

# On a theorem of Breiman and a class of random difference equations

**Citation for published version (APA):**

Denisov, D. E., & Zwart, B. (2005). *On a theorem of Breiman and a class of random difference equations*. (Report Eurandom; Vol. 2005039). Eurandom.

**Document status and date:**

Published: 01/01/2005

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

**General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

[www.tue.nl/taverne](http://www.tue.nl/taverne)

**Take down policy**

If you believe that this document breaches copyright please contact us at:

[openaccess@tue.nl](mailto:openaccess@tue.nl)

providing details and we will investigate your claim.

# On a theorem of Breiman and a class of random difference equations

Denis Denisov\* and Bert Zwart<sup>†,‡</sup>

\* EURANDOM

P.O. Box 513, 5600 MB Eindhoven, The Netherlands

<sup>†</sup> Department of Mathematics & Computer Science  
Eindhoven University of Technology

P.O. Box 513, 5600 MB Eindhoven, The Netherlands

<sup>‡</sup> CWI

P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

November 22, 2005

## Abstract

We consider the tail behavior of the product of two independent nonnegative random variables  $X$  and  $Y$ . Breiman (1965) has considered this problem assuming that  $X$  is regularly varying with index  $\alpha$  and that  $\mathbb{E}\{Y^{\alpha+\epsilon}\} < \infty$  for some  $\epsilon > 0$ . We investigate when the condition on  $Y$  can be weakened and apply our findings to analyze a class of random difference equations.

*2000 Mathematics Subject Classification:* 60H25, 60J30, 60F10.

*Keywords & Phrases:* Regular variation, subexponential distributions, random difference equations.

## 1 Introduction

Suppose that  $X$  and  $Y$  are two independent nonnegative random variables such that  $\mathbb{P}\{X > x\}$  is regularly varying of index  $-\alpha$ ,  $\alpha \geq 0$ , and that  $\mathbb{E}\{Y^{\alpha+\epsilon}\} < \infty$  for some  $\epsilon > 0$ . Then

$$\mathbb{P}\{XY > x\} \sim \mathbb{E}\{Y^\alpha\}\mathbb{P}\{X > x\}, \quad (1)$$

as  $x \rightarrow \infty$ , with  $f(x) \sim g(x)$  denoting  $f(x) = g(x)(1 + o(1))$ . This result has been stated first in Breiman [2] for  $\alpha \in [0, 1]$  and is known as Breiman's theorem; a more recent study containing a proof for all  $\alpha$  is in Cline and Samorodnitsky [5].

We are interested in extensions of (1), in particular in relaxing the condition  $\mathbb{E}\{Y^{\alpha+\epsilon}\} < \infty$ . Apart from its intrinsic interest, our motivation for this comes from the well known random affine equation

$$R \stackrel{d}{=} MR + Q. \quad (2)$$

This equation appears in many different applications, most notably in actuarial and financial mathematics. If  $P(|M| > 1) > 0$ , then  $R$  typically has a power tail and this case

is fairly well understood; classical papers are Kesten [14] and Goldie [10]. Unfortunately, not many results are available when  $Q$  is light-tailed and  $|M| \leq 1$ . Some partial results can be found in Goldie & Grübel [11] and Maulik & Zwart [19]. A relatively clean case seems to be when  $\exp Q$  is regularly varying with index  $-\alpha$  and independent of  $M$ . After taking exponents in (2) one may wonder whether the application of Breiman's theorem is justified, i.e. whether  $\mathbb{P}\{R > x\} \sim \mathbb{E}\{\exp\{\alpha MR\}\}\mathbb{P}\{Q > x\}$ . If  $|M| < 1 - \delta$  a.s. for some  $\delta > 0$  then one can show that  $\mathbb{E}\{\exp\{(\alpha + \epsilon)MX\}\} < \infty$  for some  $\epsilon > 0$ , so that Breiman's theorem (1) can indeed be applied. However, assuming the existence of such a  $\delta > 0$  is not very natural, and we are interested in to which extent the tail equivalence between  $R$  and  $Q$  remains true without invoking such an assumption. To obtain an answer to this question, the conditions under which Breiman's theorem remains true need to be relaxed along the lines above.

We now proceed with an informal presentation of our results. If  $X$  is regularly varying with index  $-\alpha$ , the most general conditions on  $Y$  under which (1) holds would be

$$\mathbb{E}\{Y^\alpha\} < \infty \text{ and } \mathbb{P}\{Y > x\} = o(\mathbb{P}\{X > x\}). \quad (3)$$

We show that this set of conditions on  $Y$  is in general not enough for (1) to hold. To obtain sufficient conditions, we make additional assumptions on the slowly varying function  $L$  in the representation  $\mathbb{P}\{X > x\} = L(x)x^{-\alpha}$ . In particular, we consider three different cases:

1. If  $\liminf_{x \rightarrow \infty} L(x) > 0$ , then (3) implies (1) without any further assumptions.
2. If  $L(x)$  is eventually decreasing to 0, then for  $x$  large enough,  $L(\exp\{x\}) = \mathbb{P}\{U > x\}$  for some long-tailed random variable  $U$ . It turns out that the additional condition  $U \in \mathcal{S}^*$  is crucial for (1) to hold if  $U$  has a finite mean. A similar type of assumption has to be made if  $L(x)$  oscillates at infinity.
3. If the condition  $\mathbb{E}\{U\} < \infty$  in the previous case does not hold (which is the case when  $\mathbb{E}\{X^\alpha\} = \infty$  and  $\lim_{x \rightarrow \infty} x^\alpha \mathbb{P}\{X > x\} = 0$ ), then we also need to invoke an additional condition to ensure validity of (1).

These three cases are respectively covered by Propositions 2.1–2.3 in Section 2. The necessity of the additional regularity conditions is illustrated by a number of counterexamples in Section 3.

The results of the present paper are related to several existing results in the literature. Several researchers independently obtained that (1) always holds if  $\mathbb{P}\{X > x\} \sim cx^{-\alpha}$  and  $\mathbb{E}\{Y^\alpha\} < \infty$ , see e.g. Lemma 2.1 in [13] and Lemma 5.1 in [19]. Our Proposition 2 is an extension of these results. Embrechts & Goldie [7] show that  $XY$  is regularly varying of index  $-\alpha$  if both  $X$  and  $Y$  are regularly varying of index  $-\alpha$ , without providing explicit asymptotics. Cline [3] contains a property of the class  $\mathcal{S}(\gamma)$  which is strongly related to the second case discussed above; we get back to this in Section 2. In addition, [3] investigates the asymptotic behavior of  $\mathbb{P}\{XY > x\}$  in various cases where  $\mathbb{P}\{XY > x\}/\mathbb{P}\{X > x\} \rightarrow \infty$ .

This paper is organized as follows. Section 2 gives a number of sufficient conditions on  $L$  in order for (1) to hold under (3). Counterexamples are provided in Section 3. In Section 4, we apply our results to obtain the tail behavior of  $R$  in the random difference equation mentioned above. Some concluding remarks are given in Section 5.

