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On a subordination result for analytic functions defined by convolution

Abstract. In this paper we discuss some subordination results for a subclass of functions analytic in the unit disk U.

1. Introduction. Let A be the class of functions $f(z)$ analytic in the unit disk $U = \{z : |z| < 1\}$ and normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We denote by $K(\alpha)$ the class of convex functions of order α, i.e.,

$$K(\alpha) = \left\{ f \in A : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ z \in U \right\}.$$

Definition 1 (Hadamard product or convolution). Given two functions $f(z)$ and $g(z)$, where $f(z)$ is defined in (1.1) and $g(z)$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

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the Hadamard product (or convolution) $f * g$ of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.2}$$

Definition 2 (Subordination). Let $f(z)$ and $g(z)$ be analytic in the unit disk U. Then $f(z)$ is said to be subordinate to $g(z)$ in U and we write

$$f(z) \prec g(z), \quad z \in U,$$

if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$, $|w(z)| < 1$ such that

$$f(z) = g(w(z)), \quad z \in U. \tag{1.3}$$

In particular, if the function $g(z)$ is univalent in U, then $f(z)$ is subordinate to $g(z)$ if

$$f(0) = g(0), \quad f(U) \subseteq g(U). \tag{1.4}$$

Definition 3 (Subordinating factor sequence). A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in U, the subordination is given by

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z), \quad z \in U, \ a_1 = 1.$$

We have the following theorem.

Theorem 1.1 (Wilf [5]). *The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0, \quad z \in U. \tag{1.5}$$

Let

$$M(\alpha) = \left\{f \in A : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha, \ z \in U\right\} \tag{1.6}$$

and let

$$M^{\delta}(b,\delta) = \left\{f \in A : \operatorname{Re}\left\{1 - \frac{2}{5} + \frac{2D^{\delta+2}f(z)}{bD^{\delta+1}f(z)}\right\} < \alpha, \ \alpha > 0, \ z \in U\right\}. \tag{1.7}$$

Here $D^\delta f(z)$ is the Ruschewey's derivative defined as

$$\begin{aligned}D^\delta f(z) &= \frac{z}{(1-z)^{\delta+1}} * f(z)\\ &= \left(z + \sum_{n=2}^{\infty} \frac{\Gamma(n+\delta)}{(n-1)!\Gamma(1+\delta)}\right) * \left(z + \sum_{n=2}^{\infty} a_n z^n\right)\\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+\delta)}{(n-1)!\Gamma(1+\delta)} a_n z^n, \quad \delta \geq -1.\end{aligned}$$

Theorem 1.2 ([3]). *If $f(z) \in A$ satisfies*

$$\begin{aligned}&\sum_{n=2}^{\infty} \{|b(1-k)(\delta+2) + 2(n-1)| + |b(1-2\alpha+k)(\delta+2)\\ &\quad + 2(n-1)|\} \frac{\Gamma(n+\delta+1)}{(n-1)!\Gamma(3+\delta)} |a_n| \leq 2|b(1-\alpha)|\end{aligned} \tag{1.8}$$

where b is a non-zero complex number, $\delta \geq -1$, $0 \leq k \leq 1$ and $\alpha > 1$, then $f(z) \in M^\delta(b, \alpha)$.

It is natural to consider the class $M^{\delta^*}(b, \alpha) \subset M^\delta(b, \alpha)$ such that

$$\begin{aligned}M^{\delta^*}(b, \alpha) = \Bigg\{ f \in A : &\sum_{n=2}^{\infty} \{|b(1-k)(\delta+2) + 2(n-1)|\\ &+ |b(1-2\alpha+k)(\delta+2) + 2(n-1)|\} \frac{\Gamma(n+\delta+1)}{(n-1)!\Gamma(3+\delta)} |a_n|\\ &\leq |b(1-\alpha)| \Bigg\}.\end{aligned} \tag{1.9}$$

Our main result in this paper is the following theorem.

Theorem 1.3. *Let $f \in M^{\delta^*}(b, \alpha)$, then*

$$\frac{B}{C}(f * g)(z) \prec g(z) \tag{1.10}$$

where

$$B = |b(1-k)(\delta+2) + 2| + |b(1-2\alpha+k)(\delta+2) + 2|$$

$$C = 2[2|b(1-\alpha)| + |b(1-k)(\delta+2) + 2| + |b(1-2\alpha+k)(\delta+2) + 2|],$$

$\delta \geq -1$, $0 \leq k \leq 1$, b is a non-zero complex number and $g(z) \in K(\alpha)$, $z \in U$. Moreover,

$$\mathrm{Re}(f(z)) > -\frac{C}{2B}. \tag{1.11}$$

The constant factor

$$\frac{B}{C} = \frac{|b(1-k)(\delta+2) + 2| + |b(1-2\alpha+k)(\delta+2) + 2|}{2[2|b(1-\alpha)| + |b(1-k)(\delta+2) + 2| + |b(1-2\alpha+k)(\delta+2) + 2|]}$$

cannot be replaced by a larger one.

