# On a theorem of Erdős, Rubin, and Taylor on choosability of complete bipartite graphs

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Submitted: April 10, 2002; Accepted: August 13, 2002.
MR Subject Classifications: 05C15, 05C65

**Abstract**

Erdős, Rubin, and Taylor found a nice correspondence between the minimum order of a complete bipartite graph that is not $r$-choosable and the minimum number of edges in an $r$-uniform hypergraph that is not 2-colorable (in the ordinary sense). In this note we use their ideas to derive similar correspondences for complete $k$-partite graphs and complete $k$-uniform $k$-partite hypergraphs.

## 1 Introduction

Let $m(r,k)$ denote the minimum number of edges in an $r$-uniform hypergraph with chromatic number greater than $k$ and $N(k,r)$ denote the minimum number of vertices in a $k$-partite graph with list chromatic number greater than $r$.

Erdős, Rubin, and Taylor [6, p. 129] proved the following correspondence between $m(r,2)$ and $N(2,r)$.

**Theorem 1** *For every $r \geq 2$, $m(r,2) \leq N(2,r) \leq 2m(r,2)$.*

This nice result shows close relations between ordinary hypergraph 2-coloring and list coloring of complete bipartite graphs. Note that $m(r,2)$ was studied in [2, 3, 4, 9, 10]. Using known bounds on $m(r,2)$, Theorem 1 yields the corresponding bounds for $N(2,r)$:

$$c\ 2^r\sqrt{\frac{r}{\ln r}} \leq N(2,r) \leq C\ 2^r r^2.$$

*This work was partially supported by the NSF grant DMS-0099608 and the Dutch-Russian Grant NWO-047-008-006.

Theorem 1 can be extended in a natural way in two directions: to complete $k$-partite graphs and to $k$-uniform $k$-partite hypergraphs. In this note we present these extensions (using the ideas of Erdős, Rubin, and Taylor).

A vertex $t$-coloring of a hypergraph $H$ is *panchromatic* if each of the $t$ colors is used on every edge of $G$. Thus, an ordinary 2-coloring is panchromatic. Some results on the existence of panchromatic colorings for hypergraphs with few edges can be found in [8]. Let $p(r,k)$ denote the minimum number of edges in an $r$-uniform hypergraph not admitting any panchromatic $k$-coloring. Note that $p(r,2) = m(r,2)$. The first extension of Theorem 1 is the following.

**Theorem 2** *For every $r \geq 2$ and $k \geq 2$, $p(r,k) \leq N(k,r) \leq k\ p(r,k)$.*

It follows from Alon's results in [1] that for some $c_2 > c_1 > 0$ and every $r \geq 2$ and $k \geq 2$,

$$\exp\{c_1 r/k\} \leq N(k,r) \leq k \exp\{c_2 r/k\}.$$

Therefore, by Theorem 2 we get reasonable bounds on $p(r,k)$ for fixed $k$ and large $r$:

$$\exp\{c_1 r/k\}/k \leq p(r,k) \leq k \exp\{c_2 r/k\}.$$

Note that the lower bound on $p(r,k)$ with $c_1 = 1/4$ follows also from Theorem 3 of the seminal paper [5] by Erdős and Lovász.

We say that a $k$-uniform hypergraph $G$ is *$k$-partite*, if $V(G)$ can be partitioned into $k$ sets so that every edge contains exactly one vertex from every part. Let $Q(k,r)$ denote the minimum number of vertices in a $k$-partite $k$-uniform hypergraph with list chromatic number greater than $r$. Note that $Q(2,r) = N(2,r)$.

**Theorem 3** *For every $r \geq 2$ and $k \geq 2$, $m(r,k) \leq Q(k,r) \leq k\ m(r,k)$.*

From [4] and [7] we know that

$$c_1 k^r \left(\frac{r}{\ln r}\right)^{1-1/\lfloor 1+\log_2 k \rfloor} \leq m(r,k) \leq c_2 k^r r^2 \log k.$$

Thus, Theorem 3 yields that

$$c_1 k^r \left(\frac{r}{\ln r}\right)^{1-1/\lfloor 1+\log_2 k \rfloor} \leq Q(k,r) \leq c_2 k^{r+1} r^2 \log k.$$

# 2 Proof of Theorem 2

Let $H = (V,E)$ be an $r$-uniform hypergraph not admitting any panchromatic $k$-coloring with $E = \{e_1, \ldots, e_{p(r,k)}\}$. Consider the complete $k$-partite graph $G = (W,A)$ with parts $W_1, \ldots, W_k$ and $W_i = \{w_{i,1}, \ldots, w_{i,|E|}\}$ for $i = 1, \ldots, k$. The ground set for lists will be $V$. Recall that every $e_i$ is an $r$-subset of $V$. For every $i = 1, \ldots, k$ and $j = 1, \ldots, |E|$, assign to $w_{i,j}$ the list $L(w_{i,j}) = e_j$.