## 2 Extensions of Breiman's theorem

In this section we investigate which assumptions, in addition to (3), are needed to guarantee (1). As described in the Introduction, we focus on additional assumptions on the

slowly varying function  $L$  in  $\mathbb{P}\{X > x\} = L(x)x^{-\alpha}$ . As a preliminary we develop a representation of slowly varying functions in terms of long-tailed distribution functions. Throughout the rest of the paper we use various properties of regularly varying functions which all appear in the monograph [1]. In addition, we use the class of subexponential distributions, denoted by  $\mathcal{S}$ , and the class of long-tailed distributions which is denoted by  $\mathcal{L}$ ; see [8] for details.

**Lemma 2.1** *Let  $L$  be slowly varying. Then  $L$  admits precisely one of the following four representations.*

- (i)  $L(x) = c(x)$ ,
- (ii)  $L(x) = c(x)/\mathbb{P}\{V > \log x\}$ ,
- (iii)  $L(x) = c(x)\mathbb{P}\{U > \log x\}$ ,
- (iv)  $L(x) = c(x)\mathbb{P}\{U > \log x\}/\mathbb{P}\{V > \log x\}$ .

In all representations,  $c(x)$  is a function converging to a constant  $c \in (0, \infty)$ .  $U$  and  $V$  are two independent long-tailed random variables with hazard rates converging to 0.

**Proof.** By the representation theorem for slowly varying functions we can write for some function  $c(x) \rightarrow c \in (0, \infty)$  and  $h(x) \rightarrow 0$ ,

$$L(x) = c(x) \exp\left\{\int_1^x h(u)/udu\right\}.$$

Write  $h(u) = h^+(u) - h^-(u)$ , with  $h^+(u)$  the positive part of  $h(u)$  and  $h^-(u)$  the negative part. Both  $h^+(u)$  and  $h^-(u)$  converge to 0. A first issue is whether  $\int_1^x h^i(u)/udu$  converges,  $i = +, -$ . If this would be the case, then this can be incorporated in the function  $c(x)$ , so without loss of generality, we can assume that either  $h^i(u) = 0$  or the corresponding integral diverges. This leads to the four cases above.

Suppose now that  $\int_1^x h^+(u)/udu$  diverges. Then there exists a long-tailed random variable  $V$  such that

$$\exp\left\{-\int_1^x h^+(u)/udu\right\} = \exp\left\{-\int_0^{\log x} h^+(e^v)dv\right\} = \mathbb{P}\{V > \log x\}.$$

A similar argument can be made for  $h^-$ . ■

We are now ready to give our first sufficient condition for (1).

**Proposition 2.1** *Assume that, in addition to (3),  $\liminf_{x \rightarrow \infty} L(x) > 0$ . Then (1) holds.*

**Proof.** The assumption on  $L$  implies that  $\alpha > 0$ . By replacing  $X$  and  $Y$  by  $X^\alpha$  and  $Y^\alpha$  if  $\alpha \neq 1$ , we can assume that  $\alpha = 1$ . We can also assume without loss of generality that  $\mathbb{P}\{Y = 0\} = 0$ . Note that the asymptotic lower bound

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}\{XY > x\}}{\mathbb{P}\{X > x\}} \geq \mathbb{E}\{Y\}$$

always holds in view of Fatou's lemma. To obtain an upper bound, write

$$\mathbb{P}\{XY > x\} = \sum_{i=1}^4 \mathbb{P}\{XY > x; Y \in A_i\},$$

with  $A_1 = [0, \epsilon)$ ,  $A_2 = [\epsilon, M)$ ,  $A_3 = [M, g(x)x)$ , and  $A_4 = [g(x)x, \infty)$ . Here,  $g(x) \downarrow 0$  is chosen such that  $\mathbb{P}\{Y > g(x)x\} = o(\mathbb{P}\{X > x\})$ . Number the four terms as  $I_1, \dots, I_4$ . Then

$$I_1 \leq \mathbb{P}\{X > x/\epsilon\} \sim \epsilon \mathbb{P}\{X > x\}.$$

Furthermore, by the uniform convergence theorem for slowly varying functions we obtain that

$$I_2 \sim \mathbb{E}\{Y; \epsilon < Y < M\} \mathbb{P}\{X > x\}.$$

The fourth term can be upper bounded as follows:

$$I_4 \leq \mathbb{P}\{Y > g(x)x\} = o(\mathbb{P}\{X > x\}).$$

Thus, it remains to consider

$$I_3 = \mathbb{P}\{X > x\} \int_M^{g(x)x} \frac{L(x/y)}{L(x)} y d\mathbb{P}\{Y \leq y\}.$$

Since  $L(x)$  is bounded away from 0 uniformly in  $x$  we are either in case (i) or (ii) of Lemma 2.1. We thus have the upper bound, for  $y \geq M \geq 1$ ,

$$\frac{L(x/y)}{L(x)} \leq \sup_{y \in [M, g(x)x]} \frac{c(x/y)}{c(x)}.$$

Consequently,

$$I_3 \leq \mathbb{P}\{X > x\} \sup_{y \in [M, g(x)x]} \frac{c(x/y)}{c(x)} \mathbb{E}\{Y; Y > M\} \sim \mathbb{P}\{X > x\} \mathbb{E}\{Y; Y > M\}.$$

Putting everything together, we obtain

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}\{XY > x\}}{\mathbb{P}\{X > x\}} \leq \epsilon + \mathbb{E}\{Y; \epsilon < Y < M\} + 2 \frac{L^-(1/\delta)}{\ell} \mathbb{E}\{Y; Y > M\}.$$

The result now follows by letting  $\epsilon \downarrow 0$  and  $M \rightarrow \infty$ . ■

We now investigate what happens if  $\liminf L(x) = 0$ . It turns out that the situation is more complicated in this case. Before we can state our results, we need to introduce a number of additional definitions. A non-negative function  $f$  is in the class  $\mathcal{S}_d$  (and in this case one calls  $f$  a subexponential density) if it satisfies the property

$$\lim_{x \rightarrow \infty} \int_0^x \frac{f(x-y)}{f(x)} f(y) dy = 2 \int_0^\infty f(u) du < \infty.$$

If  $f(x) = \mathbb{P}\{U > x\}$  for some random variable  $U$ , we say that  $U \in \mathcal{S}^*$ . Both classes  $\mathcal{S}_d$  and  $\mathcal{S}^*$  have been introduced by Klüppelberg [16, 17].

In addition, recall that a non-negative random variable  $T$  is in the class  $\mathcal{S}(\gamma)$ ,  $\gamma \geq 0$  if, as  $x \rightarrow \infty$ ,

$$\frac{\mathbb{P}\{T > x+y\}}{\mathbb{P}\{T > x\}} \rightarrow e^{-\gamma y} \quad \text{and} \quad \frac{\mathbb{P}\{T+T' > x\}}{\mathbb{P}\{T > x\}} \rightarrow 2\mathbb{E}\{e^{\gamma T}\} < \infty,$$

with  $T'$  an i.i.d. copy of  $T$ . Note that  $\mathcal{S}(0) = \mathcal{S}$ . It is shown in [17] that, for  $\gamma > 0$ ,  $T \in \mathcal{S}(\gamma)$  if and only if  $e^{\gamma x} \mathbb{P}\{T > x\}$  is in  $\mathcal{S}_d$ . A recent interesting paper on these classes of distributions is Foss & Korshunov [9].

We are now ready to state our second result.