2. Proof of the main result. Let $f(z) \in M^{\delta^*}(b, \alpha)$ and suppose that

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in K(\alpha).$$

Then by definition,

$$\frac{B}{C}(f * g)(z) = \frac{B}{C}\left(z + \sum_{n=2}^{\infty} a_n b_n z^n\right). \tag{2.1}$$

Hence, by Definition 3, to show the subordination (1.10) it is enough to prove that

$$\left\{\frac{B}{C} a_n\right\}_{n=1}^{\infty} \tag{2.2}$$

is a subordinating factor sequence with $a_1 = 1$. Therefore, by Theorem 1.1 it is sufficient to show that

$$\operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty} \frac{B}{C} a_n z^n\right\} > 0, \quad z \in U. \tag{2.3}$$

Now,

$$\begin{aligned} \operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty} \frac{B}{C} a_n z^n\right\} &= \operatorname{Re}\left\{1 + 2\frac{B}{C} a_1 z + \frac{2}{C}\sum_{n=2}^{\infty} B a_n z^n\right\} \\ &\geq 1 - 2\frac{B}{C} r - \frac{2}{C}\sum_{n=2}^{\infty} B|a_n| r^n. \end{aligned} \tag{2.4}$$

Since $\dfrac{\Gamma(n+\delta+1)}{(n-1)!\Gamma(3+\delta)}$ is a monotone non-decreasing function of $n = 2, 3, \ldots,$ we have

$$\begin{aligned} \operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty} \frac{B}{C} a_n z^n\right\} &> 1 - 2\frac{B}{C} r \\ &- \frac{2}{C}\sum_{n=2}^{\infty}\{[|b(1-k)(\delta+2) + 2(n-1)]| + |b(1-2\alpha+k)(\delta+2) + 2(n-1)|\} \\ &\times \frac{\Gamma(n+\delta+1)}{(n-1)!\Gamma(3+\delta)}|a_n| r, \quad 0 < r < 1. \end{aligned}$$

By (1.8)

$$\begin{aligned} &\sum_{n=2}^{\infty}\{|b(1-k)(\delta+2) + 2(n-1)| + |b(1-2\alpha+k)(\delta+2) + 2(n-1)|\} \\ &\qquad \times \frac{\Gamma(n+\delta+1)}{(n-1)!\Gamma(3+\delta)}|a_n| \leq 2|b(1-\alpha)|. \end{aligned}$$

Hence,

$$\begin{aligned}\operatorname{Re}\left\{1+2\sum_{n=1}^{\infty}\frac{B}{C}a_n z^n\right\} &= \operatorname{Re}\left\{1+2\frac{B}{C}a_1 z+\frac{2}{C}\sum_{n=2}^{\infty}Ba_n z^n\right\}\\ &> 1-2\frac{B}{C}r-\frac{4|b(1-\alpha)|}{C}r\\ &= 1-\frac{2B+4|b(\alpha-1)|}{C}r\\ &= 1-r>0\end{aligned}$$

$(|z|=r<1)$. Therefore, we obtain

$$\operatorname{Re}\left\{1+2\sum_{n=1}^{\infty}\frac{B}{C}a_n z^n\right\}>0$$

which is (2.3) that was to be established.

We now show that

$$\operatorname{Re}(f(z))>-\frac{C}{2B}.$$

Taking

$$g(z)=\frac{z}{1-z}\in K(\alpha),$$

(1.10) becomes

$$\frac{B}{C}f(z)\prec\frac{z}{1-z}.$$

Therefore,

$$\operatorname{Re}\left(\frac{B}{C}f(z)\right)>\operatorname{Re}\left(\frac{z}{1-z}\right). \tag{2.5}$$

Since

$$\operatorname{Re}\left(\frac{z}{1-z}\right)>-\frac{1}{2},\quad |z|<r, \tag{2.6}$$

this implies that

$$\frac{B}{C}\operatorname{Re}(f(z))>-\frac{1}{2}. \tag{2.7}$$

Hence, we have

$$\operatorname{Re}(f(z))>-\frac{C}{2B}$$

which is (1.11).

