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ON A SUBCLASS OF THE FAMILY OF DARBOUX FUNCTIONS

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Abstract. We investigate functions $f : I \to \mathbb{R}$ (where I is an open interval) such that for all $u, v \in I$ with $u < v$ and $f(u) \neq f(v)$ and each $c \in (\min(f(u), f(v)), \max(f(u), f(v)))$ there is a point $w \in (u, v)$ such that $f(w) = c$ and f is approximately continuous at w.

Let μ be the Lebesgue measure on $\mathbb{R}$. For a (Lebesgue) measurable set $A \subset \mathbb{R}$ and a point x we define the *upper* (resp. *lower*) *density* $D_u(A, x)$ (resp. $D_l(A, x)$) of A at x ([1, 6]) as

$$\limsup_{h\to 0^+} \frac{\mu(A \cap [x - h, x + h])}{2h},$$

resp.

$$\liminf_{h\to 0^+} \frac{\mu(A \cap [x - h, x + h])}{2h}.$$

A point x is said to be a *density point* of a set B if there is a Lebesgue measurable set $A \subset B$ such that $D_l(A, x) = 1$ ([1, 6, 7]).

The family T_d of all sets $A \subset \mathbb{R}$ for which the implication

$$x \in A \ \Rightarrow\ x \text{ is a density point of } A$$

holds is a topology called the *density topology* ([1, 6]). All sets in T_d are Lebesgue measurable [1] and each measurable set E contains an F_σ-set $F \in T_d$ with $\mu(E \setminus F) = 0$ ([1]).

Moreover, let T_e denote the Euclidean topology in $\mathbb{R}$. The continuity of functions from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$ is called the *approximate continuity* ([1, 6, 7]). An equivalent definition is the following: f is approximately continuous at a point x if there is a measurable set A such that $x \in A$, $D_l(A, x) = 1$ and the restriction $f|A$ is continuous at x ([1]).

The following property is analogous to the strong Świątkowski property introduced in [3, 5].

Let I be an open interval. We will say that a function $f : I \to \mathbb{R}$ has the *D_{ap}-property* ($f \in D_{ap}$) if for all $u, v \in I$ with $u < v$ and $f(u) \neq f(v)$ and

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for each $c \in (\min(f(u), f(v)), \max(f(u), f(v)))$ there is a point $w \in (u, v)$ such that $f(w) = c$ and f is approximately continuous at w.

The strong Świątkowski property has the same definition with approximate continuity replaced by continuity.

Obviously each function with the D_{ap}-property has the Darboux property.

Let ϱ_C be the metric of uniform convergence in the space D of all Darboux functions from I to $\mathbb{R}$ (i.e. $\varrho_C(f, g) = \min(1, \sup_{t\in I} |f(t) - g(t)|)$).

It is well known that there are nonzero Darboux functions $f : \mathbb{R} \to \mathbb{R}$ which vanish almost everywhere ([1, p. 6 (Th. 4.3) or p. 12 (Ex. 2.2) or p. 13 (Th. 2.4)]). Evidently each such function f belongs to the interior (with respect to ϱ_C) of the set $D \setminus D_{ap}$.

THEOREM 1. *The set D_{ap} is nowhere dense in the space (D, ϱ_C).*

Proof. Let U be a nonempty open set in (D, ϱ_C). Assume that there is a function $g \in D_{ap} \cap U$. There is an $r > 0$ such that each $\psi \in D$ with $\varrho_C(g, \psi) < r$ belongs to U. If g is constant then for a Darboux function $f : I \to [0, 1]$ vanishing almost everywhere and such that $f(I) = [0, 1]$ the sum $h = g + rf/2$ belongs to $D \setminus D_{ap}$, and so does each function $\psi \in D$ with $\varrho_C(\psi, h) < r/6$. So we assume that g is not constant. Then $g(I)$ is a nondegenerate interval. Let $J \subset \operatorname{int}(g(I))$ be an open interval of length $d(J) < r/2$, and let $(E_\alpha)_{\alpha<2^\omega}$ be a transfinite sequence of all nonempty F_σ-sets $E \subset g^{-1}(J)$ belonging to T_d with $\operatorname{diam}(g(E)) < d(J)/2$.