**Proposition 2.2** *Assume in addition to (3) that  $\alpha > 0$  and that  $L$  admits representation (iii) or (iv) of Lemma 2.1. If either  $L(e^x) \in \mathcal{S}_d$  or the auxiliary random variable  $U \in \mathcal{S}^*$ , then (1) holds.*

**Proof.** If  $L(e^x) \in \mathcal{S}_d$  then, according to Theorem 2.1 of Klüppelberg [17],  $\log X \in \mathcal{S}(\gamma)$  and the result then follows from [3]; see the discussion below Theorem 1 in that paper. For the other case, we proceed similarly as in the proof of the previous proposition. It remains to estimate  $I_3$ . Define  $c^*(x) = \sup_{y \in [M, g(x)x]} \frac{c(x/y)}{c(x)}$ . Then

$$\begin{aligned} I_3 &= \mathbb{P}\{X > x\} \int_M^{g(x)x} \frac{L(x/y)}{L(x)} y d\mathbb{P}\{Y \leq y\} \\ &\leq \mathbb{P}\{X > x\} c^*(x) \int_M^{g(x)x} \frac{\mathbb{P}\{U > \log x - \log y\}}{\mathbb{P}\{U > \log x\}} y d\mathbb{P}\{Y \leq y\}. \end{aligned}$$

We need to show that

$$\lim_{M \rightarrow \infty} \limsup_{x \rightarrow \infty} \int_M^{g(x)x} \frac{\mathbb{P}\{U > \log x - \log y\}}{\mathbb{P}\{U > \log x\}} y d\mathbb{P}\{Y \leq y\} = 0. \quad (4)$$

Denote  $s(x) = \mathbb{P}\{Y > x\} / \mathbb{P}\{X > x\} \rightarrow 0$ . Integrating by parts, we obtain

$$\begin{aligned} &\int_M^{g(x)x} \mathbb{P}\{U > \log x - \log y\} y d\mathbb{P}\{Y \leq y\} = \\ &- \mathbb{P}\{Y > g(x)x\} \mathbb{P}\{U > -\log g(x)\} x g(x) + \mathbb{P}\{Y > M\} M \mathbb{P}\{U > \log x - \log M\} \\ &+ \int_M^{xg(x)} \mathbb{P}\{U > \log x - \log y\} \mathbb{P}\{Y > y\} dy + \int_M^{xg(x)} \mathbb{P}\{Y > y\} y d_y \mathbb{P}\{U > \log x - \log y\}. \end{aligned}$$

We continue by bounding all terms on the right hand side of this expression. The first term is non-positive and can therefore be discarded. To bound the second term, note that

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}\{Y > M\} M \mathbb{P}\{U > \log x - \log M\}}{\mathbb{P}\{U > \log x\}} = \mathbb{P}\{Y > M\} M.$$

For the third term, we have,

$$\begin{aligned} &\limsup_{x \rightarrow \infty} \frac{\int_M^{xg(x)} \mathbb{P}\{U > \log x - \log y\} \mathbb{P}\{Y > y\} dy}{\mathbb{P}\{U > \log x\}} \\ &\leq \limsup_{x \rightarrow \infty} \sup_{y \geq M} s(y) \frac{\int_M^x \mathbb{P}\{U > \log x - \log y\} \mathbb{P}\{U > \log y\} d \log y}{\mathbb{P}\{U > \log x\}} \leq \sup_{y \geq M} s(y) 2\mathbb{E}\{U\}, \end{aligned}$$

since  $U \in \mathcal{S}^*$ . Finally we have, for the fourth term,

$$\begin{aligned} &\limsup_{x \rightarrow \infty} \frac{\int_M^{xg(x)} \mathbb{P}\{Y > y\} y d_y \mathbb{P}\{U > \log x - \log y\}}{\mathbb{P}\{U > \log x\}} \\ &\leq \limsup_{x \rightarrow \infty} \sup_{y \geq M} c(y) s(y) \frac{\int_M^x \mathbb{P}\{U > \log y\} d_y \mathbb{P}\{U > \log x - \log y\}}{\mathbb{P}\{U > \log x\}} \leq 2 \sup_{y \geq M} c(y) s(y), \end{aligned}$$

since  $U \in \mathcal{S}^*$  and therefore also subexponential. Putting everything together, we see that

$$\limsup_{x \rightarrow \infty} \int_M^{g(x)x} \frac{\mathbb{P}\{U > \log x - \log y\}}{\mathbb{P}\{U > \log x\}} y d\mathbb{P}\{Y \leq y\} \leq \mathbb{P}\{Y > M\} M + \sup_{y \geq M} s(y) 2\mathbb{E}\{U\} + 2 \sup_{y \geq M} c(y) s(y).$$

This converges to 0 if  $M \rightarrow \infty$ , which implies (4). ■

Note finally that the two assumptions  $U \in \mathcal{S}^*$  and  $L(e^x) \in \mathcal{S}_d$  are equivalent if case (iii) of Lemma 2.1 applies. However, in general, the two assumptions are not implied by one another.

We continue by investigating a third case, which occurs when the auxiliary random variable  $U$  has infinite mean, in which case  $U$  cannot be in  $\mathcal{S}^*$ . Put

$$m(x) = \int_0^x t^{\alpha-1} \mathbb{P}\{X > t\} dt.$$

It is clear that since  $\mathbb{E}\{X^\alpha\} = \infty$  if  $\mathbb{E}\{U\} = \infty$ , we have  $m(x) \rightarrow \infty$ .

**Proposition 2.3** *Assume in addition to (3), that  $\alpha > 0$  and that*

$$\frac{\mathbb{P}\{Y > x\}}{\mathbb{P}\{X > x\}} m(x) \rightarrow 0. \tag{5}$$

*Let  $L$  admit representation (iii) or (iv) of Lemma 2.1, with  $U \in \mathcal{D}$ . Then (1) holds.*

**Proof.** By considering  $X^\alpha$  and  $Y^\alpha$  when  $\alpha \neq 1$ , it suffices to prove the result for  $\alpha = 1$ . We proceed similarly as in the proof of the previous proposition. It remains to estimate  $I_3$ . Define  $c^*(x) = \sup_{y \in [M, g(x)x]} \frac{c(x/y)}{c(x)}$ . Then

$$\begin{aligned} I_3 &= \mathbb{P}\{X > x\} \int_M^{g(x)x} \frac{L(x/y)}{L(x)} y d\mathbb{P}\{Y \leq y\}. \\ &\leq \mathbb{P}\{X > x\} c^*(x) \int_M^{g(x)x} \frac{\mathbb{P}\{U > \log x - \log y\}}{\mathbb{P}\{U > \log x\}} y d\mathbb{P}\{Y \leq y\}. \end{aligned}$$

We split the integral into two integrals (according to the intervals  $[M, \sqrt{x}]$  and  $[\sqrt{x}, g(x)x]$ ) and estimate both of them separately, For the first integral we have

$$\begin{aligned} \int_M^{\sqrt{x}} \frac{\mathbb{P}\{U > \log x - \log y\}}{\mathbb{P}\{U > \log x\}} y d\mathbb{P}\{Y \leq y\} &\leq \frac{\mathbb{P}\{U > \log \sqrt{x}\}}{\mathbb{P}\{U > \log x\}} \int_M^{\sqrt{x}} y d\mathbb{P}\{Y \leq y\} \\ &\leq \sup_x \frac{\mathbb{P}\{U > x/2\}}{\mathbb{P}\{U > x\}} \int_M^\infty y d\mathbb{P}\{Y \leq y\}. \end{aligned}$$

