PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 92, Number 4, December 1984

ON A THEOREM OF INGHAM ON NONHARMONIC FOURIER SERIES

ROBERT M. YOUNG

ABSTRACT. A well-known result due to Ingham [3] shows that the system of complex exponentials $\{e^{i\lambda_n t}\}$ is a basic sequence in $L^2(-\pi,\pi)$ whenever $\lambda_{n+1} - \lambda_n \geq \gamma > 1$. In this note, we show that the system need not be basic if $\lambda_{n+1} - \lambda_n > 1$.

1. Introduction. Let $\{\lambda_n\}$ be an increasing sequence of real numbers. A well-known result due to Ingham [**3**] states that if $\lambda_{n+1} - \lambda_n \geq \gamma > 1$ then the series $\sum c_n e^{i\lambda_n t}$ converges in $L^2(-\pi,\pi)$ whenever $\sum |c_n|^2 < \infty$ and, moreover,

$$A \sum |c_n|^2 \leq \left\| \sum c_n e^{i\lambda_n t} \right\|^2 \leq B \sum |c_n|^2 . \tag{1}$$

Here, A and B are positive constants depending only on γ. (That the right-hand inequality is valid whenever $\gamma > 0$ appears to have been first proved by Titchmarsh [**10**].)

Ingham showed that his result is the best possible in the sense that if $\gamma = 1$ then the left-hand inequality cannot obtain. A counterexample is provided by the sequence $\{\lambda_n\}$ where

$$\lambda_n = \begin{cases} n - 1/4, & n > 0, \\ n + 1/4, & n < 0. \end{cases} \tag{2}$$

It follows readily from (1) that the system of exponentials $\{e^{i\lambda_n t}\}$ is a *basic sequence* in $L^2(-\pi,\pi)$, that is, a basis for its closed linear span S. Accordingly, each function f in S has a unique representation

$$f(t) = \sum c_n e^{i\lambda_n t} \quad \text{(in the mean)}.$$

The study of such *nonharmonic Fourier series* was initiated by Paley and Wiener [**5**] who showed that the system $\{e^{i\lambda_n t}\}$ is a basis for $L^2(-\pi,\pi)$ whenever the λ_n are sufficiently close to the integers. Since then the theory has been generalized in many ways and in many different directions (see, e.g., [**2**, **6**, **8**, **11**] and the references therein).

Condition (1), while tractable, is a stringent requirement to place on a basic sequence. Nevertheless, we show in Theorem 1 that the right-hand inequality must obtain for every basic sequence of exponentials. At present, there is no known example of such a sequence for which the left-hand inequality does not also obtain. Theorem 2 further dramatizes the strength of Ingham's result by showing that the slightly weaker separation condition $\lambda_{n+1} - \lambda_n > 1$ cannot even guarantee that the system $\{e^{i\lambda_n t}\}$ is a basic sequence in $L^2(-\pi,\pi)$.

Received by the editors December 20, 1983.
1980 *Mathematics Subject Classification*. Primary 42C99; Secondary 46B15.
Key words and phrases. Nonharmonic Fourier series, basic sequence, biorthogonal system.

0002-9939/84 $1.00 + $.25 per page

THEOREM 1. *If $\{\lambda_n\}$ is an increasing sequence of real numbers for which the system of exponentials $\{e^{i\lambda_n t}\}$ is a basic sequence in $L^2(-\pi,\pi)$, then the inequality $\|\sum c_n e^{i\lambda_n t}\|^2 \le B\sum |c_n|^2$ is valid for some constant B and all square summable sequences of scalars $\{c_n\}$.*

THEOREM 2. *There exists a sequence $\{\mu_n\}$ of real numbers satisfying $\mu_{n+1} - \mu_n > 1$ such that $\{e^{i\mu_n t}\}$ is exact in $L^2(-\pi,\pi)$ and yet not a basis.*

Recall that $\{e^{i\mu_n t}\}$ is said to be *exact* if it is complete but fails to be complete upon the removal of a single term.

2. Proof of Theorem 1. We need only show that the λ_n are separated, i.e., that $\lambda_{n+1} - \lambda_n \ge \gamma$ for some positive constant γ; the result will then follow from [**10**].

Let S be the closure in $L^2(-\pi,\pi)$ of the linear span of the system $\{e^{i\lambda_n t}\}$, and let $\{f_n\}$, $f_n \in S^*$, be the associated sequence of coefficient functionals. Then $\|e^{i\lambda_n t}\|\,\|f_n\| \le M$ for some constant M and all values of n (see, e.g., [**9**, p. 20]). Since each λ_n is real, $\|e^{i\lambda_n t}\| = 1$ and hence $\|f_n\| \le M$. Now $f_n(e^{i\lambda_n t} - e^{i\lambda_{n+1} t}) = 1$ so that $\|f_n\|\,\|e^{i\lambda_n t} - e^{i\lambda_{n+1} t}\| \ge 1$. Accordingly, $\|e^{i\lambda_n t} - e^{i\lambda_{n+1} t}\| \ge 1/M$ and the existence of γ follows.