We can find disjoint sets $G_\alpha \subset E_\alpha$ of cardinality continuum each. Indeed, using a measure preserving Borel bijection Φ between $[0, 1]$ and $[0, 1]^2$ one can assume that each $H_\alpha = \Phi(E_\alpha) \subset [0, 1]^2$ is Borel of positive planar measure. Now using the Fubini theorem one can find, inductively on α, distinct reals $x_\alpha \in [0, 1]$ such that $(H_\alpha)_{x_\alpha}$ has positive measure (in $[0, 1]$) and hence is of cardinality continuum (being a Borel set). Let $G_\alpha = \Phi^{-1}((H_\alpha)_{x_\alpha})$.

For $\alpha < 2^\omega$ let h_α be a function from G_α to J with $h_\alpha(G_\alpha) = J$. Put

$$h(x) = \begin{cases} h_\alpha(x) & \text{for } x \in G_\alpha,\ \alpha < 2^\omega, \\ g(x) & \text{elsewhere on } I. \end{cases}$$

It is obvious that $|h(x) - g(x)| < r/2$ for all $x \in I$. So for $\psi \in D$ with $\varrho_C(\psi, h) < d(J)/6$ we have $\varrho_C(\psi, g) \le \varrho_C(\psi, h) + \varrho_C(h, g) < r/6 + r/2 < r$ and $\psi \in U$.

We will prove that for each $\psi \in D$ with $\varrho_C(\psi, h) < d(J)/6$ we have $\psi \in D \setminus D_{ap}$. Indeed, if $\psi \in D_{ap}$ then there is a point $u \in I$ at which ψ is approximately continuous and $\psi(u) \in J$. Then there is a nonempty F_σ-set

$E_\alpha \in T_d$ such that $\mathrm{diam}(\psi(E_\alpha)) < d(J)/6$, which contradicts the inequality

$$\begin{aligned}\mathrm{diam}(\psi(E_\alpha)) &\ge \mathrm{diam}(\psi(G_\alpha)) \\ &\ge \mathrm{diam}(h(G_\alpha)) - \frac{2d(J)}{6} = d(J) - \frac{2d(J)}{6} = \frac{2d(J)}{3}.\end{aligned}$$

So ψ is not in D_{ap}.

For the proof that $h \in D$ fix $x, y \in I$ such that $x < y$ and $h(x) \ne h(y)$ and a $z \in (\min(h(x), h(y)), \max(h(x), h(y)))$. The following cases are possible:

(1) $z \in J$ and $(x, y) \cap \bigcup_{\alpha<2^\omega} E_\alpha = \emptyset$;
(2) $z \in J$ and $(x, y) \cap \bigcup_{\alpha<2^\omega} E_\alpha \ne \emptyset$;
(3) $z \in g(I) \setminus J = h(I) \setminus J$.

In case (1), since $g \in D_{ap}$, it follows that there is a point $u \in (x, y)$ with $g(u) = z$ which is an approximate continuity point of g. Since $g(u) = h(u) = z$, the proof is complete.

In case (2) there is an ordinal $\alpha < 2^\omega$ with $E_\alpha \subset (x, y)$. Since $h(G_\alpha) = J$, there is a point $w \in (x, y) \cap E_\alpha$ with $h(w) = z$.

In case (3) either $z \in [\max J, \max(g(I)))$ or $z \in (\min g(I), \min J]$. Assume that $z \in [\max J, \max(g(I)))$ and $\max(h(x), h(y)) = h(y)$. Then $h(z) = g(z)$ and $h(y) = g(y)$. If $h(x) = g(x)$ then by the D_{ap}-property of g there is a point $v \in (x, y)$ with $h(v) = g(v) = z$. If $h(x) \ne g(x)$ then $g(x) < \max J < z < h(y) = g(y)$ and as above there is a point $t \in (x, y)$ with $h(t) = g(t) = z$. In the other subcases of case (3) similar reasonings show that h has the Darboux property. This finishes the proof. ∎