Assume that $G$ has a coloring $f$ from the lists. Since $G$ is a complete $k$-partite graph, every color $v$ is used on at most one part. Then $f$ produces a $k$-coloring $g_f$ of $V$ as follows: we let $g_f(v)$ be equal to the index $i$ such that $v = f(w_{i,j})$ for some $j$ or be equal to 1 if there is no such $w_{i,j}$ at all. Since for every $j$ all vertices in $\{w_{1,j}, w_{2,j}, \ldots, w_{k,j}\}$ must get different colors, $g_f$ is a panchromatic $k$-coloring of $H$, a contradiction. This proves that $N(k, r) \leq k\ p(r, k)$.

Now, consider a complete $k$-partite graph $G = (W, A)$ with parts $W_1, \ldots, W_k$ and $|W| < p(r, k)$. Let $L$ be an arbitrary $r$-uniform list assignment for $W$. Let $H = (V, E)$ be the hypergraph with $V = \bigcup_{w \in W} L(w)$ and $E = \{L(w) \mid w \in W\}$. Since $|E| = |W| < p(r, k)$, there exists a panchromatic $k$-coloring $g$ of $H$. Define the coloring $f_g$ of $W$ as follows: if $w \in W_i$, choose in the edge $L(w)$ of $H$ any vertex $v$ with $g(v) = i$ and let $f_g(w) = v$. Then vertices in different $W_i$ cannot get the same color, and $f$ is a coloring from the lists of vertices in $G$. This proves that $N(k, r) \geq p(r, k)$.

# 3 Proof of Theorem 3

Let $H = (V, E)$ be an $r$-uniform hypergraph not admitting any $k$-coloring with $E = \{e_1, \ldots, e_{m(r,k)}\}$. Consider the complete $k$-partite $k$-uniform hypergraph $G = (W, A)$ with parts $W_1, \ldots, W_k$ and $W_i = \{w_{i,1}, \ldots, w_{i,|E|}\}$ for $i = 1, \ldots, k$. The ground set for lists will be $V$. Recall that every $e_i$ is an $r$-subset of $V$. For every $i = 1, \ldots, k$ and $j = 1, \ldots, |E|$, assign $w_{i,j}$ the list $L(w_{i,j}) = e_j$.

Assume that $G$ has a coloring $f$ from the lists. Note that no color $v$ is present on every $W_i$, since otherwise $G$ would have an edge with all vertices of color $v$. Thus, $f$ produces a $k$-coloring $g_f$ of $V$ as follows: we let $g_f(v)$ be equal to the smallest $i$ such that $v$ is not a color of any vertex in $W_i$. Assume that $g_f$ is not a proper coloring, i.e., that some $e_j$ is monochromatic of some color $i$ under $g_f$. But some $v' \in e_j$ must be $f(w_{i,j})$, and therefore $g_f(v') \neq i$, a contradiction. This proves that $Q(k, r) \leq k\ m(r, k)$.

Now, consider a complete $k$-partite $k$-uniform hypergraph $G = (W, A)$ with parts $W_1, \ldots, W_k$ and $|W| < Q(r, k)$. Let $L$ be an arbitrary $r$-uniform list for $W$. Let $H = (V, E)$ be the hypergraph with $V = \bigcup_{w \in W} L(w)$ and $E = \{L(w) \mid w \in W\}$. Since $|E| = |W| < Q(r, k)$, there exists a $k$-coloring $g$ of $H$. Define the coloring $f_g$ of $W$ as follows: if $w \in W_i$, choose the next number $i'$ after $i$ in the cyclic order $1, 2, \ldots, k$ such that there is a vertex $v' \in L(w)$ with $g(v') = i'$ and let $f_g(w) = v'$. Since $L(w)$ is not monochromatic in $g$, we have $i' \neq i$. On the other hand, no $v$ with $g(v) = i'$ will be used to color a $w \in W_{i'}$. Thus $f_g$ is a proper coloring of $G$. This proves that $Q(k, r) \geq m(r, k)$.

**Acknowledgment.** I thank both referees for the helpful comments.

# References

[1] N. Alon, Choice number of graphs: a probabilistic approach, *Combinatorics, Probability and Computing,* **1** (1992), 107–114.

[2] J. Beck, On 3-chromatic hypergraphs, *Discrete Math.* **24** (1978), 127–137.

[3] P. Erdős, On a combinatorial problem, I, *Nordisk. Mat. Tidskrift*, **11** (1963), 5–10.

[4] P. Erdős, On a combinatorial problem, II, *Acta Math. Hungar.*, **15** (1964), 445–447.

[5] P. Erdős, L. Lovász, Problems and Results on 3-chromatic hypergraphs and some related questions, In *Infinite and Finite Sets*, A. Hajnal et. al., editors, Colloq. Math. Soc. J. Bolyai, **11**, North Holland, Amsterdam, 609–627, 1975.

[6] P. Erdős, A.L. Rubin and H. Taylor, Choosability in graphs, Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium XXVI (1979), 125–157.

[7] A.V. Kostochka. Coloring uniform hypergraphs with few colors, submitted.

[8] A.V. Kostochka and D. R. Woodall, Density conditions for panchromatic colourings of hypergraphs, *Combinatorica,* 21 (2001), 515–541,

[9] J. Radhakrishnan and A. Srinivasan, Improved bounds and algorithms for hypergraph two-coloring, *Random Structures and Algorithms*, **16** ( 2000), 4–32.

[10] J. Spencer, Coloring $n$-sets red and blue, *J. Comb.Theory Ser. A*, **30** (1981), 112–113.