3. Proof of Theorem 2. The system $\{e^{i\lambda_n t}\}$ where the λ_n are given by (1), is known to be exact in $L^2(-\pi,\pi)$ [**4**, p. 67]. We begin by showing that it is not a basis. Suppose it were. Then we could write

$$1 = \sum c_n e^{i\lambda_n t} \quad \text{(in the mean).} \tag{3}$$

To compute the c_n, we shall make use of the Paley-Wiener space P consisting of all entire functions of exponential type at most π that are square integrable on the real axis. The inner product of two functions F and G in P is, by definition,

$$(F,G) = \int_{-\infty}^{\infty} F(x)\overline{G(x)}\,dx.$$

By virtue of the Paley-Wiener theorem, the complex Fourier transform

$$f(t) \to \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{izt}\,dt$$

is an isometric isomorphism from $L^2(-\pi,\pi)$ onto all of P. The exponentials $e^{i\lambda_n t}$ are sent to the "reproducing" functions

$$K_n(z) = \frac{\sin \pi(z-\lambda_n)}{\pi(z-\lambda_n)}$$

which then consistute a basis for P. Let $\{g_n\}$ be biorthogonal to $\{K_n\}$. When the Fourier transform is applied to (3), we obtain

$$\frac{\sin \pi z}{\pi z} = \sum c_n K_n(z)$$

where $c_n = ((\sin \pi z)/\pi z, g_n) = g_n(0)$.

Let

$$F(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right).$$

It was shown by Levinson [**4**, p. 67] that

$$F(z) = c\int_{-\pi}^{\pi} \left(\cos^{-1/2}\frac{1}{2}t\right) e^{izt}\,dt.$$

Since $\cos^{-1/2}\frac{1}{2}t$ is integrable over $(-\pi,\pi)$, it follows that $F(z)$ is bounded along the real axis and each of the functions

$$F_n(z) = F(z)/F'(\lambda_n)(z-\lambda_n)$$

therefore belongs to P. Since $(F_n, K_m) = F_n(\lambda_m) = \delta_{mn}$, it follows that $F_n = g_n$. Thus

$$c_n = F_n(0) = -1/\lambda_n F'(\lambda_n),$$

and (3) becomes

$$1 = -\sum_{n\neq 0}\frac{e^{i\lambda_n t}}{\lambda_n F'(\lambda_n)} = -2\sum_{n=1}^{\infty}\frac{\cos\lambda_n t}{\lambda_n F'(\lambda_n)} \tag{4}$$

since $zF'(z)$ is even. It is to be shown that the series on the right does not converge in $L^2(-\pi,\pi)$.

Now the values of $F'(\lambda_n)$ were determined explicitly in [**7**]:

$$F'(\lambda_n) = (-1)^n\Gamma^2\left(\frac{3}{4}\right)\frac{\Gamma(n)}{\Gamma\left(n+\frac{1}{2}\right)} \qquad (n = 1,2,3,\ldots.).$$

Using the asymptotic formula $\Gamma(n)/\Gamma(n+\frac{1}{2}) = 1/\sqrt{n} + O(1/n^{3/2})$ [**1**], we have

$$F'(\lambda_n) = A(-1)^n\left\{1/\sqrt{\lambda_n}+\varepsilon_n\right\} \quad \text{where } \varepsilon_n = O(1/n^{3/2}).$$

A straightforward calculation then shows that the difference between the series on the right in (4) and

$$\frac{1}{A}\sum_{n=1}^{\infty}\frac{(-1)^n\cos\lambda_n t}{\sqrt{\lambda_n}}$$

is uniformly convergent on $[-\pi,\pi]$. Accordingly, we need only show that this series diverges in $L^2(-\pi,\pi)$.

Let $x = \pi - t$ $(0 \le t \le \pi)$. Then

$$\sum_{n=1}^{\infty}\frac{(-1)^n\cos\lambda_n t}{\sqrt{\lambda_n}} = \sum_{n=1}^{\infty}\frac{(-1)^n\cos(\pi n - \pi/4 - \lambda_n x)}{\sqrt{\lambda_n}} = \sum_{n=1}^{\infty}\frac{\cos(\pi/4+\lambda_n x)}{\sqrt{\lambda_n}}.$$

For $N = 1,2,3,\ldots$, let $\delta_N = \pi/16N$. If $x\in[0,\delta_N]$, then $\pi/4+\lambda_n x \in [\pi/4, 3\pi/8]$ whenever $1\le n\le 2N$ and hence $\cos(\pi/4+\lambda_n x)\ge A > 0$. Thus

$$\sum_{N}^{2N}\frac{\cos(\pi/4+\lambda_n x)}{\sqrt{\lambda_n}} \ge A\sum_{N}^{2N}\frac{1}{\sqrt{n-1/4}} \ge A\frac{N+1}{\sqrt{2N-1/4}} \ge B\sqrt{N}$$

where B is a positive constant independent of N. Accordingly,

$$\left\|\sum_{N}^{2N}\frac{(-1)^n\cos\lambda_n t}{\sqrt{\lambda_n}}\right\|^2 \geq \left\|\sum_{N}^{2N}\frac{\cos(\pi/4+\lambda_n x)}{\sqrt{\lambda_n}}\right\|^2_{L^2(0,\delta_N)}$$
$$\geq \frac{1}{2\pi}B^2N\delta_N = \frac{B^2}{32}$$

for all N. Thus the series in (3) does not converge in $L^2(-\pi,\pi)$, and the system $\{e^{i\lambda_n t}\}$ fails to be a basis.