As  $M$  goes to infinity the latter goes to 0. We integrate the second integral by parts to obtain

$$\begin{aligned} &\int_{\sqrt{x}}^{g(x)x} \mathbb{P}\{U > \log x - \log y\} y d\mathbb{P}\{Y \leq y\} = \\ &- \mathbb{P}\{Y > g(x)x\} \mathbb{P}\{U > -\log g(x)\} x g(x) + \mathbb{P}\{Y > \sqrt{x}\} \sqrt{x} \mathbb{P}\{U > \log x - \log \sqrt{x}\} \\ &+ \int_{\sqrt{x}}^{xg(x)} \mathbb{P}\{U > \log x - \log y\} \mathbb{P}\{Y > y\} dy + \int_{\sqrt{x}}^{xg(x)} \mathbb{P}\{Y > y\} y dy \mathbb{P}\{U > \log x - \log y\}. \end{aligned}$$

The first term is non-positive. For the second term we have, since  $\mathbb{E}\{Y\} < \infty$ ,

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}\{Y > \sqrt{x}\} \sqrt{x} \mathbb{P}\{U > \log \sqrt{x}\}}{\mathbb{P}\{U > \log x\}} = 0.$$

The third term satisfies

$$\begin{aligned}
& \frac{\int_{\sqrt{x}}^{xg(x)} \mathbb{P}\{U > \log x - \log y\} \mathbb{P}\{Y > y\} dy}{\mathbb{P}\{U > \log x\}} = o(1) \frac{\int_{\sqrt{x}}^{xg(x)} \mathbb{P}\{U > \log x - \log y\} \frac{\mathbb{P}\{X > y\}}{m(y)} dy}{\mathbb{P}\{U > \log x\}} \\
& = \frac{o(1) \int_{\sqrt{x}}^{xg(x)} \mathbb{P}\{U > \log x - \log y\} \mathbb{P}\{U > \log y\} d \log y}{m(\sqrt{x}) \mathbb{P}\{U > \log x\}} \\
& = o(1) \frac{1}{m(\sqrt{x})} \frac{\mathbb{P}\{U > \log \sqrt{x}\}}{\mathbb{P}\{U > \log x\}} \int_{\sqrt{x}}^{xg(x)} \mathbb{P}\{U > \log x - \log y\} d \log y \\
& = o(1) \frac{m(\sqrt{x}) - m(1/g(x))}{m(\sqrt{x})} \frac{\mathbb{P}\{U > \log \sqrt{x}\}}{\mathbb{P}\{U > \log x\}} \rightarrow 0.
\end{aligned}$$

Finally, for the fourth term we have,

$$\begin{aligned}
& \frac{\int_{\sqrt{x}}^{xg(x)} \mathbb{P}\{Y > y\} y d_y \mathbb{P}\{U > \log x - \log y\}}{\mathbb{P}\{U > \log x\}} \\
& \leq o(1) \frac{\int_{\sqrt{x}}^x \mathbb{P}\{U > \log y\} d_y \mathbb{P}\{U > \log x - \log y\}}{\mathbb{P}\{U > \log x\}} \rightarrow 0,
\end{aligned}$$

since  $U \in \mathcal{S}$ . ■

Note that that it is not sufficient to assume just  $\frac{\mathbb{P}\{Y > x\}}{\mathbb{P}\{X > x\}} \rightarrow 0$ . This is illustrated with a counterexample in the next section.

For completeness, we finally state a well known result for  $\alpha = 0$ .

**Proposition 2.4** (*Embrechts & Goldie [7]*). *Suppose in addition to (3) that  $\alpha = 0$  and that in particular  $\log X \in \mathcal{S}$ . Then (1) holds.*

That it is difficult to remove the assumption  $\log X \in \mathcal{S}$  is illustrated in the next section.

### 3 Counterexamples

In the previous section, we saw that Breiman's theorem can be extended in a number of cases, but that the minimal conditions (3) were not achieved. The goal of the present section is to illustrate that it is hard or even impossible to weaken the assumptions made in Propositions 2.2–2.4. In the next three subsections, we give a counterexample related to each of these three propositions.

#### 3.1 A counterexample related to Proposition 2.2

In this section we construct independent non-negative random variables  $X$  and  $Y$  such that  $X$  is regularly varying,  $Y$  satisfies (3), but for which (1) fails.

Since  $\alpha > 0$ , we can take  $\alpha = 1$  without loss of generality. Let  $a(x)$  be a distribution tail which is long-tailed, but not in  $\mathcal{S}^*$ . Assume that  $\int_0^\infty a(x) dx < \infty$  and

$$\limsup_{x \rightarrow \infty} \int_0^x \frac{a(y)a(x-y)}{a(x)} dy = \infty.$$



(All known examples of distributions in  $\mathcal{L} \setminus \mathcal{S}$  satisfy this property.) Since  $a(x)$  is long-tailed but not in  $\mathcal{S}^*$  there exists a function  $h(x)$  such that  $h(x) \rightarrow \infty$ ,  $a(x - h(x)) \sim a(x)$  and

$$\limsup_{x \rightarrow \infty} \int_{h(x)}^{x-h(x)} \frac{a(y)a(x-y)}{a(x)} dy = \infty.$$

Set now  $L(x) = a(\log x)$ . From the previous considerations, it follows that there exists a function  $d(x)$  such that we have  $d(x) \rightarrow \infty$ ,  $d(x) = o(\sqrt{x})$ ,  $L(x/d(x)) \sim L(x)$ ,  $\int_1^\infty L(x)/xdx < \infty$ , and

$$\limsup_{x \rightarrow \infty} \int_{d(x)}^{x/d(x)} \frac{L(y)L(x/y)}{yL(x)} dy =: \limsup_{x \rightarrow \infty} r(x) = \infty.$$

Let  $x_n, n \geq 1$  be a sequence such that  $x_n \rightarrow \infty$  and  $r(x_n) \rightarrow \infty$ .

We are now ready to define  $X$  and  $Y$ . Let  $X$  be a random variable with tail  $L(x)/x$ . Observe that  $X$  has finite mean. Set for  $x \geq 0$ ,  $s(x) = \sqrt{r(x_n)}$  when  $x \in [d(x_n), d(x_{n+1}))$ . Observe that  $s(x) \rightarrow \infty$ . Let  $g(x) = \mathbb{P}\{X > x\}/(xs(x)) = L(x)/(x^2s(x))$  be the density of  $Y$ . Since  $s(x) \rightarrow \infty$  it can be shown by using Karamata's theorem that  $\mathbb{P}\{Y > x\} = o(\mathbb{P}\{X > x\})$ .

Then

$$\begin{aligned} \frac{\mathbb{P}\{XY > x; d(x) < Y < x/d(x)\}}{\mathbb{P}\{X > x\}} &= \int_{d(x)}^{x/d(x)} \frac{L(y)L(x/y)}{ys(y)L(x)} dy \\ &\geq \int_{d(x)}^{x/d(x)} \frac{L(y)L(x/y)}{ys(d(x))L(x)} dy = \frac{r(x)}{s(d(x))}. \end{aligned}$$

This is at least  $\sqrt{r(x_n)}$  at the points  $x_n, n \geq 1$ . We conclude that

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}\{XY > x; Y > d(x)\}}{\mathbb{P}\{X > x\}} = \infty,$$

implying that (1) does not hold. This illustrates that the condition  $U \in \mathcal{S}^*$  in Proposition 2.2 cannot be weakened in general. We would like to remark that in the above construction one can additionally assume that both  $a(x)$  and  $\int_x^\infty a(u)du$  are subexponential, but for which  $a(x) \notin \mathcal{S}^*$ , see Denisov *et al.* [6] for an example of such a distribution tail.