Theorem 2. *Let DB_1 be the family of all Darboux Baire 1 functions from I to $\mathbb{R}$ considered as the metric space (DB_1, ϱ_C). The set $D_{ap}B_1$ of all Baire 1 functions with the D_{ap}-property is nowhere dense in DB_1.*

Proof. Fix $f \in DB_1$ and $r \in (0, 1)$. There is an open interval $J \subset I$ with $\mathrm{diam}(f(J)) < r/16$. Let $g \in DB_1$ be such that $g(J) = [0, 1]$ and the closure $A = \mathrm{cl}(B)$ of $B = \{x \in I : g(x) > 0\}$ is nowhere dense, of measure zero and contained in $C(f) \cap J$ (see [1, p. 13 (Th. 2.4)]). Moreover, let $h = f + rg/2$. Evidently $\varrho_C(h, f) = r/2 < r$. Being the sum of two Baire 1 functions, h is also Baire 1. Since $I \setminus A \subset C(g)$ and $A \subset C(f)$, it follows that $h \in DB_1$.

To complete the proof, we will show that if $\phi \in DB_1$ and $\varrho_C(\phi, h) < r/8$, then $\psi \notin D_{ap}$. Indeed, there are $u, v \in J$ with $g(u) = 0$ and $g(v) = 1$. We have

$$\phi(u) < h(u) + \frac{r}{8} = f(u) + \frac{r}{8} \text{ and } \phi(v) > h(v) - \frac{r}{8} = f(v) + \frac{r}{2} - \frac{r}{8} = f(v) + \frac{3r}{8}.$$

Since $u, v \in J$ and $\mathrm{diam}(f(J)) < r/16$, we obtain

$$\phi(v) > f(v) + \frac{3r}{8} > f(u) - \frac{r}{16} + \frac{3r}{8} = f(u) + \frac{r}{8} + \frac{3r}{16} > \phi(u) + \frac{3r}{16}.$$

Fix $c \in (\phi(v) - r/16, \phi(v)) \subset (\phi(u), \phi(v))$. Since for $x \in J \setminus A$ we have

$$\phi(x) < h(x) + \frac{r}{8} = f(x) + \frac{r}{8} < f(v) + \frac{r}{16} + \frac{r}{8} = h(v) - \frac{r}{2} + \frac{3r}{16}$$
$$< \phi(v) + \frac{r}{8} - \frac{5r}{16} < c + \frac{r}{16} - \frac{3r}{16} = c - \frac{r}{8},$$

and $\mu(A) = 0$, there is no approximate continuity point $w \in (u, v)$ of ϕ at which $\phi(w) = c$. ∎

LEMMA 1. *If $A \in T_d$ is a nonempty F_σ-set contained in I then for each positive integer n there is a bounded approximately continuous function $f : I \to \mathbb{R}$ such that $f(A) \supset [-n, n]$.*

Proof. By Zahorski's Lemma 11 from [7] there is an approximately continuous function $g : I \to \mathbb{R}$ such that $g(A) = (0, 1]$ and $g(I \setminus A) = \{0\}$. Let $h(x) = g(x) - 1/2$ and $f(x) = 3nh(x)$ for $x \in I$. Then the function f is bounded and approximately continuous and $f(A) = (-3n/2, 3n/2] \supset [-n, n]$. ∎

THEOREM 3. *Every function $f : I \to \mathbb{R}$ is the sum of two functions from D_{ap}.*

Proof. Let (I_n) be an enumeration of all open intervals with rational endpoints contained in I. For each n we find two disjoint Cantor sets $A_{n,1}, A_{n,2} \subset I_n \setminus \bigcup_{k<n,\ i\le 2} A_{k,i}$ of positive measure, and for $n \ge 1$ and $i \le 2$ we find nonempty F_σ-sets $B_{n,i} \subset A_{n,i}$ belonging to T_d. By Lemma 1 we select approximately continuous bounded functions $g_{n,i} : I \to \mathbb{R}$ such that $g_{n,i}(B_{n,i}) \supset [-n, n]$. Put

$$g(x) = \begin{cases} g_{n,1}(x) & \text{for } x \in B_{n,1},\ n \ge 1, \\ f(x) - g_{n,2}(x) & \text{for } x \in B_{n,2},\ n \ge 1, \\ f(x) & \text{elsewhere on } I, \end{cases}$$

and

$$h(x) = \begin{cases} g_{n,2}(x) & \text{for } x \in B_{n,2},\ n \ge 1, \\ f(x) - g_{n,1}(x) & \text{for } x \in B_{n,1},\ n \ge 1, \\ 0 & \text{elsewhere on } I. \end{cases}$$

Evidently $f = g + h$.