Let $\{\varepsilon_1,\varepsilon_2,\ldots\}$ be a decreasing sequence of positive numbers such that $\varepsilon_1 < \frac{1}{4}$ and $\sum\varepsilon_n < \infty$. It is to be shown that if $\mu_n = \lambda_n - \varepsilon_n$, $\mu_{-n} = -\mu_n$ $(n = 1,2,\ldots)$, then the system $\{e^{i\mu_n t}\}$ satisfies the conclusions of the theorem.

Clearly, $\mu_{n+1} - \mu_n > 1$. Since $\sum|\lambda_n - \mu_n| < \infty$ and $\{e^{i\lambda_n t}\}$ is exact, so is $\{e^{i\mu_n t}\}$ [**6**]. It remains only to show that $\{e^{i\mu_n t}\}$ is not a basis for $L^2(-\pi,\pi)$. Suppose it were. Then the system $\{h_n(t)\}$, biorthogonal to $\{e^{i\mu_n t}\}$, would satisfy $\|e^{i\mu_n t}\|\cdot\|h_n\| \leq M$ for some constant M and all values of n. Since each μ_n is real, $\|e^{i\mu_n t}\| = 1$ and hence $\|h_n\| \leq M$. We complete the proof by showing that

$$\sum|\lambda_n - \mu_n| < \infty \Rightarrow \sum\|e^{i\lambda_n t} - e^{i\mu_n t}\| < \infty. \tag{5}$$

The convergence of $\sum\|e^{i\lambda_n t} - e^{i\mu_n t}\|\,\|h_n\|$ will then imply that $\{e^{i\lambda_n t}\}$ is a basis for $L^2(-\pi,\pi)$ (see, e.g., [**9**, p. 94]). The contradiction will prove the theorem.

To establish (5), write

$$e^{i\lambda_n t} - e^{i\mu_n t} = e^{i\lambda_n t}(1 - e^{-i\varepsilon_n t}).$$

Expanding $1 - e^{i\gamma t}$ in an everywhere-convergent Taylor series, we find

$$|e^{i\lambda_n t} - e^{i\mu_n t}| \leq \sum_{K=1}^{\infty}\frac{\varepsilon_n^K t^K}{K!},$$

and hence

$$\sum_n\|e^{i\lambda_n t} - e^{i\mu_n t}\| \leq \sum_{K=1}^{\infty}\frac{\pi^K}{K!}\left(\sum_n\varepsilon_n^K\right) \leq \sum_{K=1}^{\infty}\frac{\pi^K}{K!}\left(\sum\varepsilon_n\right)^K$$
$$= \exp(\pi\sum\varepsilon_n) - 1 < \infty.$$

This completes the proof.

REFERENCES

1. E. T. Copson, *Asymptotic expansions,* Cambridge Univ. Press, 1965.
2. S. V. Hruscev, N. K. Nikolskii and B. S. Pavlov, *Unconditional bases of exponentials and of reproducing kernels,* Lecture Notes in Math., vol. 864, Springer-Verlag, 1981, pp. 214–335.
3. A. E. Ingham, *Some trigonometrical inequalities with applications to the theory of series,* Math. Z. **41** (1936), 367–379.
4. N. Levinson, *Gap and density theorems,* Amer. Math. Soc. Colloq. Publ., vol. 26, Amer. Math. Soc., Providence, R.I., 1940
5. R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain,* Amer. Math. Soc. Colloq. Publ., vol. 19, Amer. Math. Soc., Providence, R.I., 1934.
6. R. M. Redheffer, *Completeness of sets of complex exponentials,* Adv. in Math. **24** (1977), 1–62.

7. R. M. Redheffer and R. M. Young, *Completeness and basis properties of complex exponentials,* Trans. Amer. Math. Soc. **277** (1983), 93–111.
8. L. Schwartz, *Etude des sommes d'expontielles,* Publ. Inst. Math. Univ. Strasbourg, 2nd ed., vol. V, Hermann, Paris, 1959.
9. I. Singer, *Bases in Banach spaces.* I, Springer-Verlag, Berlin and New York, 1970.
10. E. C. Titchmarsh, *A class of trigonometrical series,* J. London Math. Soc. **3** (1928), 300–304.
11. R. M. Young, *An introduction to nonharmonic Fourier series,* Academic Press, New York, 1980.

DEPARTMENT OF MATHEMATICS, OBERLIN COLLEGE, OBERLIN, OHIO 44074