To show the sharpness of the constant factor

$$\frac{B}{C}=\frac{|b(1-k)(\delta+2)+2|+|b(1-2\alpha+k)(\delta+2)+2|}{2[2|b(1-\alpha)|+|b(1-k)(\delta+2)+2|+|b(1-2\alpha+k)(\delta+2)+2|]},$$

we consider the function:

$$f_1(z) = z - \frac{2|b(1-\alpha)|}{B}z^2 = \frac{Bz - 2|b(1-\alpha)z^2}{B} \tag{2.8}$$

$(z \in U;\ \delta \geq -1;\ 0 \leq k \leq 1;\ b \in \mathbb{C} \setminus \{0\})$. Applying (1.10) with $g(z) = \dfrac{z}{1-z}$ and $f(z) = f_1(z)$ we have

$$\frac{Bz - 2b(\alpha-1)z^2}{C} \prec \frac{z}{1-z}. \tag{2.9}$$

Using the fact that

$$|\operatorname{Re} z| \leq |z|, \tag{2.10}$$

we now show that

$$\min\left\{\operatorname{Re}\frac{Bz - 2b(\alpha-1)z^2}{C} : z \in U\right\} = -\frac{1}{2}. \tag{2.11}$$

Now,

$$\begin{aligned}\left|\operatorname{Re}\frac{Bz - 2|b(1-\alpha)|z^2}{C}\right| &\leq \left|\frac{Bz - 2|b(1-\alpha)|z^2}{C}\right| \\ &= \frac{|Bz - 2|b(1-\alpha)|z^2|}{|C|} \\ &\leq \frac{B|z| + 2|b(1-\alpha)||z^2|}{C} \\ &= \frac{B + 2|b(1-\alpha)|}{C} = \frac{1}{2}\end{aligned} \tag{2.12}$$

$(|z| = 1)$. This implies that

$$\left|\operatorname{Re}\frac{Bz - 2|b(1-\alpha)|z^2}{C}\right| \leq \frac{1}{2}, \tag{2.13}$$

i.e.,

$$-\frac{1}{2} \leq \operatorname{Re}\frac{Bz - 2|b(1-\alpha)|z^2}{C} \leq \frac{1}{2}.$$

Hence,

$$\min\left\{\operatorname{Re}\left(\frac{B}{C}f_1(z)\right) : z \in U\right\} = -\frac{1}{2},$$

which completes the proof of Theorem 1.3.

3. Some applications. Taking $\delta = 1$ and $b = 1$ in Theorem 1.3, we obtain the following:

Corollary 1. *If the function $f(z)$ defined by* (1.1) *is in $M^{\delta^*}(b, \alpha)$, then*

$$\frac{|5-3\alpha|}{2|6-4\alpha|}(f * g)(z) \prec g(z) \tag{3.1}$$

($z \in U$; $\alpha > 1$, $g \in K(\alpha)$). In particular,

$$\mathrm{Re}(f(z)) > -\frac{|6-4\alpha|}{|5-3\alpha|}. \tag{3.2}$$

The constant factor

$$\frac{|5-3\alpha|}{2|6-4\alpha|}$$

cannot be replaced by any larger one.

Remark 1. By taking $\alpha = \frac{71}{45} > 1$ in Corollary 1, we obtain the result of Aouf et al. [1]

Taking $b = 1$, $\delta = 0$ in Theorem 1.3, we obtain the following:

Corollary 2. *If the function $f(z)$ defined by* (1.1) *is in $M^{\delta^*}(b, \alpha)$, then*

$$\frac{|2-\alpha|}{|5-3\alpha|}(f * g)(z) \prec g(z) \tag{3.3}$$

($z \in U$; $\alpha > 1$, $g \in K(\alpha)$). In particular,

$$\mathrm{Re}(f(z)) > -\frac{|5-3\alpha|}{2|2-\alpha|}, \quad z \in U. \tag{3.4}$$

The constant factor

$$\frac{|2-\alpha|}{|5-3\alpha|}$$

cannot be replaced by any larger one.

Remark 2. By taking $\alpha = \frac{11}{6}$ and $\alpha = \frac{20}{11}$ in Corollary 2, we obtain the results of Selvaraj and Karthikeyan [4].

Taking $b = 1$, $\delta = -1$ and $k = 0$ in Theorem 1.3, we obtain the following:

Corollary 3. *If the function $f(z)$ defined by* (1.1) *is in $M^{\delta^*}(b, \alpha)$, then*

$$\frac{|3-\alpha|}{|8-4\alpha|}(f * g)(z) \prec g(z) \tag{3.5}$$

($z \in U$; $\alpha > 1$, $g \in K(\alpha)$). In particular,

$$\mathrm{Re}(f(z)) > -\frac{|4-2\alpha|}{|3-\alpha|}, \quad z \in U. \tag{3.6}$$

The constant factor

$$\frac{|3-\alpha|}{|8-4\alpha|}$$

cannot be replaced by any larger one.

Remark 3. If we take $\alpha = \frac{7+3m}{3+m}$ in Corollary 3, $(m > 0)$ and in particular $m = 1$ (i.e., $\alpha = \frac{5}{2} > 1$), we obtain the result of Attiya et al. [2].

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