If $u < v$, $g(u) \ne g(v)$ and $c \in (\min(g(u), g(v)), \max(g(u), g(v)))$ then there is $k \ge 1$ with $k > |c|$ and $A_{k,1} \subset (u, v)$. From the construction of g it follows that there exists a point $w \in B_{k,1}$ such that $f(w) = g_{k,1}(w) = c$. Evidently g is approximately continuous at w. So $g \in D_{ap}$. Similarly we can prove that $h \in D_{ap}$. ∎

REMARK 1. Observe that in Theorem 3, if f is of Baire class $\alpha \ge 2$ (resp. Lebesgue measurable, with the Baire property) then so are the functions g, h

constructed in the proof. It is known ([4]) that each Baire 1 function is the sum of two strong Świątkowski Baire 1 functions.

THEOREM 4. *Every function $f : I \to \mathbb{R}$ is the limit of a pointwise convergent sequence of functions from D_{ap}.*

Proof. Let (I_n) be an enumeration of all open intervals with rational endpoints contained in I. For each n we find a Cantor set $A_n \subset I_n \setminus \bigcup_{k<n} A_k$ of positive measure and a nonempty F_σ-set $B_n \subset A_n$ belonging to T_d. By Lemma 1 we choose an approximately continuous function $g_n : I \to \mathbb{R}$ such that $g_n(B_n) \supset [-n, n]$. For $k \geq 1$ put

$$f_k(x) = \begin{cases} g_n(x) & \text{for } x \in B_n,\ n \geq k, \\ f(x) & \text{elsewhere on } I. \end{cases}$$

Evidently $f = \lim_{k\to\infty} f_n$. Fix $k \geq 1$. If $u < v$ and if $f_k(u) \neq f_k(v)$, and if $c \in (\min(f_k(u), f_k(v)), \max(f_k(u), f_k(v)))$, then there is $n \geq k$ with $n > |c|$ and $A_n \subset (u, v)$. From the construction of f_k it follows that there exists a point $w \in A_n$ such that $g_n(w) = f_k(w) = c$. Evidently f_k is approximately continuous at w. So $f_k \in D_{ap}$. ∎

REMARK 2. Observe that in Theorem 4, if f is of Baire class $\alpha \geq 2$ (resp. Lebesgue measurable, with the Baire property) then so are the functions f_n constructed in the proof.

The set $C_{ap}(f)$ of all approximate continuity points of an arbitrary function $f : I \to \mathbb{R}$ is a G_δ-set with respect to the density topology T_d, so it is measurable. Moreover, there are functions in D_{ap} which are not measurable.

THEOREM 5. *There is a function $f : I \to \mathbb{R}$ having the D_{ap}-property which is not measurable* (*resp. does not have the Baire property*).

Proof. Let $f : I \to \mathbb{R}$ be nonmeasurable (resp. without the Baire property). By Theorem 3 there are $g, h \in D_{ap}$ with $f = g + h$. Evidently g or h is not measurable (resp. does not have the Baire property). ∎

THEOREM 6. *There is a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ belonging to D_{ap} which uniformly converges to a function f which does not have the Darboux property.*

Proof. Let (I_n) be an enumeration of all open intervals with rational endpoints. For each $n \geq 1$ we find a Cantor set $A_n \subset I_n \setminus \bigcup_{k<n} A_k$ of positive measure and a nonempty F_σ-set $B_n \subset A_n$ belonging to T_d. By the Zahorski theorem ([1, 5]) there are approximately continuous functions $g_n : \mathbb{R} \to [0, 1]$,

