p^{\Omega}(p-\alpha)^2(1+\lambda p-\lambda)}n,$$

which completes the proof Theorem 9. □

Finally, by taking the function $f_j(z)$ given by

$$f_j(z) = z^p - \frac{p^{\Omega}(p-\alpha)(1+\lambda p-\lambda)}{(n+p)^{\Omega}(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)}z^{n+p} \qquad (j=1,2),$$

we can see that the result is sharp.

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