### 3.2 A counterexample related to Proposition 2.4

We use the same notation as in the previous subsection but now assume that  $\alpha = 0$ . Let  $a(x)$  be a distribution tail which is long-tailed, but not in  $\mathcal{S}$ . In addition assume that

$$\limsup_{x \rightarrow \infty} \int_0^x \frac{a(x-y)}{a(x)} da(y) = \infty.$$

(All known examples of distributions in  $\mathcal{L} \setminus \mathcal{S}$  satisfy this property.) Since  $a(x)$  is long-tailed there exists a function  $h(x)$  such that  $h(x) \rightarrow \infty$ ,  $a(x - h(x)) \sim a(x)$  and

$$\limsup_{x \rightarrow \infty} \int_{h(x)}^{x-h(x)} \frac{a(x-y)}{a(x)} da(y) = \infty.$$

Set now  $L(x) = a(\log x)$ . From the previous considerations, it follows that there exists a function  $d(x)$  such that we have  $d(x) \rightarrow \infty$ ,  $d(x) = o(\sqrt{x})$ ,  $L(x/d(x)) \sim L(x)$ , and

$$\limsup_{x \rightarrow \infty} \int_{d(x)}^{x/d(x)} \frac{L(x/y)}{L(x)} dL(y) =: \limsup_{x \rightarrow \infty} r(x) = \infty.$$

Let  $x_n, n \geq 1$  be a sequence such that  $x_n \rightarrow \infty$  and  $r(x_n) \uparrow \infty$ .

We are now ready to define  $X$  and  $Y$ . Let  $X$  be a random variable with tail  $L(x)$ . Set for  $x \geq 0$ ,  $s(x) = \sqrt{r(x_n)}$  when  $x \in [d(x_n), d(x_{n+1}))$ . Let  $\mathbb{P}\{Y > x\} = \mathbb{P}\{X > x\}/s(x) = L(x)/s(x)$ . Since  $s(x) \uparrow \infty$ ,  $\mathbb{P}\{Y > x\} = o(\mathbb{P}\{X > x\})$ . Conditioning on values of  $X$  we obtain

$$\begin{aligned} \frac{\mathbb{P}\{XY > x; d(x) < X < x/d(x)\}}{\mathbb{P}\{X > x\}} &= \int_{d(x)}^{x/d(x)} \frac{L(x/y)}{s(x/y)L(x)} dL(y) \\ &\geq \int_{d(x)}^{x/d(x)} \frac{L(x/y)}{s(d(x))L(x)} dL(y) = \frac{r(x)}{s(d(x))}. \end{aligned}$$

This is at least  $\sqrt{r(x_n)}$  at the points  $x_n, n \geq 1$ . We conclude that

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}\{XY > x; X > d(x)\}}{\mathbb{P}\{X > x\}} = \infty,$$

implying that (1) does not hold.

### 3.3 A counterexample related to Proposition 2.3

In this section we show that the condition  $\mathbb{P}\{Y > x\} = o(\mathbb{P}\{X > x\}/m(x))$  in Proposition 2.3 cannot be weakened to the more appealing condition  $\mathbb{P}\{Y > x\} = o(\mathbb{P}\{X > x\})$ . To construct a counterexample, we let

$$f_X(x) = \frac{1}{x^2 \log x}$$

be the density of  $X$ . In that case  $\mathbb{P}\{X > x\} \sim \frac{1}{x \log x}$  and  $\mathbb{P}\{U > x\} \sim \frac{1}{x}$ . Further, let

$$f_Y(y) = \begin{cases} f_Y(\sqrt{x_{n+1}}), & x_n \leq y < \sqrt{x_{n+1}} \\ \frac{1}{y^2 \log y \log \log y}, & \sqrt{x_{n+1}} \leq y < x_{n+1} \end{cases}$$

be the density of  $Y$ . Here,  $x_n = \exp\{e^{n^2}\}$ . It is clear that

$$\begin{aligned} \mathbb{P}\{Y > x\} &= \int_x^\infty f_Y(y) dy < \int_x^\infty \frac{1}{y^2 \log y \log \log y} dy \\ &\sim \frac{1}{x \log x \log \log x} = o(\mathbb{P}\{X > x\}). \end{aligned}$$

Also,

$$\begin{aligned} \mathbb{E}\{Y\} &= \sum_{n=0}^{\infty} \left( \int_{x_n}^{\sqrt{x_{n+1}}} + \int_{\sqrt{x_{n+1}}}^{x_{n+1}} \right) y f_Y(y) dy \\ &= \sum_{n=0}^{\infty} f_Y(\sqrt{x_{n+1}}) \frac{x_{n+1} - x_n^2}{2} + \sum_{n=0}^{\infty} (\log \log \log x_{n+1} - \log \log \log \sqrt{x_{n+1}}). \end{aligned}$$

The first sum can be bounded by

$$\sum f_Y(\sqrt{x_{n+1}})x_{n+1} = \sum \frac{1}{\log \sqrt{x_{n+1}} \log \log \sqrt{x_{n+1}}} < \sum \frac{2}{e^{n^2}} < \infty.$$

To bound the second sum, note that

$$\log \log \log x_{n+1} - \log \log \log \sqrt{x_{n+1}} = \log \frac{n^2}{n^2 - \log 2} \sim \frac{\log 2}{n^2}.$$

Thus, the second sum is finite as well and we conclude that  $\mathbb{E}\{Y\} < \infty$ . Summing up all these facts, we have:  $U$  is regularly varying random variable,  $\mathbb{P}\{Y > x\} = o(\mathbb{P}\{X > x\})$ , and  $\mathbb{E}\{Y\} < \infty$ . Now we will show that (1) does not hold.

It is sufficient to show that

$$\lim_{M \rightarrow \infty} \liminf_{x \rightarrow \infty} \int_M^{g(x)x} \frac{\mathbb{P}\{U > \log x - \log y\}}{\mathbb{P}\{U > \log x\}} y d\mathbb{P}\{Y \leq y\} > 0. \quad (6)$$

We have,

$$\begin{aligned} & \int_M^{g(x_n)x_n} \frac{\mathbb{P}\{U > \log x_n - \log y\}}{\mathbb{P}\{U > \log x_n\}} y d\mathbb{P}\{Y \leq y\} > \int_{\sqrt{x_n}}^{g(x_n)x_n} \frac{\mathbb{P}\{U > \log x_n - \log y\}}{\mathbb{P}\{U > \log x_n\}} y d\mathbb{P}\{Y \leq y\} \\ & = \log x_n \int_{\sqrt{x_n}}^{g(x_n)x_n} \frac{1}{\log x_n - \log y} \frac{dy}{y \log y \log \log y} > \log x_n \int_{\sqrt{x_n}}^{g(x_n)x_n} \frac{1}{\log x_n - \log y} \frac{dy}{y \log x_n \log \log x_n} \\ & = \frac{1}{\log \log x_n} \int_{\sqrt{x_n}}^{g(x_n)x_n} \frac{1}{\log x_n - \log y} \frac{dy}{y} = \frac{\log \log \sqrt{x_n} - \log \log \frac{1}{g(x_n)}}{\log \log x_n}. \end{aligned}$$

Now note that we can choose  $g(x)$  which tends to 0 very slowly, e.g.  $g(x) > 1/\log x$ . In that case,

$$\frac{\log \log \sqrt{x_n} - \log \log \frac{1}{g(x_n)}}{\log \log x_n} \rightarrow 1.$$

Therefore, (6) holds which implies that (1) does not hold.