$n \geq 1$, such that $g_n(B_n) = (0,1]$ and $g_n(\mathbb{R} \setminus B_n) = \{0\}$. For $n \geq 1$ let

$$f_n(x) = \begin{cases} g_k(x) & \text{for } x \in B_k,\ k > n, \\ g_k(x) & \text{if } k \leq n \text{ and } g_k(x) \neq 1/2, \\ 1/2 - 1/4^k & \text{if } k \leq n \text{ and } g_k(x) = 1/2, \\ 0 & \text{elsewhere on } \mathbb{R}, \end{cases}$$

and

$$f(x) = \begin{cases} g_k(x) & \text{if } x \in B_k \text{ and } g_k(x) \neq 1/2,\ k \geq 1, \\ 1/2 - 1/4^k & \text{if } x \in B_k \text{ and } g_k(x) = 1/2,\ k \geq 1, \\ 0 & \text{elsewhere on } \mathbb{R}. \end{cases}$$

Since $|f_n - f| \leq 1/4^n$ for $n \geq 1$, the sequence (f_n) uniformly converges to f. Fix $n \geq 1$. For each $k > n$ and each $y \in (0,1)$ there are points $x_k \in B_k$ such that f_n is approximately continuous at x_k and $f_n(x_k) = y$. So every f_n, $n \geq 1$, has the D_{ap}-property. Since $f(\mathbb{R}) = [0,1] \setminus \{1/2\}$, the function f does not have the Darboux property. ∎

REMARK 3. Theorem 6 may also be obtained from Maliszewski's theorem [3], stating that every quasicontinuous functions from Bruckner–Ceder–Weiss' class $\mathcal{U}$ is the uniform limit of some sequence of strong Świątkowski functions. However, observe that the functions f and f_n constructed in the proof of Theorem 6 are not quasicontinuous.

It is well known that a uniform limit of DB_1 functions is DB_1 ([1]).

THEOREM 7. *There is a sequence of Baire* 1 *functions* $f_n : \mathbb{R} \to \mathbb{R}$ *belonging to* D_{ap} *which uniformly converges to a function* f *without the* D_{ap}*-property.*

Proof. Choose $I_n = [a_n, b_n]$, $n \geq 1$, such that $0 < a_{n+1} < b_{n+1} < a_n < b_n < 1$ for $n \geq 1$ and $D_u(\bigcup_n I_n, 0) > 0$. For each $n \geq 1$ find $J_n = [c_n, d_n] \subset (b_{n+1}, a_n)$ and a continuous function $g_n : [b_{n+1}, a_n] \to [c_n, 1]$ such that $g_n(a_n) = g_n(b_{n+1}) = 1$ and $g_n(x) = x$ for $x \in J_n$. Let e_n be the centre of J_n, $n \geq 1$. For $n \geq 1$ let

$$f_n(x) = \begin{cases} 1 & \text{for } x \in [b_1, \infty), \\ 1 & \text{for } x \in [a_n, b_n],\ n \geq 1, \\ x & \text{for } x \in (-\infty, 0], \\ 0 & \text{for } x = e_k,\ k > n, \\ g_k(x) & \text{for } x \in [b_{k+1}, a_k],\ k \leq n, \\ g_k(x) & \text{for } x \in [b_{k+1}, c_k] \cup [d_k, a_k],\ k > n, \\ \text{linear on the intervals } [c_k, e_k] \text{ and } [e_k, d_k],\ k > n, \end{cases}$$

and

$$f(x) = \begin{cases} 1 & \text{for } x \in [b_1, \infty), \\ 1 & \text{for } x \in [a_k, b_k],\ k \geq 1, \\ x & \text{for } x \in (-\infty, 0], \\ g_k(x) & \text{for } x \in [b_{k+1}, a_k],\ \ k \geq 1. \end{cases}$$

Evidently f and $f_n,\ n \geq 1$, are continuous at all $x \neq 0$ (so they are Baire 1) and have the Darboux property. Moreover, they are not approximately continuous at $x = 0$. Since in each open interval J containing 0 there is a point $x \neq 0$ at which f_n is continuous and $f_n(x) = 0$, we see that $f_n \in D_{ap}$. As $f^{-1}(0) = \{0\}$, it follows that f does not have the D_{ap}-property. Since $|f_n - f| \leq a_n$ for $n \geq 1$ and $\lim_{n\to\infty} a_n = 0$, the sequence (f_n) uniformly converges to f. ■

The Darboux property may be defined locally ([2]).