## 4 Application to a random difference equation

Let  $R$  be a random variable satisfying (2), let  $0 \leq M \leq 1$ ,  $\mathbb{P}\{M = 1\} = 0$ , and let  $Q$  be independent of  $M$ . Throughout this section, we assume that  $0 \leq M \leq 1$ ,  $\mathbb{P}\{M = 1\} = 0$ ,  $\mathbb{P}\{Q > x\} = g(x)e^{-\alpha x}$ , with  $\alpha > 0$  and  $g(\log x)$  slowly varying. Our main interest is in obtaining the tail behavior of  $R$ . Before we proceed with our analysis, we mention some related work on this problem. Without dependence and non-negativity assumptions on  $Q$  and  $M$ , logarithmic asymptotics for  $R$  have been obtained by Goldie & Grübel [11]. Precise asymptotics in the present setting, with  $Q$  exponentially distributed, have been obtained in Maulik & Zwart [19].

The goal of this section is to relax the assumption on  $Q$  made in [19], and to give an illustration of the applicability of the results obtained in previous sections: in the next four subsections, we give applications of the four Propositions 2.1–2.4.

### 4.1 An application of Proposition 2.1

If the function  $g(x)$  is bounded away from 0 it is straightforward to obtain the tail behavior of  $R$ , as shown by the following result.

**Proposition 4.1** *Assume that  $\liminf_{x \rightarrow \infty} g(x) > 0$ . Then*

$$\mathbb{P}\{R > x\} \sim \mathbb{E}\{e^{\alpha MR}\} \mathbb{P}\{Q > x\} \quad (7)$$

*if and only if  $\mathbb{E}\{e^{\alpha MQ}\} < \infty$ .*

**Proof.** The proof is similar to the proof of Theorem 5.1 in [19]. Along the lines of Proposition 5.1 of that paper, it can be shown that  $\mathbb{E}\{e^{\alpha MR}\} < \infty$  if and only if  $\mathbb{E}\{e^{\alpha MQ}\} < \infty$ . Taking exponents on both sides in (2) we obtain  $\exp\{R\} = \exp\{MR\} \exp\{Q\}$ . We see that all conditions of Proposition 2.1, with  $L(x) = g(\log x)$ , are satisfied, providing the result. ■

As we shall see below, the case in which  $g(x) \rightarrow 0$  is more challenging.

## 4.2 An application of Proposition 2.2

If  $g(x) \rightarrow 0$ , the main difficulty is to show that  $MR$  is sufficiently light-tailed. In the setting of Proposition 2.2 this is possible under reasonable assumptions:

**Proposition 4.2** *Suppose  $Q \in \mathcal{S}(\alpha)$ ,  $\alpha > 0$ . Then  $\mathbb{P}\{R > x\} \sim \mathbb{E}\{e^{\alpha MR}\} \mathbb{P}\{Q > x\}$ .*

**Proof.** Since  $\mathbb{E}\{e^{\alpha Q}\} < \infty$  and  $R \stackrel{d}{=} MR + Q$ , the statement follows from Proposition 2.2 after we have verified that

$$\mathbb{P}\{MR > x\} = o(\mathbb{P}\{Q > x\}). \quad (8)$$

For this, we first use a similar bounding procedure as in Proposition 5.1 of [19]: Let  $M_n, n \geq 0$  and  $Q_n, n \geq 0$  be mutually independent i.i.d. copies of  $M$  and  $Q$  respectively. Then we can write  $MR \stackrel{d}{=} \sum_{k=1}^{\infty} Q_k \prod_{i=1}^k M_i$ . Define the sequence of random times  $\bar{\tau}_k, k \geq 0$ , as follows. Let  $\bar{\tau}_0 = 0$ , and, for  $k \geq 1$ ,

$$\bar{\tau}_k = \inf\{n > \bar{\tau}_{k-1} : M_n \leq \eta\}.$$

Take  $\eta$  small enough so that  $\mathbb{P}\{M > \eta\} \mathbb{E}\{e^{\alpha Q}\} < 1$  and write

$$MR \stackrel{d}{=} \sum_{k=1}^{\infty} Q_k \prod_{i=1}^k M_i = \sum_{k=1}^{\infty} \sum_{n=\bar{\tau}_{k-1}+1}^{\bar{\tau}_k} Q_n \prod_{i=1}^n M_i \leq M_1 \sum_{k=1}^{\infty} \eta^{k-1} \sum_{n=\bar{\tau}_{k-1}+1}^{\bar{\tau}_k} Q_n. \quad (9)$$

Set  $C_k = \sum_{n=\bar{\tau}_{k-1}+1}^{\bar{\tau}_k} Q_n$ . The sequence  $\{C_k, k \geq 1\}$  is i.i.d. and since  $Q_n \in \mathcal{S}(\alpha)$ , we have, using a well known result on geometric random sums (see e.g. Cline [4])

$$\mathbb{P}\{C_k > x\} \sim c_\eta \mathbb{P}\{Q > x\}, \quad (10)$$

with  $c_\eta$  a finite constant. Set  $R_\eta = \sum_{k=1}^{\infty} \eta^{k-1} C_k$ . Observe that  $R$  is stochastically dominated by  $R_\eta$  and that  $R_\eta \stackrel{d}{=} \eta R_\eta + C_1$ . This is an equation similar to the original equation (2), but with  $Q$  replaced by  $C_1$  and  $M$  replaced by  $\eta$ . We see that

$$\mathbb{E}\{e^{sR_\eta}\} = \prod_{n=1}^{\infty} \mathbb{E}\{e^{s\eta^{n-1} C_n}\},$$

from which it simply follows that there exists a  $\delta > 0$  such that  $\mathbb{E}\{e^{(\alpha+\delta)\eta R_\eta}\} < \infty$ .

Consequently, by the classical version of Breiman's theorem, we obtain that

$$\mathbb{P}\{R_\eta > x\} \sim \mathbb{E}\{e^{\alpha R_\eta}\} \mathbb{P}\{C_1 > x\} \sim \mathbb{E}\{e^{\alpha R_\eta}\} c_\eta \mathbb{P}\{Q > x\}.$$

Since  $M < 1$  a.s. and since  $\mathbb{P}\{Q > x/\zeta\} = o(\mathbb{P}\{Q > x\})$  we conclude from the tail asymptotics for  $R_\eta$  that for any  $\zeta < 1$ ,

$$\begin{aligned} \mathbb{P}\{MR_\eta > x\} &\leq \mathbb{P}\{M > \zeta\}\mathbb{P}\{R_\eta > x\} + \mathbb{P}\{R_\eta > x/\zeta\} \\ &\stackrel{\zeta \uparrow 1}{\leq} o(\mathbb{P}\{Q > x\}). \end{aligned}$$

Since  $R$  is stochastically dominated by  $R_\eta$ , we arrive at (8).  $\blacksquare$

### 4.3 An application of Proposition 2.3

To check the sufficient condition of Proposition 2.3 requires more work. We therefore focus on a special case. To save space, we leave out some of the details which are straightforward or which overlap with similar arguments given before.