A function $f : I \to \mathbb{R}$ has the *Darboux property at the point* $x \in I$ ($f \in D(x)$) if for each real $r > 0$ and for all

$$c_1 \in (\min(f(x), \liminf_{t\to x^+} f(t)), \max(f(x), \limsup_{t\to x^+} f(t)))$$

and

$$c_2 \in (\min(f(x), \liminf_{t\to x^-} f(t)), \max(f(x), \limsup_{t\to x^-} f(t)))$$

there are points $u \in (x, x + r) \cap I$ and $v \in (x - r, x) \cap I$ such that $f(u) = c_1$ and $f(v) = c_2$.

Observe that a function $f : I \to \mathbb{R}$ has the Darboux property if and only if $f \in D(x)$ for each $x \in I$ ([2]).

Similarly we can introduce the following local D_{ap}-property.

We will say that a function $f : I \to \mathbb{R}$ has the *D_{ap}-property at the point* $x \in I$ ($f \in D_{ap}(x)$) if for each real $r > 0$ and for all

$$c_1 \in (\min(f(x), \liminf_{t\to x^+} f(t)), \max(f(x), \limsup_{t\to x^+} f(t)))$$

and

$$c_2 \in (\min(f(x), \liminf_{t\to x^-} f(t)), \max(f(x), \limsup_{t\to x^-} f(t)))$$

there are points $u \in (x, x + r) \cap I$ and $v \in (x - r, x) \cap I$ at which f is approximately continuous and such that $f(u) = c_1$ and $f(v) = c_2$.

It is evident that if $f : I \to \mathbb{R}$ has the D_{ap}-property then $f \in D_{ap}(x)$ for each $x \in I$. Moreover, the function f from Theorem 7 is in $D_{ap}(x)$ for each $x \in I$, but not in D_{ap}.

Recall that a Baire 1 function $f : I \to \mathbb{R}$ has the Darboux property if and only if for each real α each of the sets $\{x : f(x) < \alpha\}$ and $\{x : f(x) > \alpha\}$ is bilaterally dense in itself (see [1]).

Let Z_{ap} denote the family of all functions $f : \mathbb{R} \to \mathbb{R}$ such that for each real α the following implications are true:

(i) if $f(x) < \alpha$ then for each $r > 0$ there are points $u \in (x-r,x)$ and $v \in (x,x+r)$ at which f is approximately continuous and $\max(f(u),f(v)) < \alpha$,

(ii) if $f(x) > \alpha$ then for each $r > 0$ there are points $u \in (x-r,x)$ and $v \in (x,x+r)$ at which f is approximately continuous and $\min(f(u),f(v)) > \alpha$.

Remark 4. *There is a function $f \in Z_{ap} \setminus D$.*

Proof. Let (I_n) be an enumeration of all open intervals with rational endpoints. For each $n \geq 1$ find two disjoint Cantor sets $A_{n,1}, A_{n,2} \subset I_n \setminus \bigcup_{k<n}$ of positive measure, and choose nonempty sets $B_{n,1} \subset A_{n,1}$ and $B_{n,2} \subset A_{n,2}$ belonging to T_d. Let

$$f(x) = \begin{cases} 1 & \text{for } x \in B_{n,1},\ n \geq 1, \\ -1 & \text{for } x \in B_{n,2},\ n \geq 1, \\ 0 & \text{elsewhere on } \mathbb{R}. \end{cases}$$

Then $f \in Z_{ap} \setminus D$. ∎

By a standard proof we obtain the following remark.

Remark 5. *If $f : \mathbb{R} \to \mathbb{R}$ belongs to D_{ap} then $f \in Z_{ap}$.*

Remark 6. *If a sequence of measurable functions $f_n : \mathbb{R} \to \mathbb{R}$ belonging to Z_{ap} uniformly converges to a function f then $f \in Z_{ap}$.*

Proof. Fix reals $r > 0$ and α. Let $x \in \mathbb{R}$ with $f(x) < \alpha$. Since (f_n) uniformly converges to f, there is k such that $|f_n(t) - f(t)| < (\alpha - f(x))/3 = s > 0$ for $n \geq k$ and $t \in \mathbb{R}$. So $f_k(x) < f(x) + s$ and from the Z_{ap}-property of f_k it follows that there is $t \in (x-r,x)$ which is an approximate continuity point of f_k such that $f_k(t) < f(x) + s$. There is a set $E \in T_d$ containing t and such that $E \subset (x-r,x)$ and $f_k(E) \subset (-\infty, f_k(t)+s)$. Being the limit of a sequence of measurable functions, f is measurable and there is an approximate continuity point u of f belonging to E. Observe that $f(u) < f_k(u) + s < f_k(t) + s + s < f(x) + 2s + s < \alpha$. In other cases the proofs are similar. ∎

Remark 7. $Z_{ap}B_1 \setminus D_{ap}B_1 \neq \emptyset$.

Proof. It suffices to observe that the function f constructed in the proof of Theorem 7 belongs to Z_{ap}. ∎

In Remark 6 the assumption of measurability of f_n, $n \geq 1$, is essential.

Example. Let (I_n) be a one-to-one enumeration of all open intervals with rational endpoints. For each $n \geq 1$, we find disjoint nowhere dense nonempty sets $A_{n,1}, \dots, A_{n,n} \in T_d$ contained in $I_n \setminus \bigcup_{k<n} \bigcup_{i \le k} A_{k,i}$. For each pair (n,k), $k \le n \ge 1$, we find a decomposition $A_{n,k} = B_{n,k} \cup C_{n,k}$ such that the sets $B_{n,k}$ and $C_{n,k}$ are nonmeasurable and $\mu^*(B_{n,k}) = \mu^*(C_{n,k}) = \mu(A_{n,k})$ (μ^* denotes the outer Lebesgue measure). For $n \ge 1$ let

$$f_n(x) = \begin{cases} 1 & \text{for } x \in A_{2k},\ k \ge n, \\ -1 & \text{for } x \in A_{2k-1},\ k \ge n, \\ 1 - 1/(2k) & \text{for } x \in B_{2k},\ k < n, \\ 1 & \text{for } x \in A_{2k} \setminus B_{2k},\ k < n, \\ -1 + 1/(2k) & \text{for } x \in B_{2k-1},\ k < n, \\ -1 & \text{for } x \in A_{2k-1} \setminus B_{2k-1},\ k < n, \\ 0 & \text{elsewhere on } \mathbb{R}. \end{cases}$$

Moreover, put

$$f(x) = \begin{cases} 1 - 1/(2n) & \text{for } x \in B_{2n},\ n \ge 1, \\ 1 & \text{for } x \in A_{2n} \setminus B_{2n},\ n \ge 1, \\ -1 + 1/(2n) & \text{for } x \in B_{2n-1},\ n \ge 1, \\ -1 & \text{for } x \in A_{2n-1} \setminus B_{2n-1},\ n \ge 1, \\ 0 & \text{elsewhere on } \mathbb{R}. \end{cases}$$

Evidently the sequence (f_n) uniformly converges to f. Each f_n is approximately continuous at all points of the sets A_k for $k \ge 2n-1$. Since each open interval contains infinitely many of the sets A_k, the function f_n is in Z_{ap}. On the other hand, if $x \in A_k$ for some $k \ge 1$ then f is not approximately continuous at x. So all approximate continuity points of f belong to $f^{-1}(0)$ and consequently f is not in Z_{ap}.

Similarly to the proof of Theorem 2 we can show that the set $Z_{ap}B_1$ of all Baire 1 functions with the Z_{ap}-property is nowhere dense in DB_1.

Since every derivative belongs to DB_1 and has the Denjoy–Clarkson property (i.e. for any open intervals J and K we have $f^{-1}(J) \cap K = \emptyset$ or $\mu(f^{-1}(J) \cap K) > 0$), each derivative has the Z_{ap}-property.

Problem. Is there a derivative $f : I \to \mathbb{R}$ which is not in D_{ap}?

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