**Proposition 4.3** *Assume that  $\mathbb{P}\{1 - M \leq x\} = h(1/x)x^\gamma$ , for some  $\gamma > 0$  and a function  $h$  which is slowly varying at infinity. Assume in addition that  $\mathbb{P}\{Q > x\} \sim \ell(x)x^{-\beta}e^{-\alpha x}$ , with  $\ell$  a slowly varying function and  $\beta \in (0, 1)$ . Then  $\mathbb{E}\{e^{\alpha MR}\} < \infty$  and  $\mathbb{P}\{R > x\} \sim \mathbb{E}\{e^{\alpha MR}\}\mathbb{P}\{Q > x\}$  if  $\beta + \gamma > 1$ .*

The proof of this proposition relies on the following lemma:

**Lemma 4.1** *Under the assumptions of the above proposition, for any  $\eta \in [0, 1)$ ,*

$$\mathbb{P}\{MQ > x \mid M > \eta\} \sim \frac{1}{\mathbb{P}\{M > \eta\}} \Gamma(1 + \gamma)(\alpha)^{-\gamma} h(x)\ell(x)x^{-\gamma-\beta}e^{-\alpha x}.$$

**Proof.** Note that the tail  $1/(1 - M)$  is regularly varying at infinity with index  $-\gamma$ . Therefore,  $\mathbb{P}\{1/(1 - M) > x\} \sim \mathbb{P}\{1/(1 - M) > x + 1\} = \mathbb{P}\{M/(1 - M) > x\}$  for  $x \rightarrow \infty$ . Consequently, if we define  $Y = (1 - M)/M$ , then  $\mathbb{P}\{Y \leq x\} \sim \mathbb{P}\{1 - M \leq x\} = h(1/x)x^\gamma$  as  $x \downarrow 0$ . Let  $w(s)$  be the LST of  $Y$ . By Fellers Tauberian theorem (see Theorem 1.7.1. in [1]), we obtain that, as  $s \rightarrow \infty$ ,

$$w(s) \sim \Gamma(1 + \gamma)h(s)s^{-\gamma}. \tag{11}$$

We see that  $M = 1/(Y + 1)$ , so that

$$\begin{aligned} \mathbb{P}\{MQ > x\} &= \mathbb{P}\{Q > x(Y + 1)\} \\ &= e^{-\alpha x}x^{-\beta}\ell(x) \int_0^\infty \frac{\ell(x(y + 1))}{\ell(x)}(y + 1)^{-\beta}e^{-\alpha xy}d\mathbb{P}\{Y \leq y\} \end{aligned}$$

Now, note that, for some  $\epsilon > 0$ ,

$$\begin{aligned} \int_0^\infty \frac{\ell(x(y + 1))}{\ell(x)}(y + 1)^{-\beta}e^{-\alpha xy}d\mathbb{P}\{Y \leq y\} &\sim \int_0^1 \frac{\ell(x(y + 1))}{\ell(x)}(y + 1)^{-\beta}e^{-\alpha xy}d\mathbb{P}\{Y \leq y\} + o(e^{-\epsilon x}) \\ &\sim \int_0^1 (y + 1)^{-\beta}e^{-\alpha xy}d\mathbb{P}\{Y \leq y\} + o(e^{-\epsilon x}) \\ &\sim \int_0^\infty (y + 1)^{-\beta}e^{-\alpha xy}d\mathbb{P}\{Y \leq y\} + o(e^{-\epsilon x}). \end{aligned}$$

In the first step, we used the fact that the contribution of the integral from the interval  $[1, \infty)$  is exponentially small, and in the second step we applied the uniform convergence theorem for slowly varying functions. From (11), we obtain, as  $x \rightarrow \infty$ ,

$$\begin{aligned} \Gamma(1 + \gamma)h(x)(\alpha x)^{-\gamma} &\sim w(\alpha x) \\ &= \int_0^\infty e^{-\alpha xy} d\mathbb{P}\{Y \leq y\} \\ &\sim \int_0^\infty (y + 1)^{-\beta} e^{-\alpha xy} d\mathbb{P}\{Y \leq y\}. \end{aligned}$$

The last equivalence can be obtained by noting that the main contribution to the asymptotics of the integral comes from  $y \in [0, \delta]$ , with  $\delta$  arbitrarily small, implying that  $(y + 1)^{-\beta}$  can be made arbitrary close to 1 for  $y \in [0, \delta]$ . Combining these results, we obtain the statement of the lemma for  $\eta = 0$ . The extension to general  $\eta$  is straightforward.  $\blacksquare$

### Proof of Proposition 4.3

Let  $M^\eta$  be a random variable distributed as  $M \mid M > \eta$ . Since  $c\ell(x)h(x)x^{-\gamma-\beta} \in \mathcal{S}_d$  for any constant  $c > 0$ , we have by Theorem 2.1 of [17] and the above lemma, that  $M^\eta Q \in \mathcal{S}(\alpha)$  for any  $\eta < 1$ . Now we proceed similarly, but slightly different as in the proof of Proposition 4.2; the notation introduced in that proof is used here as well. We can bound  $MR$  in a similar but slightly more precise way as in (9) to obtain

$$MR \stackrel{d}{=} \sum_{k=1}^{\infty} Q_k \prod_{i=1}^k M_i = \sum_{k=1}^{\infty} \sum_{n=\bar{\tau}_{k-1}+1}^{\bar{\tau}_k} Q_n \prod_{i=1}^n M_i \leq \sum_{k=1}^{\infty} \eta^{k-1} \sum_{n=\bar{\tau}_{k-1}+1}^{\bar{\tau}_k} M_n Q_n. \quad (12)$$

Define  $\bar{C}_k = \sum_{n=\bar{\tau}_{k-1}+1}^{\bar{\tau}_k} M_n Q_n$ . It is clear that  $\bar{C}_k, k \geq 1$  is an i.i.d. sequence. Let  $M_i^\eta, i \geq 1$  be an i.i.d sequence independent of everything else such that  $M_1^\eta \stackrel{d}{=} M_1 \mid M_1 > \eta$ . Then, we see that

$$\bar{C}_1 \leq \sum_{n=1}^{\bar{\tau}_1} M_n^\eta Q_n.$$

Since  $M_n^\eta Q_n \in \mathcal{S}(\alpha)$ , we can proceed as in the proof of Proposition 4.2: Since  $\bar{\tau}_1$  is independent of the sequence  $(M_n^\eta)$ ,  $\bar{C}_1$  has the same tail behavior as  $M_1^\eta Q_1$ . As before,  $R$  is stochastically smaller than  $R_\eta = \sum_{n=1}^{\infty} \eta^{n-1} \bar{C}_n$ , and as before one can derive that  $R_\eta$  has the same tail behavior of  $\bar{C}_1$ . We conclude that  $\mathbb{P}\{MR > x\} = O(\mathbb{P}\{MQ > x\})$ . This enables us to apply Proposition 2.3: Define  $X = \exp\{Q\}$ . Then

$$\begin{aligned} m(x) &= \int_0^x t^{\alpha-1} \mathbb{P}\{X > t\} dt \\ &= \int_0^x t^{-1} \ell(\log t) (\log t)^{-\beta} dt \\ &\sim \int_{\log t=0}^{\log x} \ell(\log t) (\log t)^{-\beta} d \log t \\ &\sim \frac{1}{1-\beta} (\log x)^{1-\beta} \ell(\log x), \end{aligned}$$

where we applied Karamata's theorem in the last step. Thus,  $m(x)/\mathbb{P}\{X > x\} \sim \frac{1}{1-\beta} x^\alpha \log x$ . Writing  $Y = \exp\{MR\}$  we see that, as  $x \rightarrow \infty$ ,

$$\begin{aligned} \mathbb{P}\{Y > x\} &= O(\mathbb{P}\{\exp\{MQ\} > x\}) \\ &= O(x^{-\alpha} (\log x)^{-\gamma-\beta} \ell(\log x) h(\log x)), \end{aligned}$$

which implies the condition (5) since  $\gamma + \beta > 1$ . We thus conclude from Proposition 2.3 that (1) holds for our choice of  $X$  and  $Y$ , which implies our assertion. ■

#### 4.4 An application of Proposition 2.4

If  $\alpha = 0$ ,  $Q$  is heavy-tailed. Results for regularly varying  $Q$  can be found in [12, 15, 20]. Here we focus on the case that the tail of  $Q$  is lighter than any power tail.

**Proposition 4.4** *If  $Q \in \mathcal{S}$  and in the domain of attraction of the Gumbel law, then*

$$\mathbb{P}\{R > x\} \sim \mathbb{P}\{Q > x\}.$$

This result is fundamentally different from the case in which  $Q$  is regularly varying with index  $-\alpha$ , in which case it is known from the above references that  $\mathbb{P}\{R > x\} \sim \frac{1}{1 - \mathbb{E}\{M^\alpha\}} \mathbb{P}\{Q > x\}$ .

**Proof.** By using the same arguments as in Proposition 4.2 (note that  $Q \in \mathcal{S}(0)$ ) we obtain that  $\mathbb{P}\{R > x\} \leq \mathbb{P}\{R_\eta > x\}$ , with  $R_\eta = \sum_{n=1}^{\infty} \eta^{n-1} C_n$ , with  $\mathbb{P}\{C_1 > x\} \sim c(\eta) \mathbb{P}\{Q > x\}$ . This implies that  $C_1$  is subexponential and in the domain of attraction of the Gumbel distribution as well. This allows us to apply Lemma A3.27 of [8] to obtain that  $\mathbb{P}\{R_\eta > x\} \sim \mathbb{P}\{C_1 > x\}$ . Thus,  $\mathbb{P}\{MR > x\}$  is asymptotically smaller than  $c(\eta) \mathbb{P}\{MQ > x\}$ . Since  $Q$  is in the domain of attraction of the Gumbel law, the tail of  $Q$  is also of rapid variation, implying that  $\mathbb{P}\{MQ > x\} = o(\mathbb{P}\{Q > x\})$ . Hence also  $\mathbb{P}\{MR > x\} = o(\mathbb{P}\{Q > x\})$ . The proof is now completed by applying Proposition 2.4. ■

## 5 Concluding remarks

In this paper, we derived several extensions of Breiman's theorem, by introducing specific assumptions on the slowly varying function  $L$ . An interesting question which we did not resolve is whether the condition that the slowly varying function  $L$  is also of  $\Pi$ -variation (see Chapter 3 of [1]) would be sufficient for (1) to hold. That the assumptions on  $L$  are in some sense necessary was illustrated in Section 3. In Section 4, we applied our results in Section 2 to analyze the equation  $R \stackrel{d}{=} MR + Q$ . We assumed that  $Q$  and  $M$  are independent, which may be too restrictive in some applications. It would therefore be interesting to extend Breiman's theorem to the case where  $X$  and  $Y$  are dependent. A partial result in this direction has recently been obtained in Maulik & Resnick [18].

### Acknowledgments

The research of the authors is supported by the Dutch BSIK project (*BRICKS*) and the EURO-NGI project. In addition, the research of the second author is supported by an NWO-VENI grant.

## References

- [1] Bingham, N., Goldie, C., Teugels, J. (1987). *Regular Variation*. Cambridge University Press, Cambridge.
- [2] Breiman, L. (1965). On some limit theorems similar to the arc-sin law. *Theory of Probability and its Applications* **10**, 323–331.

- [3] Cline, D. (1987). Convolution tails, product tails and domains of attraction. *Probability Theory and Related Fields* **72**, 529–557.
- [4] Cline, D. (1987). Convolutions of distributions with exponential and subexponential tails. *Journal of the Australian Mathematical Society Series A* **43**, 347–365.
- [5] Cline, D., Samorodnitsky, G. (1994). Subexponentiality of the product of independent random variables. *Stochastic Processes and their Applications* **49**, 75–98.
- [6] Denisov, D., Foss, S., Korshunov, D. (2004). Tail asymptotics for the supremum of a random walk when the mean is not finite. *Queueing Systems* **46**, 15–33.
- [7] Embrechts, P., Goldie, C. (1980). On closure and factorization properties of subexponential and related distributions. *Journal of the Australian Mathematical Society Series A* **29**, 243–256.
- [8] Embrechts, P., Klüppelberg, C., Mikosch, T. (1997). *Modelling Extremal Events*. Springer, New York.
- [9] Foss, S., Korshunov, D. (2005). Lower limits and equivalences for convolution tails. Technical report, Heriot-Watt University.
- [10] Goldie, C. (1991). Implicit renewal theory and tails of solutions of random equations. *Annals of Applied Probability* **1**, 126–166.
- [11] Goldie, C., Grübel, C. (1996). Perpetuities with thin tails. *Advances in Applied Probability* **28**, 463–480.
- [12] Grey, D.R. (1994). Regular variation in the tail behaviour of solutions of random difference equations. *Annals of Applied Probability* **4**, 169–183.
- [13] Gomes, I.M., de Haan, L., Pestana, D. (2004). Joint exceedances of the arch process. *Journal of Applied Probability* **41**, 919–926.
- [14] Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Mathematica* **131**, 207–248.
- [15] Konstantinides, D., Mikosch, T. (2005). Large deviations and ruin probabilities for solutions to stochastic recurrence equations with heavy-tailed innovations. *Annals of Probability* **33**, 1992–2035.
- [16] Klüppelberg, C. (1988). Subexponential distributions and integrated tails. *Journal of Applied Probability* **25**, 132–141.
- [17] Klüppelberg, C. (1989). Subexponential distributions and characterizations of related classes. *Probability Theory and Related Fields* **82**, 259–269.
- [18] Maulik, K., Resnick, S. (2005). Characterizations and examples of hidden regular variation. *Extremes* **7**, 31–67.
- [19] Maulik, K., Zwart, B. (2006). Tail asymptotics of exponential functionals of Lévy processes. *Stochastic Processes and their Applications*, to appear.
- [20] Resnick, S.I., Willekens, E. (1991). Moving averages with random coefficients and random coefficient autoregressive models. *Stochastic Models* **7**, 511–525.